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NORMAL FORM APPROACH TO THE ONE-DIMENSIONAL PERIODIC CUBIC NONLINEAR SCHRÖDINGER EQUATION IN ALMOST CRITICAL FOURIER–LEBESGUE SPACES

By

TADAHIRO OH AND YUZHAO WANG

Abstract. In this paper, we study the one-dimensional cubic nonlinear Schrödinger equation (NLS) on the circle. In particular, we develop a normal form approach to study NLS in almost critical Fourier–Lebesgue spaces. By applying an infinite iteration of normal form reductions introduced by the first author with Z. Guo and S. Kwon (2013), we derive a normal form equation which is equivalent to the renormalized cubic NLS for regular solutions. For rough functions, the normal form equation behaves better than the renormalized cubic NLS, thus providing a further renormalization of the cubic NLS. We then prove that this normal form equation is unconditionally globally well-posed in the Fourier–Lebesgue spaces $\mathcal{F}L^p(\mathbb{T})$, $1 \le p < \infty$. By inverting the transformation, we conclude global well-posedness of the renormalized cubic NLS in almost critical Fourier–Lebesgue spaces in a suitable sense. This approach also allows us to prove unconditional uniqueness of the (renormalized) cubic NLS in $\mathcal{F}L^p(\mathbb{T})$ for $1 \le p \le \frac{3}{2}$.

1 Introduction

1.1 Nonlinear Schrödinger equation.

We consider the following cubic nonlinear Schrödinger equation (NLS) on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\begin{cases} i\partial_t u + \partial_x^2 u \pm |u|^2 u = 0, \\ u|_{t=0} = u_0, \end{cases} \qquad (x, t) \in \mathbb{T} \times \mathbb{R}. \tag{1.1}$$

The equation (1.1) arises from various physical settings such as nonlinear optics and quantum physics. See [37] for the references therein. It is also known to be one of the simplest completely integrable PDEs [38, 1, 2, 17, 27].

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The Cauchy problem (1.1) has been studied extensively both on the real line and on the circle. See [33, 21] for the references therein. In this paper, we study the periodic cubic NLS (1.1) in the Fourier–Lebesgue spaces $\mathcal{F}L^p(\mathbb{T})$ defined via the norm:

$$\|f\|_{\mathcal{F}L^p(\mathbb{T})} := \bigg(\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^p\bigg)^{\frac{1}{p}}$$

with a usual modification when $p = \infty$. For any $2 \leq p \leq q \leq \infty$, we have the following continuous embeddings:

$$\begin{aligned}\mathcal{F}L^1(\mathbb{T}) \hookrightarrow \mathcal{F}L^{q'}(\mathbb{T}) \hookrightarrow \mathcal{F}L^{p'}(\mathbb{T}) \hookrightarrow \mathcal{F}L^2(\mathbb{T})& \\ = L^2(\mathbb{T}) \hookrightarrow \mathcal{F}L^p(\mathbb{T}) \hookrightarrow \mathcal{F}L^q(\mathbb{T}) &\hookrightarrow \mathcal{F}L^\infty(\mathbb{T}).\end{aligned}$$

The space $\mathcal{F}L^1(\mathbb{T})$ is the Wiener algebra. The space $\mathcal{F}L^\infty(\mathbb{T})$ is the space of pseudo-measures, which contains all finite Borel measures on $\mathbb{T}$ but also more singular distributions. See [25]. Our main interest is to study (1.1) in $\mathcal{F}L^p(\mathbb{T})$ for $p \gg 1$.

On the one hand, the cubic NLS (1.1) is known to be globally well-posed in $\mathcal{F}L^2(\mathbb{T}) = L^2(\mathbb{T})$ [6]. On the other hand, combining the known results [19, 21, 34], we can easily show that it is ill-posed in the Fourier–Lebesgue space $\mathcal{F}L^p(\mathbb{T})$ for $p > 2$ in a very strong sense. See Proposition 1.1 below. This necessitates us to renormalize the nonlinearity and consider the following renormalized cubic NLS:

$$\begin{cases} i\partial_t u + \partial_x^2 u \pm \big(|u|^2 - 2 \fint_{\mathbb{T}} |u|^2 dx\big)u = 0 \\ u|_{t=0} = u_0. \end{cases} \tag{1.2}$$

Note that the renormalized cubic NLS (1.2) is “equivalent” to the original cubic NLS (1.1) for smooth solutions in the following sense. For $u \in C(\mathbb{R}; L^2(\mathbb{T}))$, we define the following invertible gauge transformation $\mathcal{G}$ by

$$\mathcal{G}(u)(t) := e^{\mp 2it \fint_{\mathbb{T}} |u(t)|^2 dx} u(t)$$

with its inverse

$$\mathcal{G}^{-1}(u)(t) := e^{\pm 2it \fint_{\mathbb{T}} |u(t)|^2 dx} u(t). \tag{1.3}$$

Then, thanks to the L^2-conservation, it is easy to see that $u \in C(\mathbb{R}; L^2(\mathbb{T}))$ is a solution to (1.1) if and only if $\mathcal{G}(u)$ is a solution to the renormalized cubic NLS (1.2). This renormalization removes a certain singular component from the nonlinearity and, as a result, the renormalized cubic NLS (1.2) behaves better than the cubic NLS (1.1) outside $L^2(\mathbb{T})$. The study of (1.2) outside $L^2(\mathbb{T})$ has attracted much attention in recent years [8, 9, 19, 33, 12, 21, 34, 31, 36].

In [19], Grünrock–Herr adapted the Fourier restriction norm method to the Fourier–Lebesgue space setting and proved local well-posedness of the renormalized cubic NLS (1.2) in $\mathcal{F}L^p(\mathbb{T})$ for $1 \leq p < \infty$ by a standard contraction argument. See also the work by Christ [9]. In [36], by using the completely integrable structure of the equation, we established the following global-in-time a priori bound:

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{\mathcal{F}L^p} \leq C(\|u_0\|_{\mathcal{F}L^p}) \tag{1.4}$$

for any smooth solution u to the renormalized cubic NLS (1.2) and $2 \leq p < \infty$, which implied global well-posedness of (1.2) in $\mathcal{F}L^p(\mathbb{T})$ for $1 \leq p < \infty$.[1]

As a corollary to the local well-posedness of the renormalized cubic NLS in [19], one easily obtains the following non-existence result for the original cubic NLS (1.1) outside $L^2(\mathbb{T})$.

Proposition 1.1. *Let* $2 < p < \infty$ *and* $u_0 \in \mathcal{F}L^p(\mathbb{T}) \setminus L^2(\mathbb{T})$. *Then, for any* $T > 0$, *there exists no distributional solution* $u \in C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$ *to the cubic NLS* (1.1) *such that*

(i) $u|_{t=0} = u_0$,

(ii) *there exist smooth global solutions* $\{u_n\}_{n\in\mathbb{N}}$ *to* (1.1) *such that* $u_n \to u$ *in* $C([-T, T]; \mathcal{D}'(\mathbb{T}))$ *as* $n \to \infty$.

In [21], the first author (with Z. Guo) proved an analogous non-existence result for (1.1) in negative Sobolev spaces. The argument was based on an a priori bound for smooth solutions to the renormalized cubic NLS (1.2) in negative Sobolev spaces and exploiting a fast oscillation in (1.3). The proof of the local well-posedness in [19] yields an a priori bound for smooth solutions to the renormalized cubic NLS (1.2) in $\mathcal{F}L^p(\mathbb{T})$. Then, we can prove Proposition 1.1 by proceeding as in [21, 35]. We omit details.

In the following, we only consider the focusing case (i.e., with the + sign in (1.1) and (1.2)) for simplicity. Our main results equally apply to the defocusing case.

1.2 Main results.

In the following, we introduce two notions of weak solutions. Let $\mathcal{N}(u)$ denote the renormalized nonlinearity in (1.2):[2]

$$\begin{aligned} \mathcal{N}(u) : &= \left(|u|^2 - 2\int_{\mathbb{T}} |u|^2 dx\right) u \\ &= \sum_{n_2 \neq n_1, n_3} \widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)e^{i(n_1-n_2+n_3)x} - \sum_{n\in\mathbb{Z}} |\widehat{u}(n)|^2\widehat{u}(n)e^{inx}. \end{aligned} \tag{1.5}$$

---

[1] For $1 \leq p < 2$, one needs to use the L^2-conservation and a persistence-of-regularity argument. See Appendix A.

[2] Hereafter, we drop the factor of 2π when it plays no role.

We first recall the following notion of **weak solutions in the extended sense**.

Definition 1.2. Let $1 \le p < \infty$ and $T > 0$.

(i) We define a sequence of Fourier cutoff operators to be a sequence of Fourier multiplier operators $\{T_N\}_{N\in\mathbb{N}}$ on $\mathcal{D}'(\mathbb{T})$ with multipliers $m_N : \mathbb{Z} \to \mathbb{C}$ such that

- m_N has a compact support on $\mathbb{Z}$ for each $N \in \mathbb{N}$,
- m_N is uniformly bounded,
- m_N converges pointwise to 1, i.e., $\lim_{N\to\infty} m_N(n) = 1$ for any $n \in \mathbb{Z}$.

(ii) Let $u \in C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$. We say that $\mathcal{N}(u)$ exists and is equal to a distribution $v \in \mathcal{D}'(\mathbb{T} \times (-T, T))$ if for every sequence $\{T_N\}_{N\in\mathbb{N}}$ of (spatial) Fourier cutoff operators, we have

$$\lim_{N\to\infty} \mathcal{N}(T_N u) = v$$

in the sense of distributions on $\mathbb{T} \times (-T, T)$.

(iii) (weak solutions in the extended sense) We say that $u \in C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$ is a weak solution of the renormalized cubic NLS (1.2) in the extended sense if

- $u|_{t=0} = u_0$,
- the nonlinearity $\mathcal{N}(u)$ exists in the sense of (ii) above,
- u satisfies (1.2) in the distributional sense on $\mathbb{T} \times (-T, T)$, where the nonlinearity $\mathcal{N}(u)$ is interpreted as above.

In [8, 9], Christ introduced this notion in studying the renormalized cubic NLS (1.2) in the low regularity setting. See also [20] for a similar notion of weak solutions, where the nonlinearity is defined as a distributional limit of smoothed nonlinearities.

Next, we introduce the following notion of **sensible weak solutions**. See also [36, 14].

Definition 1.3 (sensible weak solutions)**.** Let $1 \le p < \infty$ and $T > 0$. Given $u_0 \in \mathcal{F}L^p(\mathbb{T})$, we say that $u \in C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$ is a sensible weak solution to the renormalized cubic NLS (1.2) on $[-T, T]$ if, for any sequence $\{u_{0,m}\}_{m\in\mathbb{N}}$ of smooth functions tending to u_0 in $\mathcal{F}L^p(\mathbb{T})$, the corresponding (classical) solutions u_m with $u_m|_{t=0} = u_{0,m}$ converge to u in $C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$. Moreover, we impose that there exists a distribution v such that $\mathcal{N}(u_m)$ converges to v in the space-time distributional sense, independent of the choice of the approximating sequence.

It is worth noting that, by using the equation, the convergence of u_m to u in $C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$ implies that $\mathcal{N}(u_m)$ converges to some v in the space-time distributional sense; see (2.9) below. Hence, the last part of Definition 1.3 is not quite necessary. We, however, keep it for clarity.

We point out that these notions of weak solutions in Definitions 1.2 and 1.3 are rather weak. The cubic nonlinearity $\mathcal{N}(u)$ for a weak solution u in the sense of Definitions 1.2 or 1.3 does not directly make sense as a distribution in general and we need to interpret it as a (unique) limit of smoothed nonlinearities $\mathcal{N}(T_N u)$ or the nonlinearities $\mathcal{N}(u_m)$ of smooth approximating solutions u_m. This in particular implies that weak solutions in the sense of Definitions 1.2 or 1.3 do not have to satisfy the equation even in the distributional sense.

On the one hand, sensible weak solutions are unique by definition. On the other hand, weak solutions in the extended sense are not unique in general. In fact, Christ [8] proved non-uniqueness of weak solutions in the extended sense for the renormalized cubic NLS (1.2) in negative Sobolev spaces.

Our main goal in this paper is (i) to develop further the normal form approach to study the (renormalized) cubic NLS, introduced in [22], and provide the solution theory for (1.2) in almost critical Fourier–Lebesgue spaces (Theorem 1.4) in the sense of Definitions 1.2 and 1.3 without using any auxiliary function spaces, in particular, without using the Fourier restriction norm method as in [6, 19] and (ii) to prove unconditional uniqueness of the (renormalized) cubic NLS in $\mathcal{F}L^p(\mathbb{T})$ for $1 \le p < \frac{3}{2}$ (Theorem 1.5). In proving these results, we apply an infinite iteration of normal form reductions and transform the (renormalized) cubic NLS into the so-called normal form equation. We then prove unconditional well-posedness of the normal form equation in $\mathcal{F}L^p(\mathbb{T})$ for any $1 \le p < \infty$; see Theorem 1.9 below.

We now state our main results.

Theorem 1.4. *Let* $1 \le p < \infty$. *Then, the renormalized cubic NLS* (1.2) *on* $\mathbb{T}$ *is globally well-posed in* $\mathcal{F}L^p(\mathbb{T})$

- *in the sense of weak solutions in the extended sense and*
- *in the sense of sensible weak solutions.*

When $1 \le p \le 2$, *the same global well-posedness result applies to the (unrenormalized) cubic NLS* (1.1).

This theorem follows from the local well-posedness by Grünrock–Herr [19], combined with the a priori bound (1.4) from [36]. As pointed out above, however, our main goal is to present an argument independent of the Fourier restriction norm method. We instead employ the normal form approach developed in [22]. Our approach does not involve any auxiliary function spaces and consequently allows us to prove unconditional uniqueness of the (renormalized) cubic NLS in $\mathcal{F}L^{\frac{3}{2}}(\mathbb{T})$ (Theorem 1.5). We point out that the local well-posedness in [19] only yields conditional uniqueness, namely in the class (1.6) below.

In [22], the first author (with Z. Guo and S. Kwon) proved an analogous result in $L^2(\mathbb{T})$ by implementing an infinite iteration of normal form reductions,[3] yielding unconditional uniqueness of the cubic NLS (1.1) in $H^{\frac{1}{6}}(\mathbb{T})$. The proof of Theorem 1.4 is also based on the same normal form approach. See the next subsection. Note that when p is very large, Theorem 1.4 is significantly harder to prove than the L^2-result in [22] due to a much weaker $\mathcal{F}L^p$-topology.

Given $u_0 \in \mathcal{F}L^p(\mathbb{T})$, let u be the global solution to (1.2) with $u|_{t=0} = u_0$ constructed in Theorem 1.4. Then, by the uniqueness of sensible solutions mentioned above, u must coincide with the global solution constructed in [6, 19, 36]. In particular, the solution u belongs to the class

$$C([-T,T];\mathcal{F}L^p(\mathbb{T})) \cap X_p^{0,b}([-T,T]) \tag{1.6}$$

for some $b > \frac{1}{p'}$, where $X_p^{0,b}([-T,T])$ denotes the local-in-time version of the Fourier restriction space $X_p^{0,b}$ adapted to the Fourier–Lebesgue setting. See (A.1) and (A.3) below.

As mentioned above, Theorem 1.4 does not allow us to directly[4] conclude that weak solutions constructed in Theorem 1.4 are distributional solutions to (1.2). For $1 \le p \le \frac{3}{2}$, however, Hausdorff–Young's inequality

$$\mathcal{F}L^p(\mathbb{T}) \subset \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}) \subset L^3(\mathbb{T}),$$

allows us to make sense of the cubic nonlinearity in a direct manner. In this case, we have the following uniqueness statement.

Theorem 1.5. *Let $1 \le p \le \frac{3}{2}$. Then, given any $u_0 \in \mathcal{F}L^p(\mathbb{T})$, the solution u to* (1.1) *or* (1.2) *with $u|_{t=0} = u_0$ constructed in Theorem 1.4 is unique in $C(\mathbb{R};\mathcal{F}L^p(\mathbb{T}))$.*

Namely, unconditional uniqueness holds for both the cubic NLS (1.1) and the renormalized cubic NLS (1.2) in $\mathcal{F}L^p(\mathbb{T})$, provided that $1 \le p \le \frac{3}{2}$. In [22], the first author (with Z. Guo and S. Kwon) proved unconditional uniqueness in $H^{\frac{1}{6}}(\mathbb{T})$ and Theorem 1.5 extends this result to the Fourier–Lebesgue setting. We also mention a recent work by Herr–Sohinger [24] where they proved unconditional uniqueness of the cubic NLS (1.1) in $L^p([-T,T] \times \mathbb{T})$ for $p > 3$. The main difference between unconditional uniqueness and uniqueness for sensible weak solutions is that the former does not assume that a solution comes with a sequence of smooth approximating solutions, while, by definition, sensible weak solutions are equipped with smooth approximating solutions.

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[3] In [22], we only proved well-posedness of the cubic NLS (1.1) in the sense of weak solutions in the extended sense. A small modification of the argument yields well-posedness in the sense of sensible weak solutions. See Section 2.

[4] That is, unless we use the uniqueness property of sensible solutions and conclude that they belong to the class (1.6) by comparing with the solutions constructed in [6, 19, 36].

Remark 1.6. When $p = \infty$, the Fourier–Lebesgue space $\mathcal{F}L^\infty(\mathbb{T})$ does not admit smooth approximations and hence is not suitable for well-posedness study. Given $s \in \mathbb{R}$ and $1 \le p \le \infty$, define $\mathcal{F}L^{s,p}(\mathbb{T})$ by the norm:

$$\|f\|_{\mathcal{F}L^{s,p}} := \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^p_n(\mathbb{Z})}. \tag{1.7}$$

Note that $\mathcal{F}L^p(\mathbb{T}) = \mathcal{F}L^{0,p}(\mathbb{T})$. For $s < -\frac{1}{p}$, we have

$$\mathcal{F}L^\infty(\mathbb{T}) \subset \mathcal{F}L^{s,p}(\mathbb{T})$$

and thus we may wish to study well-posedness in $\mathcal{F}L^{s,p}(\mathbb{T})$ for finite p with $s < -\frac{1}{p}$ since this space admits smooth approximations. On the other hand, the scaling critical regularity for the cubic NLS (1.1) with respect to the Fourier–Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$ is given by $s_{\text{crit}} = -\frac{1}{p}$. In particular, the cubic NLS (1.1) and the renormalized cubic NLS (1.2) are known to be ill-posed in the (super)critical regime.[5] When $s < 0$, it is easy to modify the argument in [7, 10, 12] and show that the solution map is not locally uniformly continuous in $\mathcal{F}L^{s,p}(\mathbb{T})$. Furthermore, when $s \le s_{\text{crit}} = -\frac{1}{p}$, the cubic NLS (1.1) and the renormalized cubic NLS (1.2) admit norm inflation; given any $\varepsilon > 0$, there exist a solution u to (1.1) or (1.2) and $t \in (0, \varepsilon)$ such that

$$\|u(0)\|_{\mathcal{F}L^{s,p}} < \varepsilon \quad \text{and} \quad \|u(t)\|_{\mathcal{F}L^{s,p}} > \varepsilon^{-1}.$$

See [28]. The norm inflation in particular implies discontinuity of the solution map at the trivial function[6] $u \equiv 0$. Lastly, a typical function in $\mathcal{F}L^\infty(\mathbb{T})$ is the Dirac delta function and (1.1) and (1.2) on $\mathbb{T}$ are known to be ill-posed with the Dirac delta function as initial data; see [14]. See also Kenig–Ponce–Vega [26] and Banica–Vega [4, 5] for the works on the cubic NLS (1.1) on the real line with the Dirac delta function as initial data.

Remark 1.7. Following the argument in [22], we can easily extend Theorem 1.4 to $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s > 0$ and $1 \le p < \infty$. Similarly, the unconditional uniqueness result in Theorem 1.5 can be extended to $\mathcal{F}L^{s,p}(\mathbb{T})$ for (i) $s > 0$ and $1 \le p \le \frac{3}{2}$ and (ii) $p > \frac{3}{2}$ and $s > \frac{2p-3}{3p}$. Note that in these ranges of (s, p), we have

$$\mathcal{F}L^{s,p}(\mathbb{T}) \hookrightarrow \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}) \hookrightarrow L^3(\mathbb{T}).$$

---

[5] In fact, it is shown in [28] that the cubic NLS (1.1) and the renormalized cubic NLS (1.2) are ill-posed even in the logarithmically subcritical regime.

[6] One can easily combine the argument in [28, 31] to prove norm inflation at general initial data, concluding discontinuity of the solution map at every function $\mathcal{F}L^{s,p}(\mathbb{T})$, provided that $s \le s_{\text{crit}} = -\frac{1}{p}$.

1.3 Normal form equation. The main idea for proving Theorems 1.4 and 1.5 is to apply an infinite iteration of normal form reductions to (1.2)[7] and transform the equation into a normal form equation (see (1.12) below), which may look more complicated from the algebraic viewpoint but exhibits better analytical properties than the original equation.

Let $S(t) = e^{it\partial_x^2}$ denote the linear Schrödinger propagator. We introduce the interaction representation:

$$\mathbf{u}(t) = S(-t)u(t) = e^{-it\partial_x^2}u(t). \tag{1.8}$$

On the Fourier side, we have $\widehat{\mathbf{u}}(n,t) = e^{in^2t}\widehat{u}(n,t)$. Then, (1.2) can be written as[8]

$$\begin{aligned}\partial_t \widehat{\mathbf{u}}_n &= i \sum_{\substack{n=n_1-n_2+n_3\\ n_2\neq n_1,n_3}} e^{i\Phi(\bar{n})t}\widehat{\mathbf{u}}(n_1)\overline{\widehat{\mathbf{u}}(n_2)}\widehat{\mathbf{u}}(n_3) - i|\widehat{\mathbf{u}}(n)|^2\widehat{\mathbf{u}}(n)\\ &=: \mathcal{N}_1(\mathbf{u})(n) + \mathcal{R}(\mathbf{u})(n).\end{aligned} \tag{1.9}$$

Here, the phase function $\Phi(\bar{n})$ is defined by

$$\begin{aligned}\Phi(\bar{n}) :&= \Phi(n,n_1,n_2,n_3) = n^2 - n_1^2 + n_2^2 - n_3^2\\ &= 2(n_2-n_1)(n_2-n_3) = 2(n-n_1)(n-n_3),\end{aligned} \tag{1.10}$$

where the last two equalities hold under

$$n = n_1 - n_2 + n_3.$$

From (1.10), we see that $\mathcal{N}_1$ corresponds to the non-resonant part (i.e., $\Phi(\bar{n}) \neq 0$) of the nonlinearity and $\mathcal{R}$ corresponds to the resonant part. Note that the Duhamel formulation for (1.2),

$$u(t) = S(t)u_0 + i\int_0^t S(t-t')\mathcal{N}(u)(t')dt',$$

is now expressed as a system of integral equations:

$$\widehat{\mathbf{u}}(n,t) = \widehat{u}_0(n) + \int_0^t \{\mathcal{N}_1(\mathbf{u})(n) + \mathcal{R}(\mathbf{u})(n)\}(t')dt' \tag{1.11}$$

for $n \in \mathbb{Z}$. In the following, the space $\mathcal{F}L^{\frac{3}{2}}(\mathbb{T})$ plays an important role and thus we introduce the following definition of regular solutions.

---

[7] In the following, we restrict our attention to the renormalized cubic NLS (1.2). See Subsection 2.4 for required modifications to handle the cubic NLS (1.1) in Theorem 1.5.

[8] Due to the presence of the time-dependent phase factor $e^{i\Phi(\bar{n})t}$, the non-resonant part $\mathcal{N}_1(\mathbf{u})$, viewed as a trilinear operator, is non-autonomous. For notational simplicity, however, we suppress such t-dependence when there is no confusion. We apply this convention to all the multilinear operators appearing in this paper.

Definition 1.8. We say that u and $\mathbf{u}$ are regular solutions to (1.2) and (1.9), respectively, if u and $\mathbf{u}$ are solutions to (1.2) and (1.9), respectively, such that $u \in C(\mathbb{R}; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ and $\mathbf{u} \in C(\mathbb{R}; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$, respectively.

The main idea is to apply a normal form reduction to (1.9), namely integration by parts in (1.11), to exploit the oscillatory nature of the non-resonant contribution. As in [22, 29], we implement an infinite iteration of normal form reductions and derive the following **normal form equation**:

$$\begin{aligned}\mathbf{u}(t) = \mathbf{u}(0) &+ \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(t) - \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(0) \\ &+ \int_0^t \bigg\{ \sum_{j=1}^{\infty} \mathcal{N}_1^{(j)}(\mathbf{u})(t') + \sum_{j=1}^{\infty} \mathcal{R}^{(j)}(\mathbf{u})(t') \bigg\} dt',\end{aligned} \tag{1.12}$$

where $\{\mathcal{N}_0^{(j)}\}_{j=2}^{\infty}$ are time-dependent $(2j-1)$-linear operators while $\{\mathcal{N}_1^{(j)}\}_{j=1}^{\infty}$ and $\{\mathcal{R}^{(j)}\}_{j=1}^{\infty}$ are time-dependent $(2j+1)$-linear operators. As we see in Section 3, multilinear dispersion effects are already embedded in these multilinear terms, which allows us to prove that these multilinear operators are bounded in $C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$ for any $1 \le p < \infty$. Moreover, we show that the normal form equation (1.12) is equivalent to (1.9) and the renormalized cubic NLS (1.2) in $C(\mathbb{R}; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$. See Proposition 2.1. As a consequence, we can easily prove local well-posedness of the normal form equation (1.12) in $\mathcal{F}L^p(\mathbb{T})$ by a simple contraction argument without any auxiliary function spaces.

Theorem 1.9. *Let* $1 \le p < \infty$. *Then, the normal form equation* (1.12) *is unconditionally globally well-posed in* $\mathcal{F}L^p(\mathbb{T})$.

In [22], an analogous result was shown in $L^2(\mathbb{T})$. When $p > 2$, the $\mathcal{F}L^p$-norm is weaker than the L^2-norm. In particular, when $p \gg 1$, this fact makes it much harder to show convergence of the series in the normal form equation (1.12) with respect to the $\mathcal{F}L^p$-topology.

Once we establish the relevant multilinear estimates (Proposition 2.1), the proof of unconditional local well-posedness for the normal form equation (1.12) follows from a simple contraction argument. Moreover, we show that the local existence time T depends only on the size of the initial data $\|u_0\|_{\mathcal{F}L^p}$ and consequently, we conclude that solutions exist globally in time in view of the global-in-time bound (1.4) from [36]. See also Appendix A.

Finally, note that Theorem 1.5 follows easily thanks to the equivalence of (1.2) and the normal form equation (1.12) for regular solutions belonging to $C(\mathbb{R};\mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$. The contraction argument in proving Theorem 1.9 yields the following Lipschitz bound:

$$\sup_{t\in[-T,T]} \|\mathbf{u}(t)-\mathbf{v}(t)\|_{\mathcal{F}L^p} \leq C(T,R)\|\mathbf{u}(0)-\mathbf{v}(0)\|_{\mathcal{F}L^p} \tag{1.13}$$

for any $T>0$, where $R>0$ satisfies $\|\mathbf{u}(0)\|_{\mathcal{F}L^p}, \|\mathbf{v}(0)\|_{\mathcal{F}L^p} \leq R$. Furthermore, from (1.2), (1.9), and (1.12) with (1.5) and (1.8), we obtain

$$\begin{aligned}&\int_0^t \mathcal{N}(u)(t')dt' \\ &\quad = S(t)\bigg\{\sum_{j=2}^{\infty}\mathcal{N}_0^{(j)}(S(-\,\cdot)u)(t) - \sum_{j=2}^{\infty}\mathcal{N}_0^{(j)}(u)(0) \\ &\qquad + \int_0^t \bigg[\sum_{j=1}^{\infty}\mathcal{N}_1^{(j)}(S(-\,\cdot)u)(t') + \sum_{j=1}^{\infty}\mathcal{R}^{(j)}(S(-\,\cdot)u)(t')\bigg]dt'\bigg\}.\end{aligned} \tag{1.14}$$

Then, (1.13) and (1.14) together with the multilinearity of the summands in (1.14) and the unitarity of the linear operator $S(t)$ in $\mathcal{F}L^p(\mathbb{T})$ allow us to conclude convergence of smoothed nonlinearities $\mathcal{N}(T_N u)$ or the nonlinearities $\mathcal{N}(u_m)$ of smooth approximating solutions u_m required in Definitions 1.2 and 1.3. This is a sketch of the proof of Theorem 1.4.

In Section 2, we present the proofs of the main results, assuming the bounds on the multilinear operators $\{\mathcal{N}_0^{(j)}\}_{j=2}^{\infty}$, $\{\mathcal{N}_1^{(j)}\}_{j=1}^{\infty}$, and $\{\mathcal{R}^{(j)}\}_{j=1}^{\infty}$ (Proposition 2.1). In Section 3, we implement an infinite iteration of normal form reductions as in [22] and prove Proposition 2.1.

Remark 1.10. Let $p>\frac{3}{2}$. Given $u\in C([-T,T];\mathcal{F}L^p(\mathbb{T}))$, we can not, in general, make sense of the cubic nonlinearity $\mathcal{N}(u)$ as a distribution since

$$\mathcal{F}L^p(\mathbb{T}) \not\subset L^3(\mathbb{T}).$$

In other words, we can not estimate the cubic nonlinearity without relying on some auxiliary function space. In (1.14), we re-expressed the cubic nonlinearity into series of the multilinear terms of increasing degrees. On the one hand, this transformation brings algebraic complexity. On the other hand, the right-hand side of (1.14) is convergent for $u\in C([-T,T];\mathcal{F}L^p(\mathbb{T}))$, allowing us to make sense of the right-hand side of (1.14) as a distribution. Namely, while the left-hand side of (1.14) and the right-hand side of (1.14) coincide for regular solutions $u\in C([-T,T];\mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$, the right-hand side of (1.14) provides a better formulation of the nonlinearity for rougher functions $u\in C([-T,T];\mathcal{F}L^p(\mathbb{T}))$, $\frac{3}{2}<p<\infty$. In this sense, we can view the right-hand side of (1.14) as a further renormalization of the renormalized nonlinearity $\mathcal{N}(u)$ in (1.5).

By expressing the normal form equation (1.12) in terms of the original function $u(t) = S(t)\mathbf{u}(t)$, we obtain

$$
\begin{aligned}
u(t) &= S(t)u(0) + S(t)\sum_{j=2}^{\infty} \mathsf{N}_0^{(j)}(u)(t) - S(t)\sum_{j=2}^{\infty} \mathsf{N}_0^{(j)}(u)(0) \\
&\quad + \int_0^t S(t-t')\bigg\{\sum_{j=1}^{\infty} \mathsf{N}_1^{(j)}(u)(t') + \sum_{j=1}^{\infty} \mathsf{R}^{(j)}(u)(t')\bigg\}dt',
\end{aligned}
\tag{1.15}
$$

where

$$
\begin{aligned}
\mathsf{N}_0^{(j)}(u)(t) &= \mathcal{N}_0^{(j)}(S(-\,\cdot)u)(t), \\
\mathsf{N}_1^{(j)}(u)(t) &= S(t)\mathcal{N}_1^{(j)}(S(-\,\cdot)u)(t), \\
\mathsf{R}^{(j)}(u)(t) &= S(t)\mathcal{R}^{(j)}(S(-\,\cdot)u)(t).
\end{aligned}
\tag{1.16}
$$

As we see in Section 3, the multilinear operators $S(t)\mathsf{N}_0^{(j)}(t)$, $\mathsf{N}_1^{(j)}$, and $\mathsf{R}^{(j)}$ are autonomous. The discussion above shows that the normal form equation (1.15) expressed in terms of $u(t) = S(t)\mathbf{u}(t)$ is a better model to study than the renormalized cubic NLS (1.2) (and the cubic NLS (1.1)) in the low regularity setting, which can be viewed as a further renormalization to the (renormalized) cubic NLS.

Lastly, we point out that the terms on the left-hand side of (1.16) are indeed autonomous (unlike the non-autonomous multilinear terms in (1.14)). See Section 3.

Remark 1.11. A precursor to this normal form approach first appeared in the work of Babin–Ilyin–Titi [3] in the study of KdV on $\mathbb{T}$, establishing unconditional well-posedness of the KdV in $L^2(\mathbb{T})$. See also [30]. In [22], the first author with Z. Guo and S. Kwon further developed this normal form approach and introduced an infinite iteration scheme of normal form reductions in the context of the cubic NLS on the circle. In this series of work, the viewpoint of unconditional well-posedness was first introduced in [30], while the viewpoint of the (Poincaré–Dulac) normal form reductions was first introduced in [22]. This normal form approach has also been used to prove nonlinear smoothing [13], improved energy estimates [32, 35], and construct an infinite sequence of invariant quantities under the dynamics [11].

Remark 1.12. In a recent paper [14], the first author with Forlano studied the cubic NLS on $\mathbb{R}$. In particular, by implementing an infinite iteration of normal form reductions, they proved analogues of Theorems 1.4, 1.5, and 1.9 in almost critical Fourier–Lebesgue spaces $\mathcal{F}L^p(\mathbb{R})$, $2 \leq p < \infty$, and almost critical modulation spaces $M^{2,p}(\mathbb{R})$, $2 \leq p < \infty$. Relevant multilinear estimates were studied based on the idea introduced in [29], namely, successive applications of basic trilinear estimates (called localized modulation estimates).

2 Proof of the main results

In this section, we present the proofs of the main results (Theorems 1.4, 1.5, and 1.9), assuming the validity of the transformation of the equation (1.11) to the normal form equation (1.12) and the boundedness of the multilinear operators in (1.12) (Proposition 2.1).

2.1 Series expansion of regular solutions. In Section 3, we implement an infinite iteration of normal form reductions and transform the equation (1.9) into the normal form equation (1.12) for regular solutions. The following proposition summarizes the properties of the multilinear operators in (1.12). Given $R > 0$, we use B_R to denote the ball of radius R centered at the origin in various function spaces.

Proposition 2.1. *Let* $1 \le p < \infty$ *and* $T > 0$. *Then, there exist time-dependent multilinear operators* $\{\mathcal{N}_0^{(j)}\}_{j=2}^\infty$, $\{\mathcal{N}_1^{(j)}\}_{j=1}^\infty$, *and* $\{\mathcal{R}^{(j)}\}_{j=1}^\infty$, *depending on the parameter* $K = K(R) \ge 1$ *such that any regular solution* $\mathbf{u} \in C([-T,T];\mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ *to* (1.9) *with* $\mathbf{u}(0) \in B_R \subset \mathcal{F}L^p(\mathbb{T})$ *satisfies the following normal form equation:*

$$\begin{aligned}\mathbf{u}(t) - \mathbf{u}(0) &= \sum_{j=2}^\infty \mathcal{N}_0^{(j)}(\mathbf{u})(t) - \sum_{j=2}^\infty \mathcal{N}_0^{(j)}(\mathbf{u})(0)\\ &\quad + \int_0^t \bigg\{ \sum_{j=1}^\infty \mathcal{N}_1^{(j)}(\mathbf{u})(t') + \sum_{j=1}^\infty \mathcal{R}^{(j)}(\mathbf{u})(t')\bigg\} dt'\end{aligned} \tag{2.1}$$

in $C([-T,T];\mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$. *Moreover,* $\{\mathcal{N}_0^{(j)}\}_{j=2}^\infty$ *are* $(2j-1)$*-linear operators, while* $\{\mathcal{N}_1^{(j)}\}_{j=1}^\infty$ *and* $\{\mathcal{R}^{(j)}\}_{j=1}^\infty$ *are* $(2j+1)$*-linear operators (depending on* $t \in [-T,T]$*), satisfying the following bounds on* $\mathcal{F}L^p(\mathbb{T})$*:*[9]

$$\sup_{t\in[-T,T]} \|\mathcal{N}_0^{(j)}(t)(f_1, f_2, \dots, f_{2j-1})\|_{\mathcal{F}L^p(\mathbb{T})} \le C_{0,j} \prod_{i=1}^{2j-1} \|f_i\|_{\mathcal{F}L^p(\mathbb{T})}, \tag{2.2}$$

$$\sup_{t\in[-T,T]} \|\mathcal{N}_1^{(j)}(t)(f_1, f_2, \dots, f_{2j+1})\|_{\mathcal{F}L^p(\mathbb{T})} \le C_{1,j} \prod_{i=1}^{2j+1} \|f_i\|_{\mathcal{F}L^p(\mathbb{T})}, \tag{2.3}$$

$$\sup_{t\in[-T,T]} \|\mathcal{R}^{(j)}(t)(f_1, f_2, \dots, f_{2j+1})\|_{\mathcal{F}L^p(\mathbb{T})} \le C_{0,j} \prod_{i=1}^{2j+1} \|f_i\|_{\mathcal{F}L^p(\mathbb{T})}, \tag{2.4}$$

---

[9] Here, we view $\mathcal{N}_0^{(j)} = \mathcal{N}_0^{(j)}(t)$, $\mathcal{N}_1^{(j)} = \mathcal{N}_1^{(j)}(t)$, and $\mathcal{R}^{(j)} = \mathcal{R}^{(j)}(t)$ as multilinear operators acting on $\mathcal{F}L^p(\mathbb{T})$ with a parameter $t \in [-T,T]$. The same comment applies to $\mathcal{R}_2^{(j)}$ in (2.13).

for any $f_i \in \mathcal{F}L^p(\mathbb{T})$*, where*

$$C_{0,j}(K) = C_p \frac{K^{4(1-j)}}{j!} \quad \text{and} \quad C_{1,j}(K) = C_p \frac{K^{\frac{16}{p'-1}+4(1-j)}}{j!} \tag{2.5}$$

for some absolute constant $C_p > 0$ *depending only on* p.

In Proposition 2.1, we imposed a strong regularity assumption:

$$\mathbf{u} \in C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T})).$$

This regularity assumption can be easily relaxed.

Corollary 2.2. *Let* $1 \le p < \infty$ *and* $T > 0$. *Suppose that a solution* $\mathbf{u} \in C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$ *to* (1.9) *admits a sequence of smooth approximating solutions* $\{\mathbf{u}_m\}_{m\in\mathbb{N}}$ *in the sense that* (i) $\mathbf{u}_m$ *is a smooth solution to* (1.9) *and* (ii) $\mathbf{u}_m$ *converges to* $\mathbf{u}$ *in* $C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$. *Then,* $\mathbf{u}$ *satisfies the normal form equation* (1.14) *in* $C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$.

In view of the estimates (2.2), (2.3), and (2.4), we see that the right-hand side of (2.1) is convergent for $\mathbf{u} \in C([-T, T]; \mathcal{F}L^p(\mathbb{T}))$. See also the proof of Theorem 1.9 below. By using the multilinearity of the operators, we only need to estimate the difference such as $\mathcal{N}_0^{(j)}(\mathbf{u}) - \mathcal{N}_0^{(j)}(\mathbf{u}_m)$. Note that such a difference contains $O(j)$-many terms since

$$|a^{2j-1} - b^{2j-1}| \lesssim \bigg(\sum_{k=1}^{2j-1} a^{2j-1-k} b^{k-1} \bigg) |a - b|$$

has $O(j)$ many terms. This, however, does not cause any issue thanks to the fast decay (2.5) of the coefficients $C_{0,j}$ and $C_{1,j}$. Since the proof of Corollary 2.2 is straightforward computation with (2.2), (2.3), and (2.4), we omit details.

We postpone the proof of Proposition 2.1 to Section 3. In the remaining part of this section, we present the proofs of Theorems 1.4, 1.5, and 1.9, assuming Proposition 2.1. In Subsection 2.4, we discuss the case of the (unrenormalized) NLS (1.1).

We first present the proof of Theorem 1.9.

Proof of Theorem 1.9. Given $1 \le p < \infty$, let $\mathbf{u}_0 \in \mathcal{F}L^p(\mathbb{T})$. With

$$K = K(\|\mathbf{u}_0\|_{\mathcal{F}L^p}) \ge 1$$

(to be chosen later), define the map $\Gamma_{\mathbf{u}_0}$ by

$$\begin{aligned}\Gamma_{\mathbf{u}_0}(\mathbf{u})(t) := \mathbf{u}_0 + \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(t) - \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(0)\\ + \int_0^t \bigg(\sum_{j=1}^{\infty} \mathcal{N}_1^{(j)}(\mathbf{u})(t') + \sum_{j=1}^{\infty} \mathcal{R}^{(j)}(\mathbf{u})(t') \bigg) dt',\end{aligned}$$

where the multilinear terms on the right-hand side (depending on the choice of $K \geq 1$) are as in Proposition 2.1. Let $T > 0$. Then, by Proposition 2.1, we have

$$\begin{aligned}&\|\Gamma_{\mathbf{u}_0}(\mathbf{u})\|_{C_T\mathcal{F}L^p}\\ &\quad\leq \|\mathbf{u}_0\|_{\mathcal{F}L^p} + \sum_{j=2}^{\infty} C_{0,j}(K)(\|\mathbf{u}_0\|_{\mathcal{F}L^p}^{2j-1} + \|\mathbf{u}\|_{C_T\mathcal{F}L^p}^{2j-1})\\ &\qquad + T\sum_{j=1}^{\infty}(C_{1,j}(K) + C_{0,j}(K))\|\mathbf{u}\|_{C_T\mathcal{F}L^p}^{2j+1},\end{aligned}$$

where $C_T\mathcal{F}L^p = C([-T,T];\mathcal{F}L^p(\mathbb{T}))$.

Let $R = 1 + \|\mathbf{u}_0\|_{\mathcal{F}L^p}$. Then from (2.2), (2.3), and (2.4), we have

$$\begin{aligned}&\|\Gamma_{\mathbf{u}_0}(\mathbf{u})\|_{C_T\mathcal{F}L^p}\\ &\quad\leq R + C\sum_{j=2}^{\infty}\frac{K^{4(1-j)}R^{2j-1}}{j!} + C\sum_{j=2}^{\infty}\frac{K^{4(1-j)}(2R)^{2j-2}}{j!}\|\mathbf{u}\|_{C_T\mathcal{F}L^p}\\ &\qquad + CT\bigg\{\sum_{j=1}^{\infty}\frac{K^{\frac{16}{p'-1}+4(1-j)}(2R)^{2j}}{j!} + \sum_{j=1}^{\infty}\frac{K^{4(1-j)}(2R)^{2j}}{j!}\bigg\}\|\mathbf{u}\|_{C_T\mathcal{F}L^p}\end{aligned} \tag{2.6}$$

for any $\mathbf{u} \in B_{2R} \subset C([-T,T];\mathcal{F}L^p(\mathbb{T}))$. The series in (2.6) are obviously convergent for any $K \geq 1$ thanks to the fast decay in j but, by choosing $K = K(R,p) \gg 1$ sufficiently large, we can guarantee that

$$C\sum_{j=2}^{\infty}\frac{K^{4(1-j)}R^{2j-1}}{j!} \leq \frac{1}{10} \quad\text{and}\quad C\sum_{j=2}^{\infty}\frac{K^{4(1-j)}(2R)^{2j-2}}{j!} \leq \frac{1}{10}.$$

Note that the third series in (2.6) has non-negative powers of K for $1 \leq j < \frac{4}{p'-1}$, while a power of K does not appear in the fourth series when $j = 1$. These terms can be controlled by choosing

$$T = T(K,R) = T(R) > 0$$

sufficiently small. As a result, we obtain

$$\|\Gamma_{\mathbf{u}_0}(\mathbf{u})\|_{C_T\mathcal{F}L^p} \leq \frac{11}{10}R + \frac{1}{5}\|\mathbf{u}\|_{C_T\mathcal{F}L^p} < 2R$$

for any $\mathbf{u} \in B_{2R} \subset C([-T,T];\mathcal{F}L^p(\mathbb{T}))$. A similar argument also yields the following difference estimate:

$$\|\Gamma_{\mathbf{u}_0}(\mathbf{u}) - \Gamma_{\mathbf{u}_0}(\mathbf{v})\|_{C_T\mathcal{F}L^p} \leq \frac{1}{5}\|\mathbf{u}-\mathbf{v}\|_{C_T\mathcal{F}L^p}. \tag{2.7}$$

In establishing the difference estimate (2.7), we need to estimate the differences such as $\mathcal{N}_0^{(j)}(\mathbf{u}) - \mathcal{N}_0^{(j)}(\mathbf{v})$ which contains $O(j)$-many terms as mentioned above. This does not cause any issue thanks to the fast decay (2.5) in j of the coefficients $C_{0,j}$ and $C_{1,j}$.

Therefore, by a standard contraction argument and a continuity argument,[10] we conclude that the normal form equation (1.12) is unconditionally locally well-posed in $C([-T,T];\mathcal{F}L^p(\mathbb{T}))$. Global well-posedness follows from the a priori bounds (1.4) and (A.10) on the $\mathcal{F}L^p$-norm of smooth solutions to (1.2) implying the same bound for smooth solutions to (1.9) and (1.12).

Lastly, by taking the difference of two solutions $\mathbf{u}, \mathbf{v} \in C([-T,T];\mathcal{F}L^p(\mathbb{T}))$ with different initial data $\mathbf{u}_0$ and $\mathbf{v}_0$, we have

$$\|\mathbf{u}-\mathbf{v}\|_{C_T\mathcal{F}L^p} \leq \frac{11}{10}\|\mathbf{u}_0-\mathbf{v}_0\|_{\mathcal{F}L^p} + \frac{1}{5}\|\mathbf{u}-\mathbf{v}\|_{C_T\mathcal{F}L^p},$$

which implies the Lipschitz bound (1.13) for

$$T = T(\|\mathbf{u}_0\|_{\mathcal{F}L^p}, \|\mathbf{v}_0\|_{\mathcal{F}L^p}) > 0$$

sufficiently small. By iterating the Lipschitz bound (1.13) on short intervals with the global-in-time bounds (1.4) and (A.10), we conclude that (1.13) for any $T > 0$.□

2.2 Sensible weak solutions: Proof of Theorem 1.4. In the following, we only show global well-posedness of the renormalized cubic NLS (1.2) in the sense of sensible weak solutions according to Definition 1.3. As for well-posedness in the sense of weak solutions in the extended sense according to Definition 1.2, one can simply use Proposition 2.1 and repeat the argument in [22].

Given $u_0 \in \mathcal{F}L^p(\mathbb{T})$, let $\{u_{0,m}\}_{m\in\mathbb{N}}$ be a sequence of smooth functions converging to u_0 in $\mathcal{F}L^p(\mathbb{T})$. Let u_m be the smooth solution to (1.2) with $u_m|_{t=0} = u_m$ and set $\mathbf{u}_m(t) = S(-t)u_m(t)$. Then, it follows from Proposition 2.1 that $\mathbf{u}_m$ is a solution to the normal form equation (2.1). From the Lipschitz bound (1.13), we have

$$\begin{aligned} \|u_m - u_n\|_{C_T\mathcal{F}L^p} &= \|\mathbf{u}_m - \mathbf{u}_n\|_{C_T\mathcal{F}L^p} \\ &\leq C(T)\|\mathbf{u}_m(0) - \mathbf{u}_n(0)\|_{\mathcal{F}L^p} \\ &= C(T)\|u_m(0) - u_n(0)\|_{\mathcal{F}L^p} \end{aligned} \tag{2.8}$$

---

[10] The contraction argument yields uniqueness only in $B_{2R} \subset C([-T,T];\mathcal{F}L^p(\mathbb{T}))$ and a continuity argument is needed to extend the uniqueness to the entire $C([-T,T];\mathcal{F}L^p(\mathbb{T}))$. This part of the argument is standard and thus we omit detail. See for example [11].

for all $m, n \geq 1$ and any $T > 0$. This shows that $\{u_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{R}; \mathcal{F}L^p(\mathbb{T}))$ endowed with the compact-open topology (in time) and hence converges to some u_∞ in $C(\mathbb{R}; \mathcal{F}L^p(\mathbb{T}))$.

Now, we prove uniqueness of the limit u_∞, independent of smooth approximating solutions. Given $u_0 \in \mathcal{F}L^p(\mathbb{T})$, let $\{u_m\}_{m\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be two sequences of smooth solutions such that $u_m(0), v_n(0) \to u_0$ in $\mathcal{F}L^p(\mathbb{T})$ as $m, n \to \infty$. Then, by the argument above, there exist $u_\infty, v_\infty \in C(\mathbb{R}; \mathcal{F}L^p(\mathbb{T}))$ such that $u_m \to u_\infty$ and $v_n \to v_\infty$ in $C(\mathbb{R}; \mathcal{F}L^p(\mathbb{T}))$ as $m, n \to \infty$. Then, by the triangle inequality with (1.13) and (2.8), we obtain

$$\begin{aligned}
\|u_\infty - v_\infty\|_{C_T\mathcal{F}L^p} &\leq \|u - u_m\|_{C_T\mathcal{F}L^p} + \|u_m - v_n\|_{C_T\mathcal{F}L^p} + \|v_n - v\|_{C_T\mathcal{F}L^p} \\
&\leq \|u - u_m\|_{C_T\mathcal{F}L^p} + C\|u_m(0) - v_n(0)\|_{\mathcal{F}L^p} + \|v_n - v\|_{C_T\mathcal{F}L^p} \\
&\longrightarrow 0,
\end{aligned}$$

as $m, n \to \infty$. Therefore, we have $u_\infty = v_\infty$.

Lastly, combining this convergence with (1.2), we obtain

$$\mathcal{N}(u_m) - \mathcal{N}(u_n) = -i\partial_t(u_m - u_n) - \partial_x^2(u_m - u_n) \longrightarrow 0 \tag{2.9}$$

in the distributional sense as $m, n \to \infty$. Therefore, we conclude that (1.2) is globally well-posed in the sense of sensible weak solutions.

2.3 Unconditional well-posedness of the renormalized cubic NLS.

We briefly discuss the proof of Theorem 1.5 for the renormalized cubic NLS (1.2). Given $u_0 \in \mathcal{F}L^{\frac{3}{2}}(\mathbb{T})$, let u and v be two solutions to (1.2) with

$$u|_{t=0} = v|_{t=0} = u_0$$

in $C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ for some $T > 0$. By Proposition 2.1, we see that their interaction representations

$$\mathbf{u}(t) = S(-t)u(t) \quad \text{and} \quad \mathbf{v}(t) = S(-t)v(t)$$

satisfy the normal form equation (1.12). Then, from the unconditional uniqueness for (1.12) in $C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ (Theorem 1.9) and the unitarity of the linear operator in $\mathcal{F}L^p(\mathbb{T})$, we conclude that $u = v$ in $C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$. This proves Theorem 1.5.

2.4 On the cubic NLS. We conclude this section by discussing the situation for the cubic NLS (1.1). By writing

$$\begin{aligned}
|u|^2u &= \bigg(|u|^2 - 2\int_{\mathbb{T}} |u|^2dx\bigg)u + 2\bigg(\int_{\mathbb{T}} |u|^2dx\bigg)u \\
&= \sum_{n_2\neq n_1,n_3} \widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3)e^{i(n_1-n_2+n_3)x} - \sum_{n\in\mathbb{Z}} |\widehat{u}(n)|^2\widehat{u}(n)e^{inx} \\
&\quad + 2\bigg(\int_{\mathbb{T}} |u|^2dx\bigg)\sum_{n} \widehat{u}(n)e^{inx} \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III},
\end{aligned} \tag{2.10}$$

we see that the third term III is the only difference from the case for the renormalized cubic NLS (1.2). By taking an interaction representation, we can write (1.1) as

$$\partial_t\widehat{\mathbf{u}}_n = \mathcal{N}_1(\mathbf{u})(n) + \mathcal{R}(\mathbf{u})(n) + \mathcal{R}_2(\mathbf{u})(n), \tag{2.11}$$

where $\mathcal{R}_2(\mathbf{u})(n)$ is given by

$$\mathcal{R}_2(\mathbf{u})(n) = 2i\bigg(\int_{\mathbb{T}} |\mathbf{u}|^2dx\bigg)\widehat{\mathbf{u}}(n).$$

As compared to (1.9), $\mathcal{R}_2(\mathbf{u})$ is the only difference. Note that this extra term $\mathcal{R}_2(\mathbf{u})$ imposes the restriction $p \leq 2$. As in the case of the renormalized NLS (1.2), we prove the following proposition in Section 3.

Proposition 2.3. *Let $1 \leq p \leq 2$ and $T > 0$. Then, there exist time-dependent multilinear operators $\{\mathcal{N}_0^{(j)}\}_{j=2}^{\infty}$, $\{\mathcal{N}_1^{(j)}\}_{j=1}^{\infty}$, $\{\mathcal{R}^{(j)}\}_{j=1}^{\infty}$, and $\{\mathcal{R}_2^{(j)}\}_{j=1}^{\infty}$ depending on the parameter*

$$K = K(R) \geq 1$$

such that the interaction representation $\mathbf{u}(t) = S(-t)u(t)$ of any regular solution $u \in C([-T,T];\mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ to (1.1) with $u(0) \in B_R \subset \mathcal{F}L^p(\mathbb{T})$ satisfies the following normal form equation:

$$\begin{aligned}
\mathbf{u}(t) - \mathbf{u}(0) &= \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(t) - \sum_{j=2}^{\infty} \mathcal{N}_0^{(j)}(\mathbf{u})(0) \\
&\quad + \int_0^t \bigg\{\sum_{j=1}^{\infty} \mathcal{N}_1^{(j)}(\mathbf{u})(t') + \sum_{j=1}^{\infty} \mathcal{R}^{(j)}(\mathbf{u})(t') + \sum_{j=1}^{\infty} \mathcal{R}_2^{(j)}(\mathbf{u})(t')\bigg\}dt'
\end{aligned} \tag{2.12}$$

in $C([-T,T];\mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$. Here, $\{\mathcal{N}_0^{(j)}\}_{j=2}^{\infty}$, $\{\mathcal{N}_1^{(j)}\}_{j=1}^{\infty}$, and $\{\mathcal{R}^{(j)}\}_{j=1}^{\infty}$ are as in Proposition 2.1, satisfying the bounds (2.2), (2.3), and (2.4), while $\{\mathcal{R}_2^{(j)}\}_{j=1}^{\infty}$ are $(2j+1)$-linear operators (depending on $t \in [-T,T]$), satisfying the following bound:

$$\sup_{t\in[-T,T]} \|\mathcal{R}_2^{(j)}(t)(f_1,f_2,\dots,f_{2j+1})\|_{\mathcal{F}L^p(\mathbb{T})} \leq C_{0,j}\prod_{i=1}^{2j+1} \|f_i\|_{\mathcal{F}L^p(\mathbb{T})}, \tag{2.13}$$

for any $f_i \in \mathcal{F}L^p(\mathbb{T})$, where $C_{0,j} = C_{0,j}(K) > 0$ is as in (2.5).

With Proposition 2.3, we can proceed as in the proof of Theorem 1.9 and prove the following unconditional well-posedness of the normal form equation (2.12) for the cubic NLS (1.1).

Theorem 2.4. *Let* $1 \leq p \leq 2$. *Then, the normal form equation* (2.12) *is unconditionally globally well-posed in* $\mathcal{F}L^p(\mathbb{T})$.

Then, Theorem 1.4 for $1 \leq p \leq 2$ and Theorem 1.5 for the cubic NLS (1.1) follow from arguments analogous to those presented above. We omit details.

3 Normal form reduction: Proof of Proposition 2.1

In this section, we implement an infinite iteration of normal form reductions in the Fourier–Lebesgue space $\mathcal{F}L^p(\mathbb{T})$, $1 \leq p < \infty$, and prove Proposition 2.1. The argument is presented in an inductive manner. More precisely, we start with the formulation (1.9) and refer to this case as the first step ($J = 1$). Define

$$\mathcal{N}_1(\mathbf{u}) := \sum_{n\in\mathbb{Z}} \mathcal{N}_1(\mathbf{u})(n)e^{inx} \quad \text{and} \quad \mathcal{R}(\mathbf{u}) := \sum_{n\in\mathbb{Z}} \mathcal{R}(\mathbf{u})(n)e^{inx}, \tag{3.1}$$

where $\mathcal{N}_1(\mathbf{u})(n)$ and $\mathcal{R}(\mathbf{u})(n)$ are as in (1.9). In what follows, we view $\mathcal{N}_1$ and $\mathcal{R}$ as trilinear operators.

For notational convenience, we set

$$\mathcal{R}^{(1)} := \mathcal{R} \quad \text{and} \quad \mathcal{N}^{(1)} := \mathcal{N}_1.$$

While we keep the resonant part $\mathcal{R}^{(1)}$ as it is, we divide the non-resonant part $\mathcal{N}^{(1)}$ into a "good" part $\mathcal{N}_1^{(1)}$ (nearly resonant part) and a "bad" part $\mathcal{N}_2^{(1)}$ (highly non-resonant part), depending on the size of the phase function $\Phi(\overline{n})$. On the one hand, the restriction on the phase function $\Phi(\overline{n})$ allows us to establish an effective estimate on the good part $\mathcal{N}_1^{(1)}$. On the other hand, the bad part does not allow for any good estimate. To exploit fast time oscillation, we then apply a normal form reduction to the bad part $\mathcal{N}_2^{(1)}$ and turn it into the terms $\mathcal{N}_0^{(2)}$, $\mathcal{R}^{(2)}$, and $\mathcal{N}^{(2)}$ in the second generation ($J = 2$). We can easily estimate the terms $\mathcal{N}_0^{(2)}$ and $\mathcal{R}^{(2)}$. As in the first step, we divide $\mathcal{N}^{(2)}$ into a good part $\mathcal{N}_1^{(2)}$ and a bad part $\mathcal{N}_2^{(2)}$, where the threshold is now given by the phase function for the quintilinear term $\mathcal{N}^{(2)}$. While the good part $\mathcal{N}_1^{(2)}$ allows for an effective quintilinear estimate, we apply a normal form reduction to the bad part $\mathcal{N}_2^{(2)}$ and turn it into three terms $\mathcal{N}_0^{(3)}$, $\mathcal{R}^{(3)}$, and $\mathcal{N}^{(3)}$ in the third generation ($J = 3$). We proceed in an inductive manner.

After applying normal form reductions $J-1$ times, we arrive at the three terms $\mathcal{N}_0^{(J)}$, $\mathcal{R}^{(J)}$, and $\mathcal{N}^{(J)}$. The main difficulty appears in the last term $\mathcal{N}^{(J)}$. As in the previous steps, we divide $\mathcal{N}^{(J)}$ into a good part $\mathcal{N}_1^{(J)}$ (with an effective $(2J+1)$-linear estimate) and a bad part $\mathcal{N}_2^{(J)}$. We then apply a normal form reduction to the bad part $\mathcal{N}_2^{(J)}$ and iterate this procedure indefinitely. Under some regularity assumption, we show that the error term $\mathcal{N}_2^{(J)}$ tends to 0 as $J \to \infty$.

In order to carry out the strategy described above, we need to address the following four issues:

- How do we separate $\mathcal{N}^{(J)}$ into "good" and "bad" parts?
- How do we estimate these good terms in the $\mathcal{F}L^p(\mathbb{T})$ when $p \gg 1$? As we see below, $\mathcal{N}_0^{(J)}$ is $(2J-1)$-linear, while $\mathcal{R}^{(J)}$ and $\mathcal{N}^{(J)}$ are $(2J+1)$-linear.
- Under what condition does the remainder term $\mathcal{N}_2^{(J)}$ tend to 0 as $J \to \infty$, and if so, in which sense?
- We need to show convergence of the series representation (2.1).

We address these issues in the remaining part of this section. In the following, we fix $1 \leq p < \infty$. The major part of this section is devoted to studying the renormalized cubic NLS (1.2). As for the (unrenormalized) cubic NLS (1.1), see Subsection 3.6.

3.1 Base case: $J = 1$.

Define the trilinear operators $\mathcal{N}^{(1)}$ and $\mathcal{R}^{(1)}$ by

$$\begin{aligned} \mathcal{N}^{(1)}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) &= i \sum_{n \in \mathbb{Z}} e^{inx} \sum_{\substack{n = n_1 - n_2 + n_3 \\ n_2 \neq n_1, n_3}} e^{i\Phi(\bar{n})t} \widehat{\mathbf{u}}_1(n_1) \overline{\widehat{\mathbf{u}}_2(n_2)} \widehat{\mathbf{u}}_3(n_3), \\ \mathcal{R}^{(1)}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) &= -i \sum_{n \in \mathbb{Z}} e^{inx} \widehat{\mathbf{u}}_1(n) \overline{\widehat{\mathbf{u}}_2(n)} \widehat{\mathbf{u}}_3(n), \end{aligned} \tag{3.2}$$

where $\Phi(\bar{n})$ is as in (1.10). For notational simplicity, we set

$$\mathcal{N}^{(1)}(\mathbf{u}) = \mathcal{N}^{(1)}(\mathbf{u}, \mathbf{u}, \mathbf{u}), \quad \text{etc.}$$

when all the three arguments coincide. Note that this notation is consistent with (1.9) and (3.1). Then, we can write (1.9) as

$$\partial_t \mathbf{u} = \mathcal{N}^{(1)}(\mathbf{u}) + \mathcal{R}^{(1)}(\mathbf{u}). \tag{3.3}$$

The resonant part satisfies the following trivial estimate.

Lemma 3.1. *Let $1 \leq p \leq \infty$. Then, we have*

$$\|\mathcal{R}^{(1)}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)\|_{\mathcal{F}L^p} \leq \prod_{i=1}^{3} \|\mathbf{u}_i\|_{\mathcal{F}L^p}. \tag{3.4}$$

Proof. This is clear from $\ell^p_n \subset \ell^{3p}_n$. □

Remark 3.2. (i) In the following, we establish various multilinear estimates. To simplify notations, we only state and prove estimates when all arguments agree with the understanding that they can be easily extended to multilinear estimates. Under this convention, (3.4) is written as

$$\|\mathcal{R}^{(1)}(\mathbf{u})\|_{\mathcal{F}L^p} \le \|\mathbf{u}\|^3_{\mathcal{F}L^p}.$$

We also use $\widehat{\mathbf{u}}_n = \widehat{\mathbf{u}}_n(t)$ to denote $\widehat{\mathbf{u}}(n,t)$. Moreover, given a multilinear operator $\mathcal{M}$, we simply use $\mathcal{M}(\mathbf{u})(n)$ to denote the Fourier coefficients of $\mathcal{M}(\mathbf{u})$.

(ii) The multilinear operators that appear below are non-autonomous, i.e., they depend on a parameter $t \in \mathbb{R}$. They, however, satisfy estimates uniformly in time and hence we simply suppress their time dependence. See (3.10) for example.

Next, we consider the non-resonant part $\mathcal{N}^{(1)}$ in (3.2). As it is, we can not establish an effective estimate and hence we divide it into two parts. Given $K \ge 1$ (to be chosen later) and $1 \le p < \infty$, let $\varepsilon = \varepsilon(p) > 0$ be a small positive number such that

$$p' - 1 - \varepsilon > 0. \tag{3.5}$$

In the following, we simply set

$$\varepsilon = \frac{p'-1}{2} > 0 \tag{3.6}$$

such that (3.5) is satisfied. Furthermore, we set

$$\theta = \frac{4p'}{p'-1-\varepsilon} > 0. \tag{3.7}$$

We write $\mathcal{N}^{(1)}$ in (3.2) as

$$\mathcal{N}^{(1)} = \mathcal{N}^{(1)}_1 + \mathcal{N}^{(1)}_2, \tag{3.8}$$

where $\mathcal{N}^{(1)}_1$ is the restriction of $\mathcal{N}^{(1)}$ onto A_1 (on the Fourier side), where

$$A_1 = \bigcup_n A_1(n)$$

with[11]

$$\begin{aligned} A_1(n) := \{(n, n_1, n_2, n_3) : n = n_1 - n_2 + n_3,\ n_1, n_3 \ne n, \\ |\Phi(\bar{n})| = |2(n-n_1)(n-n_3)| \le (3K)^\theta\} \end{aligned} \tag{3.9}$$

and $\mathcal{N}^{(1)}_2 := \mathcal{N}^{(1)} - \mathcal{N}^{(1)}_1$. Then, the "good" part $\mathcal{N}^{(1)}_1$ satisfies the following trilinear estimate.

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[11] Clearly, the number 3^θ in (3.9) does not make any difference at this point. However, we insert it to match with (3.25). See also (3.18).

Lemma 3.3. *Let* $\mathcal{N}_1^{(1)}$ *be as in* (3.8)*. Then, we have*

$$\|\mathcal{N}_1^{(1)}(\mathbf{u})\|_{\mathcal{F}L^p} \lesssim K^{\frac{2\theta}{p'}} \|\mathbf{u}\|^3_{\mathcal{F}L^p}, \tag{3.10}$$

where θ *is as in* (3.7)*.*

As in the $p = 2$ case studied in [22], the following divisor estimate [23] plays an important role in the following. Given an integer n, let $d(n)$ denote the number of divisors of m. Then, we have

$$d(n) \lesssim e^{c\frac{\log n}{\log\log n}} (= o(n^\delta) \text{ for any } \delta > 0). \tag{3.11}$$

Remark 3.4. With (3.6) and (3.7), we have

$$K^{\frac{2\theta}{p'}} = K^{\frac{16}{p'-1}},$$

which appears in (2.5) of Proposition 2.1.

Proof. Fix $n, \mu \in \mathbb{Z}$ with $|\mu| \leq (3K)^\theta$. Then, it follows from the divisor estimate (3.11) that there are at most $(3K)^{0+}$ many choices for n_1 and n_3 (and hence for n_2 from $n = n_1 - n_2 + n_3$) satisfying

$$\mu = 2(n - n_1)(n - n_3). \tag{3.12}$$

Hence, we have

$$\sup_n \Bigg(\sum_{|\mu| \leq (3K)^\theta} \sum_{\substack{n = n_1 - n_2 + n_3 \\ n_2 \neq n_1, n_3 \\ \mu = \Phi(\bar{n})}} 1 \Bigg) \lesssim \sum_{|\mu| \leq (3K)^\theta} (3K)^{0+} \lesssim (3K)^{2\theta}.$$

Then, by Hölder's inequality, we have

$$\begin{aligned}
\|\mathcal{N}_1^{(1)}(\mathbf{u})\|_{\mathcal{F}L^p} &= \Bigg(\sum_n \Bigg| \sum_{|\mu| \leq (3K)^\theta} \sum_{\substack{n = n_1 - n_2 + n_3 \\ n_2 \neq n_1, n_3 \\ \mu = \Phi(\bar{n})}} \widehat{\mathbf{u}}_{n_1} \overline{\widehat{\mathbf{u}}}_{n_2} \widehat{\mathbf{u}}_{n_3} \Bigg|^p \Bigg)^{\frac{1}{p}} \\
&\leq \Bigg\{ \sum_n \Bigg(\sum_{|\mu| \leq (3K)^\theta} \sum_{\substack{n = n_1 - n_2 + n_3 \\ n_2 \neq n_1, n_3 \\ \mu = \Phi(\bar{n})}} 1 \Bigg)^{\frac{p}{p'}} \Bigg(\sum_{n_1, n_3 \in \mathbb{Z}} |\widehat{\mathbf{u}}_{n_1}|^p |\widehat{\mathbf{u}}_{n_1 + n_3 - n}|^p |\widehat{\mathbf{u}}_{n_3}|^p \Bigg) \Bigg\}^{\frac{1}{p}} \\
&\lesssim K^{\frac{2\theta}{p'}} \|\mathbf{u}\|^3_{\mathcal{F}L^p}.
\end{aligned}$$

This proves (3.10). □

Now, we apply a normal form reduction to the remaining highly non-resonant part $\mathcal{N}_2^{(1)}$. More precisely, we differentiate $\mathcal{N}_2^{(1)}$ by parts (i.e., the product rule on differentiation in a reversed order) and write

$$\begin{aligned}
\mathcal{N}_2^{(1)}(\mathbf{u})(n) &= \sum_{A_1(n)^c} \partial_t\Big(\frac{e^{i\Phi(\bar{n})t}}{\Phi(\bar{n})}\Big)\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}}_{n_2}\widehat{\mathbf{u}}_{n_3} \\
&= \sum_{A_1(n)^c} \partial_t\Big[\frac{e^{i\Phi(\bar{n})t}}{\Phi(\bar{n})}\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}}_{n_2}\widehat{\mathbf{u}}_{n_3}\Big] - \sum_{A_1(n)^c} \frac{e^{i\Phi(\bar{n})t}}{\Phi(\bar{n})}\partial_t(\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}}_{n_2}\widehat{\mathbf{u}}_{n_3}) \\
&= \partial_t\Big[\sum_{A_1(n)^c} \frac{e^{i\Phi(\bar{n})t}}{\Phi(\bar{n})}\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}}_{n_2}\widehat{\mathbf{u}}_{n_3}\Big] - \sum_{A_1(n)^c} \frac{e^{i\Phi(\bar{n})t}}{\Phi(\bar{n})}\partial_t(\widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}}_{n_2}\widehat{\mathbf{u}}_{n_3}) \\
&=: \partial_t\mathcal{N}_0^{(2)}(\mathbf{u})(n) + \widetilde{\mathcal{N}}^{(2)}(\mathbf{u})(n).
\end{aligned} \tag{3.13}$$

The boundary term $\mathcal{N}_0^{(2)}$ can be estimated in a straightforward manner. Using the equation (1.9), we can express $\widetilde{\mathcal{N}}^{(2)}(\mathbf{u})(n)$ as a quintilinear form:

$$\begin{aligned}
\widetilde{\mathcal{N}}^{(2)}(\mathbf{u})(n) &= -\sum_{A_1(n)^c} \frac{e^{i\Phi(\bar{n})t}}{\Phi(\bar{n})}\Big\{\mathcal{R}(\mathbf{u})(n_1)\overline{\widehat{\mathbf{u}}}_{n_2}\widehat{\mathbf{u}}_{n_3} \\
&\qquad\qquad + \widehat{\mathbf{u}}_{n_1}\overline{\mathcal{R}(\mathbf{u})(n_2)}\widehat{\mathbf{u}}_{n_3} + \widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}}_{n_2}\mathcal{R}(\mathbf{u})(n_3)\Big\} \\
&= -\sum_{A_1(n)^c} \frac{e^{i\Phi(\bar{n})t}}{\Phi(\bar{n})}\Big\{\mathcal{N}_1(\mathbf{u})(n_1)\overline{\widehat{\mathbf{u}}}_{n_2}\widehat{\mathbf{u}}_{n_3} \\
&\qquad\qquad + \widehat{\mathbf{u}}_{n_1}\overline{\mathcal{N}_1(\mathbf{u})(n_2)}\widehat{\mathbf{u}}_{n_3} + \widehat{\mathbf{u}}_{n_1}\overline{\widehat{\mathbf{u}}}_{n_2}\mathcal{N}_1(\mathbf{u})(n_3)\Big\} \\
&=: \mathcal{R}^{(2)}(\mathbf{u})(n) + \mathcal{N}^{(2)}(\mathbf{u})(n).
\end{aligned} \tag{3.14}$$

In view of (3.2), we regard $\mathcal{R}^{(2)}(\mathbf{u})(n)$ and $\mathcal{N}^{(2)}(\mathbf{u})(n)$ on the right-hand side as quintilinear forms. As in the first step, we will need to divide $\mathcal{N}^{(2)}$ into good and bad parts and apply another normal form reduction to the bad part. Before proceeding further, we first recall the notion of ordered trees introduced in [22]. This allows us to express multilinear terms in a concise manner.

Remark 3.5. We formally exchanged the order of the sum and the time differentiation in the first term at the third equality. This can be easily justified in the distributional sense (see Lemma 5.1 in [22]) and also in the classical sense if

$$\mathbf{u} \in C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T})) \subset C([-T, T]; L^3(\mathbb{T})).$$

See [22].

3.2 Notations: index by trees.

In this subsection, we recall the notion of ordered trees and relevant definitions from [22].

Definition 3.6. (i) Given a partially ordered set $\mathcal{T}$ with partial order $\leq$, we say that $b \in \mathcal{T}$ with $b \leq a$ and $b \neq a$ is a child of $a \in \mathcal{T}$, if $b \leq c \leq a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that a is the parent of b.

(ii) A tree $\mathcal{T}$ is a finite partially ordered set satisfying the following properties.

- Let $a_1, a_2, a_3, a_4 \in \mathcal{T}$. If $a_4 \leq a_2 \leq a_1$ and $a_4 \leq a_3 \leq a_1$, then we have $a_2 \leq a_3$ or $a_3 \leq a_2$.
- A node $a \in \mathcal{T}$ is called terminal, if it has no child. A non-terminal node $a \in \mathcal{T}$ is a node with exactly three children denoted by a_1, a_2, and a_3.
- There exists a maximal element $r \in \mathcal{T}$ (called the root node) such that $a \leq r$ for all $a \in \mathcal{T}$. We assume that the root node is non-terminal.
- $\mathcal{T}$ consists of the disjoint union of $\mathcal{T}^0$ and $\mathcal{T}^\infty$, where $\mathcal{T}^0$ and $\mathcal{T}^\infty$ denote the collections of non-terminal nodes and terminal nodes, respectively.

The number $|\mathcal{T}|$ of nodes in a tree $\mathcal{T}$ is $3j + 1$ for some $j \in \mathbb{N}$, where

$$|\mathcal{T}^0| = j \quad \text{and} \quad |\mathcal{T}^\infty| = 2j + 1.$$

Let us denote the collection of trees in the jth generation by $T(j)$:

$$T(j) := \{\mathcal{T} : \mathcal{T} \text{ is a tree with } |\mathcal{T}| = 3j + 1\}.$$

Note that $\mathcal{T} \in T(j)$ contains j parental nodes.

(iii) (ordered tree) We say that a sequence $\{\mathcal{T}_j\}_{j=1}^J$ is a chronicle of J generations, if

- $\mathcal{T}_j \in T(j)$ for each $j = 1, \dots, J$,
- $\mathcal{T}_{j+1}$ is obtained by changing one of the terminal nodes in $\mathcal{T}_j$ into a non-terminal node (with three children), $j = 1, \dots, J - 1$.

Given a chronicle $\{\mathcal{T}_j\}_{j=1}^J$ of J generations, we refer to $\mathcal{T}_J$ as an **ordered tree** of the Jth generation. We denote the collection of the ordered trees of the Jth generation by $\mathfrak{T}(J)$. Note that the cardinality of $\mathfrak{T}(J)$ is given by

$$|\mathfrak{T}(J)| = 1 \cdot 3 \cdot 5 \cdots (2J - 1) = (2J - 1)!! =: c_J. \tag{3.15}$$

The notion of ordered trees comes with associated chronicles; it encodes not only the shape of a tree but also how it "grew". This property will be convenient in encoding successive applications of the product rule for differentiation. In the following, we simply refer to an ordered tree $\mathcal{T}_J$ of the Jth generation but it is understood that there is an underlying chronicle $\{\mathcal{T}_j\}_{j=1}^J$.

Given a tree $\mathcal{T}$, we associate each terminal node $a \in \mathcal{T}^\infty$ with the Fourier coefficient (or its complex conjugate) of the interaction representation $\mathbf{u}$ and sum over all possible frequency assignments. In order to do this, we introduce the index function $\mathbf{n}$ assigning frequencies to all the nodes in $\mathcal{T}$ in a consistent manner.

Definition 3.7 (index function)**.** Given an ordered tree $\mathcal{T}$ (of the Jth generation for some $J \in \mathbb{N}$), we define an index function $\mathbf{n} : \mathcal{T} \to \mathbb{Z}$ such that,

(i) $n_a = n_{a_1} - n_{a_2} + n_{a_3}$ for $a \in \mathcal{T}^0$, where a_1, a_2, and a_3 denote the children of a,

(ii) $\{n_a, n_{a_2}\} \cap \{n_{a_1}, n_{a_3}\} = \emptyset$ for $a \in \mathcal{T}^0$,

(iii) $|\mu_1| := |2(n_r - n_{r_1})(n_r - n_{r_3})| > (3K)^\theta$,[12] where r is the root node,

where we identified $\mathbf{n} : \mathcal{T} \to \mathbb{Z}$ with $\{n_a\}_{a \in \mathcal{T}} \in \mathbb{Z}^{\mathcal{T}}$. We use

$$\mathfrak{N}(\mathcal{T}) \subset \mathbb{Z}^{\mathcal{T}}$$

to denote the collection of such index functions $\mathbf{n}$.

Remark 3.8. Note that $\mathbf{n} = \{n_a\}_{a \in \mathcal{T}}$ is completely determined once we specify the values n_a for $a \in \mathcal{T}^\infty$.

Given an ordered tree $\mathcal{T}_J$ of the Jth generation with the chronicle $\{\mathcal{T}_j\}_{j=1}^J$ and associated index functions $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)$, we use superscripts to denote "generations" of frequencies.

Fix $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_J)$. Consider $\mathcal{T}_1$ of the first generation. Its nodes consist of the root node r and its children r_1, r_2, and r_3. We define the first generation of frequencies by

$$(n^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}) := (n_r, n_{r_1}, n_{r_2}, n_{r_3}).$$

The ordered tree $\mathcal{T}_2$ of the second generation is obtained from $\mathcal{T}_1$ by changing one of its terminal nodes $a = r_k \in \mathcal{T}_1^\infty$ for some $k \in \{1, 2, 3\}$ into a non-terminal node. Then, we define the second generation of frequencies by

$$(n^{(2)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

Note that we have $n^{(2)} = n_k^{(1)} = n_{r_k}$ for some $k \in \{1, 2, 3\}$. As we see later, this corresponds to introducing a new set of frequencies after the first differentiation by parts.

After $j - 1$ steps, the ordered tree $\mathcal{T}_j$ of the jth generation is obtained from $\mathcal{T}_{j-1}$ by changing one of its terminal nodes $a \in \mathcal{T}_{j-1}^\infty$ into a non-terminal node. Then, we define the jth generation of frequencies by

$$(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

Note that these frequencies satisfy (i) and (ii) in Definition 3.7.

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[12] Recall that we are on $A_1(n)^c$. See (3.9).

Lastly, we use μ_j to denote the corresponding phase factor introduced at the jth generation. Namely, we have

$$\begin{aligned} \mu_j &= \mu_j(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) \\ &:= (n^{(j)})^2 - (n_1^{(j)})^2 + (n_2^{(j)})^2 - (n_3^{(j)})^2 \\ &= 2(n_2^{(j)} - n_1^{(j)})(n_2^{(j)} - n_3^{(j)}) \\ &= 2(n^{(j)} - n_1^{(j)})(n^{(j)} - n_3^{(j)}), \end{aligned} \tag{3.16}$$

where the last two equalities hold thanks to (i) in Definition 3.7.

Remark 3.9. For simplicity of notation, we may drop the minus signs, the complex number i, and the complex conjugate sign in the following when they do not play an important role.

3.3 Second generation: $J = 2$.

With the ordered tree notion introduced in the previous subsection, we now rewrite (3.14) as

$$\begin{aligned} \widetilde{\mathcal{N}}^{(2)}(\mathbf{u})(n) &= \sum_{\mathcal{T}_1 \in \mathfrak{T}(1)} \sum_{b \in \mathcal{T}_1^\infty} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_1) \\ \mathbf{n}_r = n}} \mathbf{1}_{A_1(n)^c} \frac{e^{i\mu_1 t}}{\mu_1} \mathcal{R}^{(1)}(\mathbf{u})(n_b) \prod_{a \in \mathcal{T}_1^\infty \setminus \{b\}} \widehat{\mathbf{u}}_{n_a} \\ &\quad + \sum_{\mathcal{T}_2 \in \mathfrak{T}(2)} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2) \\ \mathbf{n}_r = n}} \mathbf{1}_{A_1(n)^c} \frac{e^{i(\mu_1 + \mu_2)t}}{\mu_1} \prod_{a \in \mathcal{T}_2^\infty} \widehat{\mathbf{u}}_{n_a} \\ &=: \mathcal{R}^{(2)}(\mathbf{u})(n) + \mathcal{N}^{(2)}(\mathbf{u})(n). \end{aligned} \tag{3.17}$$

In the first equality, we use (1.9) and replace $\partial_t \widehat{\mathbf{u}}_{n_b}$ by $\mathcal{R}^{(1)}(\mathbf{u})(n_b)$ and $\mathcal{N}^{(1)}(\mathbf{u})(n_b)$. Strictly speaking, the new phase factor may be $\mu_1 - \mu_2$ when the time derivative falls on the complex conjugate. However, for our analysis, it makes no difference and hence we simply write it as $\mu_1 + \mu_2$. We apply the same convention for subsequent steps.

Putting (3.13) and (3.17) together, we have

$$\mathcal{N}_2^{(1)}(\mathbf{u})(n) = \partial_t \mathcal{N}_0^{(2)}(\mathbf{u})(n) + \mathcal{R}^{(2)}(\mathbf{u})(n) + \mathcal{N}^{(2)}(\mathbf{u})(n).$$

The boundary term $\mathcal{N}_0^{(2)}(\mathbf{u})$ and the "resonant" term $\mathcal{R}^{(2)}$ can be bounded in a straightforward manner.

Lemma 3.10. *Let* $1 \le p < \infty$. *Then, we have*

$$\begin{aligned}\|\mathcal{N}_0^{(2)}(\mathbf{u})\|_{\mathcal{F}L^p} &\lesssim K^{-4}\|\mathbf{u}\|_{\mathcal{F}L^p}^3,\\ \|\mathcal{R}^{(2)}(\mathbf{u})\|_{\mathcal{F}L^p} &\lesssim K^{-4}\|\mathbf{u}\|_{\mathcal{F}L^p}^5.\end{aligned}$$

For the proof of Lemmas 3.10, see Lemma 3.12 and 3.13 with $J = 2$.

With $\theta > 0$ as in (3.7), we decompose the frequency space[13] of $\mathcal{N}^{(2)}$ for fixed $\mathcal{T}_2 \in \mathfrak{T}(2)$ into

$$A_2 := \{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_2) : |\mu_1 + \mu_2| \le (5K)^\theta\}, \tag{3.18}$$

and its complement A_2^c. Then we decompose $\mathcal{N}^{(2)}$ as

$$\mathcal{N}^{(2)} = \mathcal{N}_1^{(2)} + \mathcal{N}_2^{(2)}, \tag{3.19}$$

where

$$\mathcal{N}_1^{(2)} := \mathcal{N}^{(2)}|_{A_2}$$

is defined as the restriction of $\mathcal{N}^{(2)}$ on A_2 and

$$\mathcal{N}_2^{(2)} := \mathcal{N}^{(2)} - \mathcal{N}_1^{(2)}.$$

Thanks to the restriction (3.18) on the frequencies, we can estimate the first term $\mathcal{N}_1^{(2)}$.

Lemma 3.11. *Let* $1 \le p < \infty$. *Then, we have*

$$\|\mathcal{N}_1^{(2)}(\mathbf{u})\|_{\mathcal{F}L^p} \lesssim K^{\frac{2\theta}{p'}-4}\|\mathbf{u}\|_{\mathcal{F}L^p}^5.$$

For the proof of Lemma 3.11, see Lemma 3.14 with $J = 2$.

As we do not have a good control on the operator $\mathcal{N}_2^{(2)}$, we apply another normal form reduction to $\mathcal{N}_2^{(2)}$. On the support of $\mathcal{N}_2^{(2)}$, we have

$$|\mu_1| > (3K)^\theta \quad \text{and} \quad |\mu_1 + \mu_2| > (5K)^\theta. \tag{3.20}$$

---

[13] If we fix $\mathcal{T}_2 \in \mathfrak{T}(2)$, then the frequency space of $\mathcal{N}^{(2)}$ for this fixed $\mathcal{T}_2$ in (3.17) is given by

$$\{(n_a, a \in \mathcal{T}_2^\infty) : \mathbf{n} = \{n_a\}_{a\in\mathcal{T}_2} \in \mathfrak{N}(\mathcal{T}_2)\}.$$

In view of Remark 3.8, we can then identify the frequency space of $\mathcal{N}^{(2)}$ for this fixed $\mathcal{T}_2$ with $\mathfrak{N}(\mathcal{T}_2)$.

By applying differentiation by parts once again, we have

$$\begin{aligned}
&\mathcal{N}_2^{(2)}(\mathbf{u})(n)\\
&= \partial_t\Bigg[\sum_{\mathcal{T}_2\in\mathfrak{T}(2)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{j=1}^2 A_j^c}\frac{e^{i(\mu_1+\mu_2)t}}{\mu_1(\mu_1+\mu_2)}\prod_{a\in\mathcal{T}_2^\infty}\widehat{\mathbf{u}}_{n_a}\Bigg]\\
&\quad+\sum_{\mathcal{T}_2\in\mathfrak{T}(2)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{j=1}^2 A_j^c}\frac{e^{i(\mu_1+\mu_2)t}}{\mu_1(\mu_1+\mu_2)}\partial_t\bigg(\prod_{a\in\mathcal{T}_2^\infty}\widehat{\mathbf{u}}_{n_a}\bigg)\\
&= \partial_t\Bigg[\sum_{\mathcal{T}_2\in\mathfrak{T}(2)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{j=1}^2 A_j^c}\frac{e^{i(\mu_1+\mu_2)t}}{\mu_1(\mu_1+\mu_2)}\prod_{a\in\mathcal{T}_2^\infty}\widehat{\mathbf{u}}_{n_a}\Bigg]\\
&\quad+\sum_{\mathcal{T}_2\in\mathfrak{T}(2)}\sum_{b\in\mathcal{T}_2^\infty}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_2)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{j=1}^2 A_j^c}\frac{e^{i(\mu_1+\mu_2)t}}{\mu_1(\mu_1+\mu_2)}\mathcal{R}^{(1)}(\mathbf{u})(n_b)\prod_{a\in\mathcal{T}_{J-1}^\infty\setminus\{b\}}\widehat{\mathbf{u}}_{n_a}\\
&\quad+\sum_{\mathcal{T}_3\in\mathfrak{T}(3)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_3)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{j=1}^2 A_j^c}\frac{e^{i(\mu_1+\mu_2+\mu_3)t}}{\mu_1(\mu_1+\mu_2)}\prod_{a\in\mathcal{T}_3^\infty}\widehat{\mathbf{u}}_{n_a}\\
&=: \partial_t\mathcal{N}_0^{(3)}(\mathbf{u})(n)+\mathcal{R}^{(3)}(\mathbf{u})(n)+\mathcal{N}^{(3)}(\mathbf{u}),
\end{aligned}\tag{3.21}$$

where the summations are restricted to (3.20). As for the last term $\mathcal{N}^{(3)}(\mathbf{u})$, we need to decompose it into $\mathcal{N}_1^{(3)}(\mathbf{u})$ and $\mathcal{N}_2^{(3)}(\mathbf{u})$, according to the further restriction

$$A_3 := \{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_3) : |\mu_1+\mu_2+\mu_3| \le (7K)^\theta\}.\tag{3.22}$$

On the one hand, the modulation restrictions (3.9), (3.18), and (3.22) allow us to estimate operators $\mathcal{N}_0^{(3)}$, $\mathcal{R}^{(3)}$, and $\mathcal{N}_1^{(3)}$; see Lemmas 3.10 and 3.11 below. On the other hand, we apply another normal form reduction to $\mathcal{N}_2^{(3)}$. In this way, we iterate normal form reductions in an indefinite manner.

3.4 General step: *J*th generation.

In this subsection, we consider the general *J*th step of normal form reductions. Before doing so, let us first go over the first two steps studied in Subsections 3.1 and 3.3. Write (3.3) as

$$\partial_t\mathbf{u} = \mathcal{R}^{(1)}(\mathbf{u})+\mathcal{N}_1^{(1)}(\mathbf{u})+\mathcal{N}_2^{(1)}(\mathbf{u}).$$

The first two terms on the right-hand side admit good estimates; see Lemmas 3.1 and 3.3. We then applied the first step of normal form reductions to the troublesome term $\mathcal{N}_2^{(1)}(\mathbf{u})$ and obtained

$$\partial_t \mathbf{u} = \partial_t \mathcal{N}_0^{(2)}(\mathbf{u}) + \sum_{j=1}^{2} \mathcal{R}^{(j)}(\mathbf{u}) + \sum_{j=1}^{2} \mathcal{N}_1^{(j)}(\mathbf{u}) + \mathcal{N}_2^{(2)}(\mathbf{u}).$$

See (3.13), (3.17), and (3.19). Note that only the last term $\mathcal{N}_2^{(2)}(\mathbf{u})$ can not be estimated in a direct manner. By applying a normal form reduction once again, we obtained

$$\partial_t \mathbf{u} = \sum_{j=2}^{3} \partial_t \mathcal{N}_0^{(j)}(\mathbf{u}) + \sum_{j=1}^{3} \mathcal{R}^{(j)}(\mathbf{u}) + \sum_{j=1}^{3} \mathcal{N}_1^{(j)}(\mathbf{u}) + \mathcal{N}_2^{(3)}(\mathbf{u}). \tag{3.23}$$

See (3.21). Once again, all the terms in (3.23), except for the last term $\mathcal{N}_2^{(3)}(\mathbf{u})$, admit good estimates; see Lemmas 3.12, 3.13, and 3.14 below. We then apply the third step of normal form reductions to $\mathcal{N}_2^{(3)}(\mathbf{u})$. We can formally iterate this process. In particular, after applying normal form reductions $J-1$ times, we would arrive at

$$\partial_t \mathbf{u} = \sum_{j=2}^{J} \partial_t \mathcal{N}_0^{(j)}(\mathbf{u}) + \sum_{j=1}^{J} \mathcal{R}^{(j)}(\mathbf{u}) + \sum_{j=1}^{J} \mathcal{N}_1^{(j)}(\mathbf{u}) + \mathcal{N}_2^{(J)}(\mathbf{u}). \tag{3.24}$$

In the following, we define each term on the right-hand side of (3.24) properly. With μ_j as in (3.16), define $\widetilde{\mu}_j$ by

$$\widetilde{\mu}_j := \sum_{k=1}^{j} \mu_k.$$

We then set

$$A_j := \{ |\widetilde{\mu}_j| \le ((2j+1)K)^{\theta} \}, \tag{3.25}$$

where $\theta > 0$ is as in (3.7). Given $j \in \mathbb{N}$, we define $\mathcal{N}_2^{(j)}(\mathbf{u})(n)$ by

$$\mathcal{N}_2^{(j)}(\mathbf{u})(n) = \sum_{\mathcal{T}_j \in \mathfrak{T}(j)} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} \mathbf{1}_{\bigcap_{k=1}^{j} A_k^c} \frac{e^{i \widetilde{\mu}_j t}}{\prod_{k=1}^{j-1} \widetilde{\mu}_k} \prod_{a \in \mathcal{T}_j^{\infty}} \widehat{\mathbf{u}}_{n_a}. \tag{3.26}$$

Note that this definition is consistent with $\mathcal{N}_2^{(1)}$, $\mathcal{N}_2^{(2)}$, and $\mathcal{N}_2^{(3)}$ that we saw in the previous subsections. By applying a normal form reduction to (3.26) with (3.3), we obtain

$$\begin{aligned}
\mathcal{N}_2^{(j)}(\mathbf{u})(n) &= \partial_t\Bigg[\sum_{\mathcal{T}_j\in\mathfrak{T}(j)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_j)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{k=1}^j A_k^c}\frac{e^{i\widetilde{\mu}_j t}}{\prod_{k=1}^j\widetilde{\mu}_k}\prod_{a\in\mathcal{T}_j^\infty}\widehat{\mathbf{u}}_{n_a}\Bigg]\\
&\quad+\sum_{\mathcal{T}_j\in\mathfrak{T}(j)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_j)\\ \mathbf{n}_r=n}}\sum_{b\in\mathcal{T}_j^\infty}\mathbf{1}_{\bigcap_{k=1}^j A_k^c}\frac{e^{i\widetilde{\mu}_j t}}{\prod_{k=1}^j\widetilde{\mu}_k}\mathcal{R}^{(1)}(\mathbf{u})(n_b)\prod_{a\in\mathcal{T}_j^\infty\setminus\{b\}}\widehat{\mathbf{u}}_{n_a}\\
&\quad+\sum_{\mathcal{T}_j\in\mathfrak{T}(j)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_j)\\ \mathbf{n}_r=n}}\sum_{b\in\mathcal{T}_j^\infty}\mathbf{1}_{\bigcap_{k=1}^j A_k^c}\frac{e^{i\widetilde{\mu}_j t}}{\prod_{k=1}^j\widetilde{\mu}_k}\mathcal{N}^{(1)}(\mathbf{u})(n_b)\prod_{a\in\mathcal{T}_j^\infty\setminus\{b\}}\widehat{\mathbf{u}}_{n_a}\\
&= \partial_t\Bigg[\sum_{\mathcal{T}_j\in\mathfrak{T}(j)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_j)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{k=1}^j A_k^c}\frac{e^{i\widetilde{\mu}_j t}}{\prod_{k=1}^j\widetilde{\mu}_k}\prod_{a\in\mathcal{T}_j^\infty}\widehat{\mathbf{u}}_{n_a}\Bigg]\\
&\quad+\sum_{\mathcal{T}_j\in\mathfrak{T}(j)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_j)\\ \mathbf{n}_r=n}}\sum_{b\in\mathcal{T}_j^\infty}\mathbf{1}_{\bigcap_{k=1}^j A_k^c}\frac{e^{i\widetilde{\mu}_j t}}{\prod_{k=1}^j\widetilde{\mu}_k}\mathcal{R}^{(1)}(\mathbf{u})(n_b)\prod_{a\in\mathcal{T}_j^\infty\setminus\{b\}}\widehat{\mathbf{u}}_{n_a}\\
&\quad+\sum_{\mathcal{T}_{j+1}\in\mathfrak{T}(j+1)}\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j+1})\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{k=1}^j A_k^c}\frac{e^{i\widetilde{\mu}_{j+1} t}}{\prod_{k=1}^j\widetilde{\mu}_k}\prod_{a\in\mathcal{T}_{j+1}^\infty}\widehat{\mathbf{u}}_{n_a}\\
&=: \partial_t\mathcal{N}_0^{(j+1)}(\mathbf{u})(n)+\mathcal{R}^{(j+1)}(\mathbf{u})(n)+\mathcal{N}^{(j+1)}(\mathbf{u})(n).
\end{aligned}\tag{3.27}$$

Here, we formally exchanged the order of the sum and the time differentiation, which can be justified. See Remark 3.5. As in Subsections 3.1 and 3.3, we divide $\mathcal{N}^{(j+1)}$ into

$$\mathcal{N}^{(j+1)} = \mathcal{N}_1^{(j+1)} + \mathcal{N}_2^{(j+1)},\tag{3.28}$$

where $\mathcal{N}_1^{(j+1)}(\mathbf{u})$ is the restriction of $\mathcal{N}^{(j+1)}(\mathbf{u})$ onto A_{j+1} and

$$\mathcal{N}_2^{(j+1)}(\mathbf{u}) := \mathcal{N}^{(j+1)}(\mathbf{u}) - \mathcal{N}_1^{(j+1)}(\mathbf{u}).$$

This allows us to define all the terms appearing in (3.24) in an inductive manner by applying a normal form reduction to $\mathcal{N}_2^{(j+1)}$.

In the remaining part of this subsection, we estimate the multilinear operators $\mathcal{N}_0^{(j)}$, $\mathcal{R}^{(j)}$, and $\mathcal{N}_1^{(j)}$.

Lemma 3.12. *Let* $1 \le p < \infty$. *Then, there exists* $C_p > 0$ *such that*

$$\|\mathcal{N}_0^{(j)}(\mathbf{u})\|_{\mathcal{F}L^p} \le C_p\frac{K^{4(1-j)}}{((2j-1)!!)^2}\|\mathbf{u}\|_{\mathcal{F}L^p}^{2j-1}\tag{3.29}$$

for any integer $j \ge 2$ *and* $K \ge 1$.

Proof. From (3.27) (with $j+1$ replaced by j), we have

$$\mathcal{N}_0^{(j)}(\mathbf{u})(n) = \sum_{\mathcal{T}_{j-1}\in\mathfrak{T}(j-1)} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c} \frac{e^{i\widetilde{\mu}_{j-1}t}}{\prod_{k=1}^{j-1}\widetilde{\mu}_k} \prod_{a\in\mathcal{T}_{j-1}^\infty} \widehat{\mathbf{u}}_{n_a}.$$

Then, by Hölder's inequality with (3.15), we have

$$\begin{aligned}
&\|\mathcal{N}_0^{(j)}(\mathbf{u})\|_{\mathcal{F}L^p}\\
&\le \Bigg\| \sum_{\mathcal{T}_{j-1}\in\mathfrak{T}(j-1)} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c}}{\prod_{k=1}^{j-1}|\widetilde{\mu}_k|^{p'}} \Bigg)^{\frac{1}{p'}} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \prod_{a\in\mathcal{T}_{j-1}^\infty} |\widehat{\mathbf{u}}_{n_a}|^p \Bigg)^{\frac{1}{p}} \Bigg\|_{\ell^p_n}\\
&\le \sup_{\substack{\mathcal{T}_{j-1}\in\mathfrak{T}(j-1)\\ n\in\mathbb{Z}}} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c}}{\prod_{k=1}^{j-1}|\widetilde{\mu}_k|^{p'}} \Bigg)^{\frac{1}{p'}}\\
&\qquad \times \sum_{\mathcal{T}_{j-1}\in\mathfrak{T}(j-1)} \Bigg\| \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \prod_{a\in\mathcal{T}_{j-1}^\infty} |\widehat{\mathbf{u}}_{n_a}|^p \Bigg)^{\frac{1}{p}} \Bigg\|_{\ell^p_n}\\
&\le (2j-3)!! \sup_{\substack{\mathcal{T}_{j-1}\in\mathfrak{T}(j-1)\\ n\in\mathbb{Z}}} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c}}{\prod_{k=1}^{j-1}|\widetilde{\mu}_k|^{p'}} \Bigg)^{\frac{1}{p'}} \|\mathbf{u}\|_{\mathcal{F}L^p}^{2j-1}.
\end{aligned} \tag{3.30}$$

In the last step, we used

$$\Bigg(\sum_n \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \prod_{a\in\mathcal{T}_{j-1}^\infty} |\widehat{\mathbf{u}}_{n_a}|^p \Bigg)^{\frac{1}{p}} = \|\mathbf{u}\|_{\mathcal{F}L^p}^{2j-1}.$$

We claim that

$$\sup_{\substack{\mathcal{T}_{j-1}\in\mathfrak{T}(j-1)\\ n\in\mathbb{Z}}} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1})\\ \mathbf{n}_r=n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c}}{\prod_{k=1}^{j-1}|\widetilde{\mu}_k|^{p'}} \Bigg)^{\frac{1}{p'}} \le B_p^{j-1} K^{4(1-j)}((2j-1)!!)^{-4}, \tag{3.31}$$

where $B_p > 0$ is a constant depending only on p. Then, by setting

$$C_p := \sup_{j\ge 2} \Bigg(\frac{B_p^{j-1}}{(2j-1)!!} \Bigg) < \infty,$$

we see that (3.29) follows from (3.30) and (3.31).

It remains to prove (3.31). First, note that given any small $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sup_{\substack{\mathcal{T}_{j-1}\in\mathfrak{T}(j-1)\\ n\in\mathbb{Z}}} \{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{j-1}) : \mathbf{n}_r = n,\, |\widetilde{\mu}_k| = \alpha_k,\, k = 1,\dots,j-1\} \\ \le C^{j-1}\prod_{k=1}^{j-1}|\alpha_k|^{\varepsilon}. \tag{3.32}$$

See Lemma 8.16 in [35] for an analogous statement. It follows from the divisor estimate (3.11) that for fixed $n^{(k)}$ and μ_k, there are at most $O(|\mu_k|^{0+})$ many choices for $n_1^{(k)}$, $n_2^{(k)}$, and $n_3^{(k)}$. Noting that $|\mu_k| \le |\alpha_k| + |\alpha_{k-1}|$, we can iterate this argument from $k = 1$ to $j-1$ and obtain (3.32).

From (3.25) and (3.32) with (3.7), we have

$$\begin{aligned}\text{LHS of (3.31)} &\le C^{j-1}\prod_{k=1}^{j-1}\Bigg(\sum_{\substack{|\widetilde{\mu}_k|>((2k+1)K)^{\theta}\\ k=1,\dots,j-1}}\frac{1}{|\widetilde{\mu}_k|^{p'-\varepsilon}}\Bigg)^{\frac{1}{p'}}\\ &\le C^{j-1}\prod_{k=1}^{j-1}\bigg(\int_{((2k+1)K)^{\theta}}^{\infty}t^{-p'+\varepsilon}\,dt\bigg)^{\frac{1}{p'}}\\ &= B_p^{(j-1)}K^{4(1-j)}((2j-1)!!)^{-4}.\end{aligned}$$

Recalling that ε in (3.6) depends only on p, we see that B_p and hence C_p depend only on $1 \le p < \infty$. This completes the proof of Lemma 3.12. □

As a consequence of Lemma 3.12 with Lemma 3.1, we obtain the following estimate on $\mathcal{R}^{(j)}$.

Lemma 3.13. *Let* $1 \le p < \infty$. *Then, there exists* $C_p > 0$ *such that*

$$\|\mathcal{R}^{(j)}(\mathbf{u})\|_{\mathcal{F}L^p} \le C_p\frac{(2j-1)K^{4(1-j)}}{((2j-1)!!)^2}\|\mathbf{u}\|_{\mathcal{F}L^p}^{2j+1} \tag{3.33}$$

for any $j\in\mathbb{N}$ *and* $K \ge 1$.

Proof. When $j = 1$, this is precisely Lemma 3.1. Let $j \ge 2$. Note that $\mathcal{R}^{(j)}(\mathbf{u})$ is nothing but $\mathcal{N}_0^{(j)}(\mathbf{u})$ by replacing $\widehat{\mathbf{u}}_{n_b}$ with $\mathcal{R}^{(1)}(\mathbf{u})(n_b)$ for $b\in\mathcal{T}_j^{\infty}$ and summing over $b\in\mathcal{T}_j^{\infty}$. Then, (3.33) follows from Lemma 3.12 with Lemma 3.1 and noting that given $\mathcal{T}_j\in\mathfrak{T}(j-1)$, we have $\#\{b : b\in\mathcal{T}_{j-1}^{\infty}\} = 2j-1$. This extra factor $2j-1$ does not cause a problem thanks to the fast decaying constant in (3.33). □

Lastly, we estimate $\mathcal{N}_1^{(j)}(\mathbf{u})$, namely, the restriction of $\mathcal{N}^{(j)}$ onto A_j.

Lemma 3.14. *Let* $1 \le p < \infty$*. Then, there exists* $C_p > 0$ *such that*

$$\|\mathcal{N}_1^{(j)}(\mathbf{u})\|_{\mathcal{F}L^p} \le C_p \frac{K^{\frac{2\theta}{p'}+4(1-j)}}{((2j-1)!!)^2} \|\mathbf{u}\|_{\mathcal{F}L^p}^{2j+1}, \tag{3.34}$$

for any $j \in \mathbb{N}$ *and* $K \ge 1$*.*

Proof. From (3.27) (with $j+1$ replaced by j), we have

$$\mathcal{N}_1^{(j)}(\mathbf{u})(n) = \sum_{\mathcal{T}_j \in \mathfrak{T}(j)} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} \mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c \cap A_j} \frac{e^{i\widetilde{\mu}_j t}}{\prod_{k=1}^{j-1} \widetilde{\mu}_k} \prod_{a \in \mathcal{T}_j^\infty} \widehat{\mathbf{u}}_{n_a}.$$

Proceeding as in (3.30) with Hölder's inequality, we have

$$\begin{aligned} \|\mathcal{N}_1^{(j)}(\mathbf{u})\|_{\mathcal{F}L^p} &\le \sup_{\substack{\mathcal{T}_j \in \mathfrak{T}(j) \\ n \in \mathbb{Z}}} \bigg(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c \cap A_j}}{\prod_{k=1}^{j-1} |\widetilde{\mu}_k|^{p'}} \bigg)^{\frac{1}{p'}} \\ &\qquad \times \sum_{\mathcal{T}_j \in \mathfrak{T}(j)} \bigg\| \bigg(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} \prod_{a \in \mathcal{T}_j^\infty} |\widehat{\mathbf{u}}_{n_a}|^p \bigg)^{\frac{1}{p}} \bigg\|_{\ell_n^p} \\ &\le (2j-1)!! \sup_{\substack{\mathcal{T}_j \in \mathfrak{T}(j) \\ n \in \mathbb{Z}}} \bigg(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c \cap A_j}}{\prod_{k=1}^{j-1} |\widetilde{\mu}_k|^{p'}} \bigg)^{\frac{1}{p'}} \|\mathbf{u}\|_{\mathcal{F}L^p}^{2j-1}. \end{aligned} \tag{3.35}$$

We claim that there exists $B_p > 0$ such that

$$\sup_{\substack{\mathcal{T}_j \in \mathfrak{T}(j) \\ n \in \mathbb{Z}}} \bigg(\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{j-1} A_k^c \cap A_j}}{\prod_{k=1}^{j-1} |\widetilde{\mu}_k|^{p'}} \bigg)^{\frac{1}{p'}} \le B_p^{j-1} (2j+1)^{1+\frac{2\theta}{p'}} K^{\frac{2\theta}{p'}+4(1-j)} ((2j-1)!!)^{-4}. \tag{3.36}$$

Then, the desired estimate (3.34) follows from (3.35) and (3.36) by setting

$$C_p := \sup_{j \ge 2} \bigg(\frac{B_p^{j-1}(2j+1)^{1+\frac{2\theta}{p'}}}{(2j-1)!!} \bigg).$$

It remains to prove (3.36). As compared to (3.31) in the proof of Lemma 3.12, the main difference is that the summation in (3.36) is over $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j)$ rather than $\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{j-1})$. Note that

$$\sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} = \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_{j-1}) \\ \mathbf{n}_r = n}} \sum_{b \in \mathcal{T}_{j-1}^\infty} \sum_{n_b = n_1^{(j)} - n_2^{(j)} + n_3^{(j)}}. \tag{3.37}$$

With $n_b = n^{(j)}$, let μ_j be as in (3.16). Then, thanks to the restriction A_j in (3.25), we see that for fixed $\widetilde{\mu}_{j-1}$ there are at most $((2j+1)K)^\theta$ many choices of μ_j. Moreover, we have $|\mu_j| \leq |\widetilde{\mu}_{j-1}| + ((2j+1)K)^\theta$. Then, by the divisor estimate (3.11), we conclude that

$$\sum_{b\in\mathcal{T}_{j-1}^\infty} \sum_{n_b = n_1^{(j)} - n_2^{(j)} + n_3^{(j)}} \mathbf{1}_{A_j} \lesssim (2j+1)((2j+1)K)^\theta(((2j+1)K)^\theta + |\widetilde{\mu}_{j-1}|)^{0+}. \tag{3.38}$$

Thus (3.36) follows from (3.31) together with (3.37) and (3.38). □

3.5 On the error term $\mathcal{N}_2^{(J)}$ and the proof of Proposition 2.1.

We first prove that the remainder term $\mathcal{N}_2^{(J)}(\mathbf{u})$ in (3.24) tends to zero as $J \to \infty$ under some regularity assumption on $\mathbf{u}$.

Lemma 3.15. *Let* $\mathcal{N}_2^{(J)}$ *be as in* (3.26) *with* $j = J$ *and* $T > 0$. *Then, given* $\mathbf{u} \in C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$, *we have*

$$\sup_{t\in[-T,T]} \|\mathcal{N}_2^{(J)}(\mathbf{u})\|_{\mathcal{F}L^\infty} \longrightarrow 0, \tag{3.39}$$

as $J \to \infty$.

Proof. By Young's inequality, we have

$$\|\mathcal{N}^{(1)}(\mathbf{u})\|_{\mathcal{F}L^\infty} + \|\mathcal{R}^{(1)}(\mathbf{u})\|_{\mathcal{F}L^\infty} \lesssim \|\mathbf{u}\|^3_{\mathcal{F}L^{\frac{3}{2}}}. \tag{3.40}$$

From (3.28) (with $j+1$ replaced by J), we have

$$\mathcal{N}_2^{(J)}(\mathbf{u}) = \mathcal{N}^{(J)}(\mathbf{u}) - \mathcal{N}_1^{(J)}(\mathbf{u}). \tag{3.41}$$

Then, by rewriting (3.27) (with $j+1$ replaced by J), we have

$$\begin{aligned}
\mathcal{N}^{(J)}(\mathbf{u})(n) &= \sum_{\mathcal{T}_J\in\mathfrak{T}(J)} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_J)\\ \mathbf{n}_r=n}} \mathbf{1}_{\bigcap_{k=1}^{J-1} A_k^c} \frac{e^{i\widetilde{\mu}_J t}}{\prod_{k=1}^{J-1}\widetilde{\mu}_k} \prod_{a\in\mathcal{T}_J^\infty} \widehat{\mathbf{u}}_{n_a} \\
&= \sum_{\mathcal{T}_{J-1}\in\mathfrak{T}(J-1)} \sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J-1})\\ \mathbf{n}_r=n}} \sum_{b\in\mathcal{T}_{J-1}^\infty} \mathbf{1}_{\bigcap_{k=1}^{J-1} A_k^c} \frac{e^{i\widetilde{\mu}_J t}}{\prod_{k=1}^{J-1}\widetilde{\mu}_k} \\
&\qquad\qquad \times (\mathcal{N}^{(1)} + \mathcal{R}^{(1)})(\mathbf{u})(n_b) \prod_{a\in\mathcal{T}_{J-1}^\infty\setminus\{b\}} \widehat{\mathbf{u}}_{n_a}.
\end{aligned}$$

Proceeding as in the proof of Lemma 3.12 with (3.15), (3.31), and (3.40), we have

$$\begin{aligned}\|\mathcal{N}^{(J)}(\mathbf{u})\|_{\mathcal{F}L^\infty} &\lesssim |\mathfrak{T}^\infty_{J-1}| \sum_{\mathcal{T}_{J-1}\in\mathfrak{T}(J-1)} \sup_{\substack{b\in\mathcal{T}^\infty_{J-1}\\ n\in\mathbb{Z}}} \Bigg\{\Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J-1})\\ \mathbf{n}_r=n}} \frac{\mathbf{1}_{\bigcap_{k=1}^{J-1} A_k^c}}{\prod_{k=1}^{J-1}|\widetilde{\mu}_k|^{p'}}\Bigg)^{\frac{1}{p'}}\\ &\qquad \times \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J-1})\\ \mathbf{n}_r=n}} |(\mathcal{N}^{(1)}+\mathcal{R}^{(1)})(\mathbf{u})(n_b)|^p \prod_{a\in\mathcal{T}^\infty_{J-1}\setminus\{b\}} |\widehat{\mathbf{u}}_{n_a}|^p\Bigg)^{\frac{1}{p}}\Bigg\}\\ &\le B_p^{J-1}K^{-4(J-1)}\,((2J-1)!!)^{-3}\,\|\mathbf{u}\|^3_{\mathcal{F}L^{\frac{3}{2}}}\cdot\\ &\qquad \times \sup_{\substack{b\in\mathcal{T}^\infty_{J-1}\\ n\in\mathbb{Z}}} \Bigg(\sum_{\substack{\mathbf{n}\in\mathfrak{N}(\mathcal{T}_{J-1})\\ \mathbf{n}_r=n}} \prod_{a\in\mathcal{T}^\infty_{J-1}\setminus\{b\}} |\widehat{\mathbf{u}}_{n_a}|^p\Bigg)^{\frac{1}{p}}\\ &\lesssim B_p^{J-1}K^{-4(J-1)}((2J-1)!!)^{-2}\|\mathbf{u}\|^3_{\mathcal{F}L^{\frac{3}{2}}}\,\|\mathbf{u}\|^{2J}_{\mathcal{F}L^p}\end{aligned} \tag{3.42}$$

for any $1 \le p < \infty$. Therefore, (3.39) follows from (3.41) with Lemma 3.14 and (3.42) with $p = \frac{3}{2}$ by taking $J \to \infty$. □

We briefly discuss the proof of Proposition 2.1.

Proof of Proposition 2.1. In view of Lemmas 3.12, 3.13, and 3.14, it suffices to verify that any solution $\mathbf{u} \in C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ to (1.9) satisfies the normal form equation (2.1). By integrating (3.24) in time, we have

$$\begin{aligned}\mathbf{u}(t) - \mathbf{u}(0) &= \sum_{j=2}^{J} \mathcal{N}_0^{(j)}(\mathbf{u})(t) - \sum_{j=2}^{J} \mathcal{N}_0^{(j)}(\mathbf{u})(0)\\ &\qquad + \int_0^t \Bigg\{\sum_{j=1}^{J} \mathcal{N}_1^{(j)}(\mathbf{u})(t') + \sum_{j=1}^{J} \mathcal{R}^{(j)}(\mathbf{u})(t')\Bigg\}dt' + \int_0^t \mathcal{N}_2^{(J)}(\mathbf{u})(t')dt'.\end{aligned}$$

By letting $J \to \infty$, we deduce from Lemma 3.15 that the normal form equation (2.1) holds in $C([-T, T]; \mathcal{F}L^\infty(\mathbb{T}))$.

Given $J \ge 2$, set

$$\begin{aligned}X_J = \mathbf{u}(t) - \mathbf{u}(0) - \Bigg[&\sum_{j=2}^{J} \mathcal{N}_0^{(j)}(\mathbf{u})(t) - \sum_{j=2}^{J} \mathcal{N}_0^{(j)}(\mathbf{u}_0)\\ &+ \int_0^t \Bigg\{\sum_{j=1}^{J} \mathcal{N}_1^{(j)}(\mathbf{u})(t') + \sum_{j=1}^{J} \mathcal{R}^{(j)}(\mathbf{u})(t')\Bigg\}dt'\Bigg].\end{aligned}$$

On the one hand, it follows from Lemmas 3.12, 3.13, and 3.14 that X_J converges to some X_∞ in $C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ as $J \to \infty$. See (2.6). On the other hand, we know that X_J converges to 0 in $C([-T, T]; \mathcal{F}L^\infty(\mathbb{T}))$. Therefore, by the uniqueness of the limit, we conclude that X_J tends to 0 in $C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$ as $J \to \infty$. This shows that the normal form equation (2.1) holds in $C([-T, T]; \mathcal{F}L^{\frac{3}{2}}(\mathbb{T}))$. □

3.6 On the cubic NLS. We conclude this section by briefly discussing the case of the (unrenormalized) cubic NLS (1.1). The only difference appears from the extra term $\mathcal{R}_2$ in (2.11). When $j = 1$, we simply set $\mathcal{R}_2^{(1)}(\mathbf{u})(n) = \mathcal{R}_2(\mathbf{u})(n)$. When we apply a normal form reduction and substitute $\partial_t \mathbf{u}$ by the equation (2.11), there is an extra term due to $\mathcal{R}_2$. By repeating the computation in (3.27), we have

$$\begin{aligned}\mathcal{N}_2^{(j)}(\mathbf{u})(n) &= \partial_t \mathcal{N}_0^{(j+1)}(\mathbf{u})(n) + \mathcal{R}^{(j+1)}(\mathbf{u})(n) + \mathcal{N}^{(j+1)}(\mathbf{u})(n) \\ &\quad + \sum_{\mathcal{T}_j \in \mathfrak{T}(j)} \sum_{\substack{\mathbf{n} \in \mathfrak{N}(\mathcal{T}_j) \\ \mathbf{n}_r = n}} \sum_{b \in \mathcal{T}_j^\infty} \mathbf{1}_{\bigcap_{k=1}^j A_k^c} \frac{e^{i\widetilde{\mu}_j t}}{\prod_{k=1}^j \widetilde{\mu}_k} \mathcal{R}_2(\mathbf{u})(n_b) \prod_{a \in \mathcal{T}_j^\infty \setminus \{b\}} \widehat{\mathbf{u}}_{n_a} \\ &=: \partial_t \mathcal{N}_0^{(j+1)}(\mathbf{u})(n) + \mathcal{R}^{(j+1)}(\mathbf{u})(n) + \mathcal{N}^{(j+1)}(\mathbf{u})(n) + \mathcal{R}_2^{(j+1)}(\mathbf{u})(n).\end{aligned}$$

Proposition 2.3 follows exactly as for Proposition 2.1 once we note the following bound on $\mathcal{R}_2^{(j)}$.

Lemma 3.16. *Let* $1 \le p \le 2$. *Then, there exists* $C_p > 0$ *such that*

$$\|\mathcal{R}_2^{(j)}(\mathbf{u})\|_{\mathcal{F}L^p} \le C_p \frac{(2j-1)K^{4(1-j)}}{((2j-1)!!)^2} \|\mathbf{u}\|_{\mathcal{F}L^p}^{2j+1}$$

for any $j \in \mathbb{N}$ *and* $K \ge 1$.

Proof. This lemma follows from Lemma 3.12 as in the proof of Lemma 3.13 once we note that

$$\|\mathcal{R}_2(\mathbf{u})\|_{\mathcal{F}L^p} \le \|\mathbf{u}\|_{\mathcal{F}L^p}^3$$

when $1 \le p \le 2$. □

Appendix A On the persistence of regularity in $\mathcal{F}L^p(\mathbb{T})$, $1 \le p < 2$

We first recall the basic definitions and properties of the Fourier restriction norm spaces $X_p^{s,b}(\mathbb{T} \times \mathbb{R})$ adapted to the Fourier–Lebesgue spaces. Let $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ be the vector space of C^∞-functions $u : \mathbb{R}^2 \to \mathbb{C}$ such that

$$u(x,t) = u(x+1,t) \quad \text{and} \quad \sup_{(x,t) \in \mathbb{R}^2} |t^\alpha \partial_t^\beta \partial_x^\gamma u(x,t)| < \infty$$

for any $\alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}$.

Definition A.1. Let $s, b \in \mathbb{R}$, $1 \le p \le \infty$. We define the space $X_p^{s,b}(\mathbb{T} \times \mathbb{R})$ as the completion of $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ with respect to the norm

$$\|u\|_{X_p^{s,b}(\mathbb{T} \times \mathbb{R})} = \|\langle n \rangle^s \langle \tau + n^2 \rangle^b \widehat{u}(n,\tau)\|_{\ell_n^p L_\tau^p(\mathbb{Z} \times \mathbb{R})}. \tag{A.1}$$

For brevity, we simply denote $X^{s,b}_p(\mathbb{T}\times\mathbb{R})$ by $X^{s,b}_p$. Recall the following characterization of the $X^{s,b}_p$-norm in terms of the interaction representation $\mathbf{u}(t) = S(-t)u(t)$:

$$\|u\|_{X^{s,b}_p} = \|\mathbf{u}\|_{\mathcal{F}L^{s,p}_x \mathcal{F}L^{b,p}_t},$$

where the iterated norm is to be understood in the following sense:

$$\|\mathbf{u}\|_{\mathcal{F}L^{s,p}_x \mathcal{F}L^{b,p}_t} := \|\langle n\rangle^s \langle\tau\rangle^b \widehat{\mathbf{u}}(n,\tau)\|_{\ell^p_n L^p_\tau} = \big\| \|\langle n\rangle^s \widehat{\mathbf{u}}(n,t)\|_{\mathcal{F}L^{b,p}_t}\big\|_{\ell^p_n}.$$

Here, $\mathcal{F}L^{s,p}_x(\mathbb{T})$ is as in (1.7) and $\mathcal{F}L^{b,p}_t(\mathbb{R})$ is defined by the norm:

$$\|f\|_{\mathcal{F}L^{b,p}(\mathbb{R})} := \|\langle\tau\rangle^b \widehat{f}(\tau)\|_{L^p_\tau(\mathbb{R})}.$$

Note that these spaces are separable when $p < \infty$.

For any $1 \le p < \infty$ and $s \in \mathbb{R}$, we have

$$X^{s,b}_p \hookrightarrow C(\mathbb{R};\mathcal{F}L^{s,p}(\mathbb{T})), \quad \text{if } b > \frac{1}{p'} = 1 - \frac{1}{p}. \tag{A.2}$$

This is a consequence of the dominated convergence theorem along with the following embedding relation: $\mathcal{F}L^{b,p}_t \hookrightarrow \mathcal{F}L^1_t \hookrightarrow C_t$, where the second embedding is the Riemann–Lebesgue lemma.

Given an interval $I \subset \mathbb{R}$, we also define the local-in-time version $X^{s,b}_p(I)$ of the $X^{s,b}_p$-space as the collection of functions u such that

$$\|u\|_{X^{s,b}_p(I)} := \inf\{\|v\|_{X^{s,b}_p} : v|_I = u\} \tag{A.3}$$

is finite.

Lastly, we recall the following linear estimates. See [15] for the proof.

Lemma A.2. (i) (Homogeneous linear estimate). *Given* $1 \le p \le \infty$ *and* $s, b \in \mathbb{R}$, *we have*

$$\|S(t)f\|_{X^{s,b}_p([0,T])} \lesssim \|f\|_{\mathcal{F}L^{s,p}}$$

for any $0 < T \le 1$.

(ii) (Nonhomogeneous linear estimate). *Let* $s \in \mathbb{R}$, $1 \le p < \infty$, *and*

$$-\frac{1}{p} < b' \le 0 \le b \le 1 + b'.$$

Then, we have

$$\left\| \int_0^t S(t-t')F(t')dt' \right\|_{X^{s,b}_p([0,T])} \lesssim T^{1+b'-b}\|F\|_{X^{s,b'}_p([0,T])} \tag{A.4}$$

for any $0 < T \le 1$.

The nonhomogeneous linear estimate (A.4) is based on (2.21) in [18]. While $p > 1$ is assumed in [18], the estimate also holds true when $p = 1$.

The following trilinear estimate is the key ingredient for establishing the persistence of regularity in $\mathcal{F}L^p(\mathbb{T})$, $1 \leq p < 2$.

Lemma A.3. *Let* $1 \leq p \leq 2$. *Then, there exists small* $\varepsilon > 0$ *(independent of* p*) such that*

$$\| |u|^2 u \|_{X_p^{0,-\frac{1}{2}+2\varepsilon}([0,T])} \lesssim \|u\|^2_{X_2^{0,\frac{1}{2}+\varepsilon}([0,T])} \|u\|_{X_p^{0,\frac{1}{2}+\varepsilon}([0,T])} \tag{A.5}$$

for any $0 < T \leq 1$.

Proof. By a standard argument, it suffices to prove (A.5) without a time restriction:

$$\| |u|^2 u \|_{X_p^{0,-\frac{1}{2}+2\varepsilon}} \lesssim \|u\|^2_{X_2^{0,\frac{1}{2}+\varepsilon}} \|u\|_{X_p^{0,\frac{1}{2}+\varepsilon}}. \tag{A.6}$$

We first estimate the non-resonant contribution from I in (2.10). We follow the argument in [19]. Let $\sigma_0 = \tau + n^2$ and $\sigma_j = \tau_j + n_j^2$, $j = 1, 2, 3$. Then, (A.6) follows once we prove

$$\bigg\| \frac{1}{\langle \sigma_0 \rangle^{\frac{1}{2}-2\varepsilon}} \sum_{\substack{n=n_1-n_2+n_3 \\ n\neq n_1,n_3}} \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \frac{f_j(n_j, \tau_j)}{\langle \sigma_j \rangle^{\frac{1}{2}+\varepsilon}} d\tau_1 d\tau_2 \bigg\|_{\ell^p_n L^p_\tau} \lesssim \bigg(\prod_{j=1}^{2} \|f_j\|_{\ell^2_n L^2_\tau} \bigg) \|f_3\|_{\ell^p_n L^p_\tau}.$$

By Cauchy–Schwarz and Young's inequalities, it suffices to prove

$$\bigg\| \frac{1}{\langle \sigma_0 \rangle^{1-4\varepsilon}} \sum_{\substack{n=n_1-n_2+n_3 \\ n\neq n_1,n_3}} \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \frac{1}{\langle \sigma_j \rangle^{1+2\varepsilon}} d\tau_1 d\tau_2 \bigg\|_{\ell^\infty_n L^\infty_\tau} < \infty. \tag{A.7}$$

From (1.10), we have

$$\prod_{j=0}^{4} \frac{1}{\langle \sigma_j \rangle^{\varepsilon}} \lesssim \frac{1}{\langle (n-n_1)(n-n_3) \rangle^{\varepsilon}}.$$

Then, by estimating the convolutions in τ_j (see Lemma 4.2 in [16]) and applying (1.10), we have

LHS of (A.7)

$$\lesssim \bigg\| \frac{1}{\langle \sigma_0 \rangle^{1-3\varepsilon}} \sum_{\substack{n=n_1-n_2+n_3 \\ n\neq n_1,n_3}} \frac{1}{\langle n-n_1 \rangle^{\varepsilon} \langle n-n_3 \rangle^{\varepsilon}} \frac{1}{\langle \tau+n^2-2(n-n_1)(n-n_3) \rangle^{1+\varepsilon}} \bigg\|_{\ell^\infty_n L^\infty_\tau}$$

$$\lesssim \bigg\| \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{\langle k \rangle^{\varepsilon}} \frac{1}{\langle \tau+n^2-2k \rangle^{1+\varepsilon}} d(k) \bigg\|_{\ell^\infty_n L^\infty_\tau} < \infty,$$

where we used the divisor estimate (3.11) in the last step.

Next, we estimate the contribution from the resonant parts $\mathbb{I}$ and $\mathbf{I}$ in (2.10). By Young's inequality followed by Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|\mathbb{I}\|_{X_p^{0,-\frac{1}{2}+2\varepsilon}} &\lesssim \bigg\| \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{u}(n,\tau_1)\overline{\widehat{u}(n,\tau_2)}\widehat{u}(n,\tau_3)d\tau_1 d\tau_2 \bigg\|_{\ell_n^p L_\tau^p} \\ &\lesssim \|\widehat{u}\|^2_{\ell_n^\infty L_\tau^1}\|\widehat{u}\|_{\ell_n^p L_\tau^p} \\ &\lesssim \|u\|^2_{X_2^{0,\frac{1}{2}+\varepsilon}}\|u\|_{X_p^{0,\frac{1}{2}+\varepsilon}}.\end{aligned}$$

With (A.2), we have

$$\begin{aligned}\|\mathbf{I}\|_{X_p^{0,-\frac{1}{2}+2\varepsilon}} &\lesssim \|u\|^2_{L_t^\infty L_x^2}\|\widehat{u}(n,\tau)\|_{\ell_n^p L_\tau^p} \\ &\lesssim \|u\|^2_{X_2^{0,\frac{1}{2}+\varepsilon}}\|u\|_{X_p^{0,\frac{1}{2}+\varepsilon}}.\end{aligned}$$

This completes the proof of Lemma A.3. □

When $p = 2$, Lemmas A.2 and A.3 allow us to prove local well-posedness of (1.1) in $L^2(\mathbb{T})$, where the local existence time is given by

$$T = T(\|u_0\|_{L^2}) \sim (1 + \|u_0\|_{L^2})^{-\theta} > 0 \tag{A.8}$$

for some $\theta > 0$. For $1 \le p < 2$, by applying Lemmas A.2 and A.3, we can easily prove local well-posedness of (1.1) in $\mathcal{F}L^p(\mathbb{T})$, where the local existence time T is given as in (A.8), namely, it depends only on the L^2-norm of initial data u_0. In this case, a contraction argument yields

$$\sup_{t\in[0,T]} \|u(t)\|_{\mathcal{F}L^p} \le C\|u_0\|_{\mathcal{F}L^p} \tag{A.9}$$

for some absolute constant $C > 0$. Then, by iterating the local argument with (A.8) and the L^2-conservation, we conclude from (A.8) and (A.9) that

$$\sup_{t\in[0,\tau]} \|u(t)\|_{\mathcal{F}L^p} \le C^{(1+\|u_0\|_{L^2})^\theta \tau}\|u_0\|_{\mathcal{F}L^p} \tag{A.10}$$

for any $\tau > 0$. This proves global well-posedness of (1.1) in $\mathcal{F}L^p(\mathbb{T})$, $1 \le p < 2$, with the growth bound (A.10) on the $\mathcal{F}L^p$-norm of solutions. A similar argument yields global well-posedness of the renormalized cubic NLS (1.2) in $\mathcal{F}L^p(\mathbb{T})$, $1 \le p < 2$.

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REFERENCES

[1] M. Ablowitz, D. Kaup, D. Newell and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53** (1974), 249–315.

[2] M. Ablowitz and Y. Ma, *The periodic cubic Schrödinger equation*, Stud. Appl. Math. **65** (1981), 113–158.

[3] A. Babin, A. Ilyin and E. Titi, *On the regularization mechanism for the periodic Korteweg–de Vries equation*, Comm. Pure Appl. Math. **64** (2011), 591–648.

[4] V. Banica and L. Vega, *The initial value problem for the binormal flow with rough data*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), 1423–1455.

[5] V. Banica and L. Vega, *Singularity formation for the 1-D cubic NLS and the Schrödinger map on* $\mathbb{S}^2$, Commun. Pure Appl. Anal. **17** (2018), 1317–1329.

[6] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), 107–156.

[7] N. Burq, P. Gérard and N. Tzvetkov, *An instability property of the nonlinear Schrödinger equation on* $\mathbb{S}^d$, Math. Res. Lett. **9** (2002), 323–335.

[8] M. Christ, *Nonuniqueness of weak solutions of the nonlinear Schrödinger equation*, arXiv:math/0503366v1 [math.AP].

[9] M. Christ, *Power series solution of a nonlinear Schrödinger equation*, in *Mathematical Aspects of Nonlinear Dispersive Equations*, Princeton University Press, Princeton, NJ, 2007, pp. 131–155.

[10] M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*, Amer. J. Math. **125** (2003), 1235–1293.

[11] J. Chung, Z. Guo, S. Kwon and T. Oh, *Normal form approach to global well-posedness of the quadratic derivative nonlinear Schrödinger equation on the circle*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), 1273–1297.

[12] J. Colliander and T. Oh, *Almost sure well-posedness of the periodic cubic nonlinear Schrödinger equation below* $L^2(\mathbb{T}^2)$, Duke Math. J. **161** (2012), 367–414.

[13] M. B. Erdoğan and N. Tzirakis, *Global smoothing for the periodic KdV evolution*, Int. Math. Res. Not. IMRN **2013** (2013), 4589–4614.

[14] J. Forlano and T. Oh, *Normal form approach to the one-dimensional cubic nonlinear Schrödinger equation in Fourier amalgam spaces*, preprint.

[15] J. Forlano, T. Oh and Y. Wang, *Stochastic cubic nonlinear Schrödinger equation with almost space-time white noise*, J. Aust. Math. Soc. **109** (2020), 44–67.

[16] J. Ginibre, Y. Tsutsumi and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151** (1997), 384–436.

[17] B. Grébert and T. Kappeler, *The Defocusing NLS Equation and its Normal Form*, European Mathematical Society, Zürich, 2014.

[18] A. Grünrock, *An improved local well-posedness result for the modified KdV equation*, Int. Math. Res. Not. IMRN **2004** (2004), 3287–3308.

[19] A. Grünrock and S. Herr, *Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data*, SIAM J. Math. Anal. **39** (2008), 1890–1920.

[20] M. Gubinelli, *Rough solutions for the periodic Korteweg–de Vries equation*, Commun. Pure Appl. Anal. **11** (2012), 709–733.

[21] Z. Guo and T. Oh, *Non-existence of solutions for the periodic cubic nonlinear Schrödinger equation below* L^2, Int. Math. Res. Not. IMRN **2018** (2018), 1656–1729.

[22] Z. Guo, S. Kwon amd T. Oh, *Poincaré–Dulac normal form reduction for unconditional well-posedness of the periodic cubic NLS*, Comm. Math. Phys. **322** (2013), 19–48.

[23] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, The Clarendon Press, Oxford University Press, New York, 1979.

[24] S. Herr and V. Sohinger, *Unconditional uniqueness results for the nonlinear Schrödinger equation*, Commun. Contemp. Math. **21** (2019), Article no. 1850058.

[25] Y. Katznelson, *An Introduction to Harmonic Analysis*, Cambridge University Press, Cambridge, 2004.

[26] C. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001), 617–633.

[27] R. Killip, M. Vişan and X. Zhang, *Low regularity conservation laws for integrable PDE*, Geom. Funct. Anal. **28** (2018), 1062–1090.

[28] N. Kishimoto, *A remark on norm inflation for nonlinear Schrödinger equations*, Commun. Pure Appl. Anal. **18** (2019), 1375–1402.

[29] S. Kwon, T. Oh and H. Yoon, *Normal form approach to unconditional well-posedness of nonlinear dispersive PDEs on the real line*, Ann. Fac. Sci. Toulouse Math. (6) **29** (2020), 649–720.

[30] S. Kwon and T. Oh, *On unconditional well-posedness of modified KdV*, Int. Math. Res. Not. IMRN **2012** (2012), 3509–3534.

[31] T. Oh, *A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces*, Funkcial. Ekvac. **60** (2017), 259–277.

[32] T. Oh, P. Sosoe and N. Tzvetkov, *An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation*, J. Éc. Polytech. Math. **5** (2018), 793–841.

[33] T. Oh and C. Sulem, *On the one-dimensional cubic nonlinear Schrödinger equation below L2*, Kyoto J. Math. **52** (2012), 99–115.

[34] T. Oh and Y. Wang, *On the ill-posedness of the cubic nonlinear Schrödinger equation on the circle*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **64** (2018), 53–84.

[35] T. Oh and Y. Wang, *Global well-posedness of the periodic cubic fourth order NLS in negative Sobolev spaces*, Forum Math. Sigma **6** (2018), e5, 80 pp.

[36] T. Oh and Y. Wang, *Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces*, J. Differential Equations **269** (2020), 612–640.

[37] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation. Self-Focusing and Wave Collapse*, Springer, New York, 1999.

[38] V.E. Zakharov and A.B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media,* Soviet Physics JETP **34** (1972), no. 1, 62–69.; translated from Ž. Èksper. Teoret. Fiz. **61** (1971), no. 1, 118–134.

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