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2-MINIMAL SUBGROUPS OF ORTHOGONAL GROUPS

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In memory of Jan Saxl

Abstract. For a finite group G, a subgroup P of G is 2-minimal if $B < P$, where $B = N_G(S)$ for some Sylow 2-subgroup S of G, and B is contained in a unique maximal subgroup of P. For fields of odd characteristic, this paper contains a detailed and explicit description of all the 2-minimal subgroups of the finite general orthogonal groups, and certain of their subgroups.

1. Introduction

This is one of a series of papers whose aim is to classify the 2-minimal subgroups of the finite non-abelian simple groups and certain of their automorphism groups. The motivation for such a programme comes from a number of directions. The two most prominent being that 2-subgroups of non-abelian simple groups often play an important role and the other being that 2-minimal subgroups are generalizations, and counterparts, of the minimal parabolic subgroups in groups of Lie type defined over fields of characteristic 2. This latter point is why we only consider orthogonal groups of odd characteristic as the structure of minimal parabolic subgroups is well understood. For the sporadic simple groups the 2-minimal subgroups were essentially analyzed by Ronan and Stroth [16]. The alternating and symmetric groups were covered in [8] by Lempken, Parker and Rowley, see also Magaard [10]. More recently the case of the projective special linear and projective special unitary groups were dealt with in [13] by Parker and Rowley. The 2-minimal subgroups of the projective symplectic groups are presented in [14]. In this paper we square up to the orthogonal groups.

To complete the study of the 2-minimal subgroups of the finite simple groups it remains to determine such subgroups in the exceptional groups. This is the subject of the paper in preparation [15].

Suppose G is a finite group. Let p be a prime, S a Sylow p-subgroup of G and $B = N_G(S)$. A subgroup P of G which properly contains B is called a *p-minimal subgroup* of G (with respect to B) if and only if

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B is contained in a unique maximal subgroup of P. Put

$$\mathcal{M}(G,B) = \{P \mid B < P \leqslant G \text{ and } P \text{ is } p\text{-minimal}\}.$$

It is the set $\mathcal{M}(G,B)$ that we shall study when $p = 2$ and G is a subgroup of the general orthogonal group $\mathrm{GO}^{\epsilon}_n(q)$ which contains $\Omega^{\epsilon}_n(q)$. We shall use $\mathrm{GO}^{\epsilon}_n(q)$, where either $\epsilon = +$ or $\epsilon = -$, to denote the n-dimensional general orthogonal group of ϵ-type over $\mathrm{GF}(q)$. The subgroup of $\mathrm{GO}^{\epsilon}_n(q)$ consisting of all elements of determinant 1 whose spinor norm is a square in $\mathrm{GF}(q)$ is denoted by $\Omega^{\epsilon}_n(q)$. Section 3 gives an expanded account of properties of these groups which are particularly relevant to this paper. We shall further assume that q is odd since, as noted earlier when q is even $\mathcal{M}(G,B)$ consists of precisely the minimal parabolic subgroups of G. A wider discussion on p-minimal subgroups is to be found in [13].

Just as for the other classical groups in [13, 14], low dimensional phenomena and various congruences of q leave their imprint on the general case. Before stating the classification of 2-minimal subgroups for the orthogonal groups, we briefly discuss the various sets of 2-minimal subgroups that we shall encounter. First we have the sets $\mathcal{O}_{1,7}(G,B)$ and $\mathcal{O}_{3,5}(G,B)$, described in Definition 6.3, where the subscripts indicate congruence conditions on q – so $\mathcal{O}_{1,7}(G,B)$ means we have $q \equiv 1,7 \pmod 8$ while for $\mathcal{O}_{3,5}(G,B)$ we have $q \equiv 3,5 \pmod 8$. These particular sets of 2-minimal subgroups emerge in Lemmas 6.1 and 6.2 with $\mathcal{O}_{1,7}(G,B)$ arising because certain orthogonal groups are themselves 2-minimal while the groups in $\mathcal{O}_{3,5}(G,B)$ have structure which is of monomial type. In Definition 5.6 we find the set $\mathcal{D}_2(G,B)$ where $\mathcal{D}$ stands for dihedral. Here $n = 2m+2$ (see Lemma 3.8 for the relevant subcase) and these 2-minimal subgroups are the legacy of 2-dimensional orthogonal groups (recall that $\mathrm{GO}^{\epsilon}_2(q) \cong \mathrm{Dih}(2(q-\epsilon))$, the dihedral group of order $2(q-\epsilon)$). Definition 6.6 introduces the sets $\mathcal{G}^+_4(G,B)$ and $\mathcal{G}_3(G,B)$. The former set contains the 2-minimal subgroups which are the spawn of $\mathrm{GO}^+_4(q)$, while the latter is the spawn of $\mathrm{GO}^{\epsilon}_3(q)$, but only when $q = 5^a$ with a odd. We also note that the parity of n also comes into play for $\mathcal{O}_{1,7}(G,B)$, $\mathcal{O}_{3,5}(G,B)$ and $\mathcal{G}_3(G,B)$. The menagerie of 2-minimal subgroups for the orthogonal groups is completed by $\mathcal{N}(G,B)$ given in Definition 5.5. The 2-minimal subgroups in $\mathcal{N}(G,B)$ are generic and are the 2-minimal subgroups of a certain so-called θ-subgroup of the orthogonal group. These subgroups are introduced immediately after Lemma 3.8. So here we see gathered toral, linker and fuser 2-minimal subgroups - this coven of 2-minimal subgroups will be described in Section 2. Most of the definitions related to the subgroups of the orthogonal groups can be found in Section 3.

In particular, here we mention that the notation ${}^{\pm}\mathrm{GO}_n^{\pm}(q)$ is explained just before Lemma 3.1 and the subgroups S_1, S_0 and S_{-1} of S are introduced immediately before Lemma 3.9.

We may now state our main theorem which can be paraphrased as saying that the 2-minimal subgroups of the orthogonal groups are in the families of subgroups discussed above whenever $n \geqslant 10$.

Theorem A. *Suppose that $n \geqslant 5$ and $G = \mathrm{GO}_n^{\epsilon}(q)$ where $q = p^a$ is odd. Let $S \in \mathrm{Syl}_2(G)$ and $B = N_G(S)$. Assume that H is a subgroup of G which contains $\Omega_n^{\epsilon}(q)$. If $P \in \mathcal{M}(H, N_H(S \cap H))$, then there exists*

$$\widehat{P} \in \mathcal{O}_{1,7}(G,B) \cup \mathcal{O}_{3,5}(G,B) \cup \mathcal{N}(G,B) \cup \mathcal{D}_2(G,B) \cup \mathcal{G}_4^+(G,B) \cup \mathcal{G}_3(G,B)$$

such that $P = \widehat{P} \cap H$ or $n \in \{7,8,9\}$, $q \equiv 3,5 \pmod 8$ and either

(i) *$H = \Omega_9^{\epsilon}(q)$ or ${}^{\epsilon}\mathrm{GO}_9^{\epsilon}(q) \cong 2 \times \Omega_9^{\epsilon}(q)$ and $\langle P, B\rangle = \mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(4) \wr 2 \times S_{-1} \in \mathcal{G}_4^+(G,B)$ and*

$$\mathcal{M}(\langle P,B\rangle \cap H, N_H(S \cap H)) = \{P, P^x\}$$

where $x \in B$ is a reflection of type $-\epsilon$.

(ii) *$n = 8$, P acts irreducibly on V and either*

  (a) *$H = \Omega_8^+(q)$ and $\langle P, B\rangle = \mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(4) \wr 2 \in \mathcal{G}_4^+(G,B)$ and*

$$\mathcal{M}(\langle P,B\rangle \cap H, N_H(S \cap H)) = \{P, P^x\}$$

  where $x \in B$ is a reflection.

  (b) *$H = {}^{\tau}\mathrm{GO}_8^+(q)$, $\tau = \pm$ and $\langle P, B\rangle = \mathrm{GO}_1^{\tau}(q) \wr \mathrm{Sym}(4) \wr 2 \in \mathcal{G}_4^+(G,B)$ and*

$$\mathcal{M}(\langle P,B\rangle \cap H, N_H(S \cap H)) = \{P, P^x\}$$

  where $x \in B$ is a reflection of type $-\tau$.

(iii) *$H \cong \Omega_7^{\epsilon}(q)$ or ${}^{\epsilon}\mathrm{GO}_7^{\epsilon}(q) \cong 2 \times \Omega_7^{\epsilon}(q)$, $\langle P, B\rangle = \langle \widehat{P}_1, \widehat{P}_2\rangle$ where $\widehat{P}_1 = \mathrm{GO}_1^{\theta^3\epsilon}(q) \wr \mathrm{Sym}(4) \times S_0 \times S_{-1} \in \mathcal{G}_4^+(G,B)$ and $\widehat{P}_2 = S_1 \times \mathrm{GO}_1^{\theta^3\epsilon}(q) \wr \mathrm{Sym}(3) \in \mathcal{O}_{3,5}(G,B)$ with $\theta = \pm$ and $q \equiv \theta \pmod 4$. Furthermore,*

$$\mathcal{M}(\langle P,B\rangle \cap H, N_H(S \cap H)) = \{\widehat{P}_1 \cap H, \widehat{P}_2 \cap H, P, P^x\}$$

where x is a reflection in B of type $-\epsilon$.

We emphasise that in all cases the precise structure of P in Theorem A is known, and moreover explicit matrices can be written down to describe these groups. The use of equal signs in our results is intended to highlight this point via the explicit decomposition of the group action on the standard orthogonal space. For $n \leqslant 4$, a detailed account of all the 2-minimal subgroups is provided in Section 4 and, in fact, for $G = \mathrm{GO}_4^-(q)$, the statement in Theorem A is still valid. As is usual, small cases tend not to follow the crowd and the results in Section 4 are too diverse to summarize here. Aside from the obvious corollary that holds when $n > 9$, we point out a further case which is of an especially pleasing form.

Corollary 1.1. *Suppose that* $n \geqslant 6$, $n = 2k$ *and* $q^k \equiv \epsilon \pmod 4$. *Then*

$$\mathcal{M}(G,B) = \mathcal{N}(G,B) \cup \mathcal{G}_4^+(G,B).$$

We now describe the contents of this paper. Section 2 begins by introducing various pieces of notation, some of this being needed for the description of Sylow 2-subgroups of symmetric groups. Since symmetric groups permeate many of our arguments and the structure of the 2-minimal subgroups of the orthogonal groups, in Theorem 2.2 we state the classification of their 2-minimal subgroups. Continuing in this vein we lay bare, in Theorem 2.4, the 2-minimal subgroups of groups of the form $D \wr \mathrm{Sym}(n/2)$ where D is a dihedral group. We note that such wreath products correspond to the natural action of the symmetric group $\mathrm{Sym}(n/2)$ on $n/2$ points. More specifically, when we construct wreath products $X \wr Y$ with Y an explicit subgroup of $\mathrm{Sym}(k)$, then we use this k-point action in the definition of the wreath product. Subgroups of the orthogonal groups of this shape lead to the 2-minimal subgroups in the set $\mathcal{N}(G,B)$. After looking at abelian subgroups of certain 2-groups in Lemma 2.5, we itemize results which hold for any p-minimal subgroup (p any prime). These results all play an important role in the inductive arguments used to prove Theorems A. As preparation for analysing the orthogonal groups, Section 3 sets up appropriate notation and defines certain subgroups which will lead us to the 2-minimal subgroups. Particularly important results here are Lemmas 3.8, 3.11 and 3.12. We remark that Lemma 3.8 is responsible for the overall direction of the proof of Theorem A.

Section 4 is devoted to finding the 2-minimal subgroups in the orthogonal groups of dimensions 3, 4 and 5 – the main conclusions being given in Theorems 4.1, 4.2, Corollary 4.3, Lemmas 4.5, 4.7 and 4.8. Then Section 5 analyses the 2-minimal subgroups of θ-subgroups, namely those isomorphic to $D \wr \mathrm{Sym}(n/2)$ where D is a dihedral group.

Definitions 5.5 and 5.6 describe the resulting 2-minimal subgroups. In Section 6 the bulk of the 2-minimal subgroups are catalogued in Definitions 6.3 and 6.6. Finally, in Section 7, the proof of Theorem A is given where we see Lemma 3.8 acting in concert with Proposition 3.14. This proposition distills the results of Kantor [6], Liebeck and Saxl [9] and Maslova [11] to describe the maximal subgroups of H of odd index where $\Omega_n^\epsilon(q) \leqslant H \leqslant \mathrm{GO}_n^\epsilon(q)$. These subgroups, of course, will host the proper 2-minimal subgroups. Here we encounter the orthogonal embeddings of the Coxeter groups of type E_7 and E_8 (the groups H_3, H_4 and F_4 having made guest appearances in Section 4). These embeddings, interestingly, do not yield any exotic 2-minimal subgroups – their 2-minimal subgroups being subsumed in those arising from smaller dimensions (see Lemma 4.9). We remark that some authors write $\mathrm{W}(\mathrm{E}_7)$, for example, to emphasise the fact that these group are also Weyl groups. Our last section, by way of illustrating Theorem A, displays in detail the 2-minimal subgroups for $\mathrm{GO}_{10}^-(q)$, $\mathrm{GO}_{10}^+(q)$ and $\Omega_7^+(5)$.

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2. Preliminary Results

We begin with some notation. For ℓ a positive integer we use ℓ_2 to denote the largest power of 2 which divides ℓ, $\ell_{2'} = \ell/\ell_2$ and $\Pi(\ell)$ is defined to be the set of all odd prime powers, excluding 1, which divide ℓ. We meet ℓ_2 mostly in the proof of Lemma 3.8 and the description of certain subfield subgroups in, for example, Theorem 4.1. Importantly, $\Pi(\ell)$ appears in the definition of toral 2-minimal subgroups.

There will be frequent brushes with symmetric groups, often as quotients of various subgroups of the general orthogonal groups. Letting $X = \mathrm{Sym}(\Omega) = \mathrm{Sym}(n)$ where $\Omega = \{1, \ldots, n\}$, we describe the two types of 2-minimal subgroups of X*–linkers* and *fusers*. First, let T be a Sylow 2-subgroup of X, and

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r} \text{ where } n_1 > n_2 > \cdots > n_r \geqslant 0$$

be the 2-adic decomposition of n. Set $I = \{1, \ldots, r\}$. Then T has r orbits $\Omega_1, \ldots, \Omega_r$ on Ω with $|\Omega_i| = 2^{n_i}$, $i \in I$. We may choose notation so as $\Omega_1 = \{1, \ldots, 2^{n_1}\}$, $\sigma_1 = 1$ and $\Omega_i = \{\sigma_i, \ldots, \sigma_i + 2^{n_i} - 1\}$ where, for $i \geqslant 2$, $\sigma_i = 1 + \sum_{j=1}^{i-1} 2^{n_j}$ is the minimal integer in Ω_i. Then

$$T = T_{n_1} \times T_{n_2} \times \cdots \times T_{n_r}$$

where $T_{n_i} \in \mathrm{Syl}_2(\mathrm{Sym}(\Omega_i))$, $i \in I$. Moreover, T_{n_i} is the iterated wreath product of n_i copies of T_1 the cyclic group of order 2 and $T = N_X(T)$

– see [4, Satz 15.3, p. 378] for further details. We additionally denote the alternating group on Ω by $\mathrm{Alt}(\Omega) = \mathrm{Alt}(n)$.

We next introduce two collections of block systems of Ω. Let $i \in I$. For $j \in \{1, \dots, n_i - 1\}$, the first collection is $\Sigma_{n_i;j}$ which consists of all T-invariant block systems of Ω_i of sets of size 2^k where $k \in \{0, \dots, n_i\}\backslash\{j\}$. For the second we choose $i, j \in I$, with $i < j$ (so $n_j < n_i$) and set $\Lambda_{n_i+n_j} = \Omega_i \cup \Omega_j$. Let Γ_i be the collection of all T-invariant block systems on Ω_i and Γ_j the collection of all T-invariant block systems on Ω_j. Then we define $\Sigma_{n_i+n_j}$ to be the collection of T-invariant systems of subsets of $\Lambda_{n_i+n_j}$ which are the union of one block system from Γ_i and one from Γ_j with the proviso that the blocks of the two chosen block systems have equal numbers of elements.

Definition 2.1. *For $i \in I$ and $j \in \{1, \dots, n_i - 1\}$,*

$$X(n_i; j) = \mathrm{Stab}_{\mathrm{Sym}(\Omega_i)}(\Sigma_{n_i;j}) \times (\prod_{\ell \in I\backslash\{i\}} T_{n_\ell}),$$

and for $i, j \in I$ with $i < j$,

$$X(n_i + n_j) = \mathrm{Stab}_{\mathrm{Sym}(\Lambda_{n_i+n_j})}(\Sigma_{n_i+n_j}) \times (\prod_{k \in I\backslash\{i,j\}} T_{n_k}).$$

The subgroups $X(n_i, j)$ and $X(n_i + n_j)$ are, respectively, the *linker* and *fuser* 2-minimal subgroups of X, and [8, Theorems 1.1 and 1.2] establishes the following result.

Theorem 2.2. *Suppose that $X = \mathrm{Sym}(\Omega)$, $Y = \mathrm{Alt}(\Omega)$ and $T \in \mathrm{Syl}_2(X)$. Then*

$$\mathcal{M}(X,T) = \{X(n_i; j), X(n_k+n_\ell) \mid i, k, \ell \in I, k < \ell \text{ and } j \in \{1, \dots n_i-1\}\}$$

and, if $|\Omega| > 9$, then $\mathcal{M}(Y, T \cap Y) = \{P \cap Y \mid P \in \mathcal{M}(X,T)\}$.

As we will discover, many 2-minimal subgroups of orthogonal groups reside in subgroups isomorphic to $N = D \wr \mathrm{Sym}(n/2)$ where D is a dihedral group. As $O_2(N)$ is contained in all the 2-minimal subgroups of N, it suffices to consider the case when D has order twice an odd number. The 2-minimal subgroups of such groups have been described in [14] and we give a summary of the salient points of this description. To begin with we look at $E \wr \mathrm{Sym}(n) = E \wr X$, where E is cyclic of odd order and E_0 is the base group. Then E_0 is isomorphic to a direct product of n copies of E and so we write

$$E_0 = \langle e_1, \dots, e_n \rangle$$

with X acting on the generators of E_0 naturally by permuting the subscripts. Take $s \in \Pi(|E|)$ a prime, let s^b be the largest power of s in

$\Pi(|E|)$ and put $\bar{s} = |E|/s^b$. Then, for $\alpha \in \Omega$ and $s^c \in \Pi(|E|)$, set

$$u_\alpha = e_\alpha^{\bar{s}}, \text{ and } w_\alpha = u_\alpha^{s^{b-c}}$$

and note that $\langle u_\alpha \mid \alpha \in \Omega\rangle \in \mathrm{Syl}_s(E_0)$ and w_α has order s^c.

For $j \in I$, $s^c \in \Pi(|E|)$, following [14, Defintion 2.7] we define

$$U(n_j; s^c; n_j) = \langle (w_{\sigma_j} - w_{\sigma_j+1})^t \mid t \in T_{n_j}\rangle = \langle (w_{\sigma_j} - w_{\sigma_j+1})^t \mid t \in T\rangle.$$

Taking $D = \mathrm{Dih}(2|E|)$ and assuming that n is even, the group $D \wr \mathrm{Sym}(n/2)$ can be identified with an odd index subgroup of $E \wr (2 \wr \mathrm{Sym}(n/2)) \leqslant E \wr \mathrm{Sym}(n)$ (see [14, Lemma 2.6]). Thus the 2-minimal subgroups of $D \wr \mathrm{Sym}(n/2)$ are among the 2-minimal subgroups of $E \wr \mathrm{Sym}(n)$ and we select these in the next definition.

Definition 2.3.

$$\begin{aligned} \mathcal{F}(L,T) &= \{X(n_i + n_j) \mid i, j \in I, i < j\}; \\ \mathcal{L}(L,T) &= \{X(n_i; j) \mid i \in I, j \in \{2, \ldots, n_i - 1\}\} \textit{ and} \\ \mathcal{T}(L,T) &= \{U(n_i; s^c; n_i)T \mid i \in I, \textit{ and } s^c \in \Pi(|E|)\}. \end{aligned}$$

The notation $U(n_i; s^c; n_i)$ is consistent with that in [14] where it is used in a slightly more general setting. Notice that

$$U(n_i; s^c; n_i)T = \mathrm{Dih}(2s^c) \wr T_{n_i-1} \times \prod_{j \in I\setminus\{i\}} T_{n_j}.$$

We can now state [14, Theorem 2.8].

Theorem 2.4. *Suppose that D is a dihedral group of order $2e$ with e odd. Let $L = D \wr \mathrm{Sym}(n/2)$, $T \in \mathrm{Syl}_2(L)$. Then $T = N_L(T)$ and*

$$\mathcal{M}(L,T) = \mathcal{F}(L,T) \cup \mathcal{L}(L,T) \cup \mathcal{T}(L,T).$$

Our preoccupation with dihedral groups continues in the next lemma.

Lemma 2.5. *Suppose that D is a dihedral group of order 2^{j+1} at least 8 and R_k is a Sylow 2-subgroup of $\mathrm{Sym}(k)$. Let $H = D \wr R_k$. If C is an abelian subgroup of H of order at least 2^{jk}, then C is contained in the base group of H, $|C| = 2^{jk}$ and is a maximal order abelian subgroup of H.*

Proof. We argue by induction on k. If $k = 1$, there is not a lot to do. So suppose that $k > 1$ and let C be an abelian subgroup of H of order 2^{jk}. Let $\mathrm{Sym}(k) = \mathrm{Sym}(\Omega)$ and let F be the base group of H. Supposing $C \not\leqslant F$, we seek a contradiction. Let $c \in C\setminus F$ and let $\ell = |\mathrm{Fix}_\Omega(\pi)|$ where π is the permutation of Ω induced by c. Then, as $C \leqslant C_H(c)$

preserves the fixed points of c on Ω, there are Sylow 2-subgroups R_ℓ and $R_{k-\ell}$ such that

$$c \in C \leqslant W = D \wr (R_{k-\ell} \times R_\ell) \cong (D \wr R_{k-\ell}) \times (D \wr R_\ell).$$

Suppose $\ell \neq 0$. Then, since $C \leqslant W$ and by induction every abelian subgroup of order $2^{j(k-\ell)}$ in $D\wr R_{k-\ell}$ and of order $2^{j\ell}$ in $D\wr R_\ell$ is contained in the base group of the respective wreath products, we obtain $C \leqslant F$ and $|C| = 2^{jk}$ in this case. Thus $\ell = 0$ and so π is fixed-point-free with at most $k/2$ orbits on Ω. Therefore $C_F(c)$ is contained in a direct product of at most $k/2$ dihedral groups and so the largest abelian subgroup of $C_F(c)$ has order at most $2^{jk/2}$. Since the largest abelian subgroup of R_k has order $2^{k/2}$, $|CF/F| \leqslant 2^{k/2}$. Hence

$$|C| \leqslant 2^{k/2}|C \cap F| = 2^{k/2}|C_F(c) \cap C| \leqslant 2^{jk/2+k/2} < |C|,$$

as $j > 1$. With this contradiction the lemma is proved. □

For the remainder of this section G is a finite group, p is a prime, $S \in \mathrm{Syl}_p(G)$ and $B = N_G(S)$.

Lemma 2.6. *Suppose that K is a normal subgroup of G and $P \in \mathcal{M}(G, B)$. Then either*

(i) *$P \in \mathcal{M}(BK, B)$; or*
(ii) *$PK/K \in \mathcal{M}(G/K, BK/K)$ and $P \in \mathcal{M}(N_G(S \cap K), B)$.*

Proof. See Lemma 3.8 of [13].

□

Lemma 2.7. *Suppose that K is a normal subgroup of G and $G = BKC_G(K)$. Assume that $N_K(S \cap K) = B \cap K$ and $P \in \mathcal{M}(G, B)$. Then $P \in \mathcal{M}(BK, B) \cup \mathcal{M}(BC_G(K), B)$.*

Proof. See Lemma 3.9 of [13]. □

Our next lemma is a variation upon Lemma 2.7.

Lemma 2.8. *Suppose that $G = KLB$ with K and L normal subgroups of G, and let $P \in \mathcal{M}(G, B)$. Then*

$$P \in \mathcal{M}(BK, B) \cup \mathcal{M}(BL, B) \cup \mathcal{M}(N_G(S \cap KL), B).$$

Proof. Since $(P \cap K)(P \cap L)$ is normalized by P,

$$P = N_P(S \cap KL)(P \cap K)(P \cap L)$$

by the Frattini Argument. Since $B \leqslant N_P(S \cap KL)$ and $B(P \cap K)$ and $B(P \cap L)$ are groups, the p-minimality of P implies that $P \in \mathcal{M}(BK, B) \cup \mathcal{M}(BL, B) \cup \mathcal{M}(N_G(S \cap KL), B)$. □

Lemma 2.9. *Suppose that K is a normal subgroup of G and $R = S \cap K$. Assume that $P \in \mathcal{M}(K, N_K(R))$ and PB is a group. If $B \cap K = N_K(R)$, then $PB \in \mathcal{M}(G, B)$.*

Proof. This is Lemma 3.7 in [13]. □

Lemma 2.10. *Suppose that K is a normal subgroup of G, $B \cap K = N_K(S \cap K)$ and $P \in \mathcal{M}(K, B \cap K)$. If $Q \leqslant P$ is such that $P = (B \cap K)Q$ and $B = (B \cap K)C_B(Q)$, then $PB \in \mathcal{M}(G, B)$ and $P = PB \cap K$.*

Proof. Observe that $C_B(Q)$ normalizes $B \cap K$ and hence normalizes $(B \cap K)Q = P$. Thus, as $B = (B \cap K)C_B(Q)$, B normalizes P and so PB is a group. The result now follows from Lemma 2.9. □

The final result in this section is used in the closing stages of the proof of Theorem A where the orthogonal groups have dimension at least 6 and the 2-minimal subgroup acts irreducibly but not primitively on the natural module.

Lemma 2.11. *Suppose that $B = S \in \mathrm{Syl}_2(G)$, $G = KLB$ where KL is normal in G and $N_G(K) = N_G(L)$ has index 2 in G. Set $B_0 = N_B(K)$, and further assume that K and L are G-conjugate with $K \cap L \leqslant B$. If $P \in \mathcal{M}(G, B)$, then either $P \leqslant N_G(B \cap KL)$ or B_0 is contained in a unique maximal subgroup of $(P \cap K)B_0$ and of $(P \cap L)B_0$.*

Proof. Let $P_0 = P \cap N_G(K)$ and $b \in B \backslash B_0$. Observe that $B \nleqslant N_G(K)$ and, as $|G : N_G(K)| = 2$, $P = BP_0 = \langle b \rangle P_0$. Since $N_G(K) = N_G(L)$, P_0 normalizes both $P_0 \cap K$ and $P_0 \cap L$. Because $(P_0 \cap K)^b = P_0 \cap L$ and $b^2 \in B_0 \leqslant P_0$, $(P_0 \cap K)(P_0 \cap L)$ is normalized by $P_0\langle b \rangle = P$. Also, as KL is normal in G and K and L are normal in KL,

$$B \cap KL = (B \cap K)(B \cap L) \in \mathrm{Syl}_2(KL)$$

and $B \cap KL$ is normalized by B. Furthermore, $B \cap KL \in \mathrm{Syl}_2((P_0 \cap K)(P_0 \cap L))$. Therefore, the Frattini Argument yields

$$P = (P_0 \cap K)(P_0 \cap L)N_P(B \cap KL).$$

Since $(P_0 \cap K)(P_0 \cap L)B \leqslant P$ and $B \leqslant N_P(B \cap KL)$, the 2-minimality of P forces either $P = (P_0 \cap K)(P_0 \cap L)B$ or $P = N_P(B \cap KL)$. The second possibility is one of our conclusions so we may assume that

$$P = (P_0 \cap K)(P_0 \cap L)B = (P \cap K)(P \cap L)B.$$

Put $W = L \cap KB_0$. We claim that $W \leqslant B_0$. Since B_0 is a 2-group and $B_0K/K \geqslant WK/K \cong W/(W \cap K)$, $W/(W \cap K)$ is a 2-group. Hence, as $W \cap K \leqslant L \cap K \leqslant B$ by hypothesis, we have that W is a 2-group. Because W is normalized by B_0 and $B_0 \in \mathrm{Syl}_2(KB_0)$, we then deduce that $W \leqslant B_0$, as claimed. Now assume B_0 is not contained in a unique maximal subgroup of $(P \cap K)B_0$ and argue for a contradiction. In this case there exist maximal subgroups U_1 and U_2 of $(P \cap K)B_0$ both containing B_0 with $U_1 \neq U_2$. Since $U_i \leqslant P$, the Dedekind Law gives

$$U_i = U_i \cap B_0(P \cap K) = B_0(U_i \cap K)$$

for $i = 1, 2$. Assume that $P_0 = (U_1 \cap K)(U_1 \cap K)^b B_0$. Again by the Dedekind Law

$$\begin{aligned} P_0 \cap KB_0 &= (U_1 \cap K)(U_1 \cap K)^b B_0 \cap KB_0 \\ &= (U_1 \cap K)B_0((U_1 \cap K)^b \cap KB_0) \\ &= U_1((U_1 \cap K)^b \cap KB_0). \end{aligned}$$

Now

$$(U_1 \cap K)^b \cap KB_0 \leqslant K^b \cap KB_0 = L \cap KB_0 = W \leqslant B_0$$

and therefore

$$P_0 \cap KB_0 = U_1 B_0 = U_1.$$

Since $(P \cap K)B_0 = (P_0 \cap K)B_0 = P_0 \cap KB_0$ and $(P \cap K)B_0 > U_1$, we have a contradiction.

Therefore $(U_1 \cap K)(U_1 \cap K)^b B_0 = (U_1 \cap K)^b U_1$ and similarly $(U_2 \cap K)(U_2 \cap K)^b B_0 = (U_2 \cap K)^b U_2$ are proper subgroups of P_0 which are normalized by B. But then the 2-minimality of P implies $\langle U_1, U_2 \rangle \neq (P \cap K)B_0$, the desired contradiction. Thus we conclude that B_0 is contained in a unique maximal subgroup of $(P \cap K)B_0$. Conjugating by b yields B_0 is contained is a unique maximal subgroup of $(P \cap L)B_0$. $\square$

3. On certain subgroups of Orthogonal Groups

In this section we describe and develop notation for various subgroups of the orthogonal groups which will be of service in the next sections. We begin by quickly establishing various pieces of notation for the orthogonal groups. The reader is referred to [1] and [17] for the standard background material and definitions.

Let (V, Q) be a non-degenerate orthogonal space of dimension n which is defined over $\mathrm{GF}(q)$ where q is a power of an odd prime. A vector $v \in V$ is *singular* if and only if $Q(v) = 0$ and a subspace $W \leqslant V$ is *totally singular* if and only if every vector of W is singular. The maximum dimension of a totally singular subspace of V is either w or $w-1$ if V has even dimension $n = 2w$ and of dimension w if V has odd dimension $n = 2w + 1$ (see [1, 21.2]). The dimension of such a space is called the *Witt index* of (V, Q). In the case when $n = 2w$ is even, we say that (V, Q) is of $+$*-type* if the Witt index is w and otherwise it is of $-$*-type*. Suppose that n is odd and let W be a non-degenerate hyperplane of $+$-type in V. Then the type of V is determined by the values of Q on the non-zero vectors in $W^\perp$. Specifically V is of *$+$-type*, respectively, *$-$-type* precisely when $Q(u)$ is a square, respectively, a non-square for all non-zero $u \in W^\perp$.

From here on we suppress explicit mention of the form Q and so the isometry group of (V, Q) is simply denoted

$$\mathrm{GO}(V) = \{g \in \mathrm{GL}(V) \mid Q(ug) = Q(u) \text{ for all } u \in V\}.$$

Since the forms are uniquely determined by their type and dimension we also use the notation $\mathrm{GO}_n^\epsilon(q)$ to denote these isometry groups where here ϵ is either $+$ or $-$ and represents the type of the form. Notice when n is odd, as the corresponding forms differ only by conjugation and scalar multiplication, we have $\mathrm{GO}_n^+(q) \cong \mathrm{GO}_n^-(q)$ whereas when n is even these groups are not isomorphic. However, for our investigations it is important that we distinguish between $\mathrm{GO}_n^+(q)$ and $\mathrm{GO}_n^-(q)$ even in the case n is odd especially when these groups are embedded in larger orthogonal groups.

Recall that a *reflection* of V preserves the form and has one eigenvalue -1 and the remaining eigenvalues are equal to 1. There are two conjugacy classes of reflections in $\mathrm{GO}_n^\epsilon(q)$ and if x is a reflection the conjugacy class to which x belongs is determined by the type of the -1 eigenspace $[V, x]$. For such a reflection x, we have $V = [V, x] \perp C_V(x)$ and so x is totally determine by indicating a non-zero vector which is negated by x. We mostly stick to the convention that reflections of $+$-type are denoted by x and reflections of $-$-type are denoted by y. When the type of a reflection is arbitrary, we use x.

The group $\mathrm{GO}_n^\epsilon(q)$ is generated by its reflections [1, (22.7)]. The subgroup $\mathrm{SO}(V) = \mathrm{SO}_n^\epsilon(q)$ consists of the elements of $\mathrm{GO}_n^\epsilon(q)$ which have determinant 1 and this subgroup has index 2 in $\mathrm{GO}_n^\epsilon(q)$. The *spinor norm* is a homomorphism $\mathcal{S}$ from $\mathrm{GO}_n^\epsilon(q)$ to the multiplicative group of $\mathrm{GF}(q)$ mod its squares $\mathrm{GF}(q)^2$ which is defined as follows. Let $g \in \mathrm{GO}_n^\epsilon(q)$ and write $g = x_1 \dots x_k$ as a product of reflections with $[V, x_i] = \langle v_i \rangle$. Then $\mathcal{S}(g) = Q(v_1) \dots Q(v_k)\mathrm{GF}(q)^2$. For $g \in \mathrm{GO}_n^\epsilon(q)$, we often write that $\mathcal{S}(g) = +$ if $\mathcal{S}(g) \in \mathrm{GF}(q^2)$ and $\mathcal{S}(g) = -$ is $\mathcal{S}(g)$ is not a square. Restricting $\mathcal{S}$ to $\mathrm{SO}(V) = \mathrm{SO}_n^\epsilon(q)$, by considering the product of two non-conjugate reflections, we see that $\mathrm{SO}_n^\epsilon(q)$ itself has a subgroup of index 2. We denote this group by $\Omega(V) = \Omega_n^\epsilon(q)$. Except for very small fields this group is perfect and when $n \geqslant 5$ it is quasisimple. With the previously established notation, we may now define three subgroups of index 2 in $\mathrm{GO}_n^\epsilon(q)$ containing the subgroup $\Omega_n^\epsilon(q)$. If x is a reflection and $[V, x]$ has ν-type, then ${}^\nu\mathrm{GO}_n^\epsilon(q)$ is generated by x and $\Omega_n^\epsilon(q)$ and the third subgroup containing $\Omega_n^\epsilon(q)$ is just $\mathrm{SO}_n^\epsilon(q)$. Notice that if x is a reflection with $[V,x]$ of $+$-type, then $\mathcal{S}(x) = 1$ and so ${}^{+}\mathrm{GO}_n^{\epsilon}(q)$ is the kernel of the spinor norm.

We now suppose that $n \geqslant 2$ and let $G = \mathrm{GO}_n^{\epsilon}(q)$. Also throughout this section H is a subgroup of G containing $\Omega_n^{\epsilon}(q)$. So H is one of the five groups $\mathrm{GO}_n^{\epsilon}(q)$, ${}^{+}\mathrm{GO}_n^{\epsilon}(q)$, ${}^{-}\mathrm{GO}_n^{\epsilon}(q)$, $\mathrm{SO}_n^{\epsilon}(q)$ and $\Omega_n^{\epsilon}(q)$.

In the instances where W is a non-degenerate subspace of V, we have $V = W \perp W^{\perp}$ and we denote the subgroup of $\mathrm{GO}(V)$ which fixes every vector of $W^{\perp}$ (and, as a consequence, leaves W invariant) by $\mathrm{GO}(W)$.

Lemma 3.1. *Suppose that W is a non-degenerate subspace of V of dimension at least 2 and that, if the dimension of W is 2, assume that W has vectors of both $+$- and $-$-type. Then $G = \mathrm{GO}(W)H$.*

Proof. The hypothesis on W guarantees $\mathrm{GO}(W)$ contains representatives from both conjugacy classes of reflections. Hence the result follows from our earlier discussion about subgroups of index 2 in G. □

The Sylow 2-subgroups of $\mathrm{GO}_n^{\epsilon}(q)$ are described explicitly by Carter and Fong [2]. However this description does not exhibit the action on the quadratic space V which reveal the 2-minimal subgroups of $\mathrm{GO}_n^{\epsilon}(q)$ in a geometric way. Hence we start by giving an elementary description of a subgroup, more suited to our needs, which contains Sylow 2-subgroups of $\mathrm{GO}_n^{\epsilon}(q)$. These descriptions depend upon the congruence of q modulo 4. So, once and for all, we define $\theta = \pm 1$ so that

$$\theta \equiv q \pmod 4.$$

We will often use θ in our orthogonal group notation, and in these cases we write $\theta = \pm$ rather than $\theta = \pm 1$. By the same token, when considering $\mathrm{GO}_n^{\epsilon}(q)$, we may use $\epsilon = \pm 1$ whenever appropriate. We recall that $\mathrm{GO}_2^{\epsilon}(q) \cong \mathrm{Dih}(2(q-\epsilon))$ and that $\mathrm{SO}_2^{\epsilon}(q)$ is cyclic of order $q-\epsilon$. Therefore $\mathrm{GO}_2^{\theta}(q)$ has non-abelian Sylow 2-subgroups and they have larger order than the Sylow 2-subgroups of $\mathrm{GO}_2^{-\theta}(q)$ which are elementary abelian of order 4. Let V_{θ} be a 2-dimensional space of θ-type. We also recall that $\mathrm{GO}_1^{+}(q) \cong \mathrm{GO}_1^{-}(q)$ has order 2 and is generated by a single reflection.

Lemma 3.2. *Suppose that $G = \mathrm{GO}_2^{\epsilon}(q)$. If x is a reflection in G and $x^* \in x^G \backslash \{x\}$ with $[x, x^*] = 1$, then $\theta = \epsilon$ (so $q \equiv \epsilon \pmod 4$). In particular, V can be written as an orthogonal sum of two isometric spaces if and only if $\theta = \epsilon$ and, in this case, V can be written as a perpendicular sum of two $+$-spaces and of two $-$-spaces.*

Proof. Suppose that x is a reflection in G, $x^* \in x^G \backslash \{x\}$ and $[x, x^*] = 1$. Assume that $q \equiv -\epsilon \pmod 4$. Then $G \cong \mathrm{Dih}(2(q-\epsilon))$ which has three conjugacy classes of involutions and Sylow 2-subgroups of order

4. Hence x and x^* are not conjugate, a contradiction. Hence $q \equiv \epsilon \pmod 4$ which means that $\epsilon = \theta$.

Assume that $V = W_1 \perp W_2$ is a decomposition of V as an orthogonal sum of two non-degenerate spaces. Then there exist reflections $x, y \in G$ such that $W_1 = C_V(y) = [V, x]$ and $W_2 = C_V(x) = [V, y]$. Thus x and y commute. If W_1 is isometric to W_2, then x and y are conjugate which means that $q \equiv \epsilon \pmod 4$. Conversely, assuming that $q \equiv \epsilon \pmod 4$, then a Sylow 2-subgroup S of G contains two conjugacy classes of fours group with representatives F_+ and F_- where notation is chosen so that F_τ has two reflections of type τ. The reflections in F_τ provide a decomposition of V into an orthogonal sum of two one-spaces of type τ, as required. $\square$

Definition 3.3. *Let U be a subspace of V which is of maximal dimension subject to being a perpendicular sum of spaces isometric to V_θ. Writing $U = U_1 \perp \cdots \perp U_m$ where each U_i is isometric to V_θ, we call $U_1 \perp \cdots \perp U_m \perp U^\perp$ a* θ-decomposition of V. *We define $V_{-1} = U^\perp$.*

A θ-decomposition, uniquely specifies U in Definition 3.3, however a given subspace U as in the definition usually hosts a multitude of θ-decompositions. Notice that by the definition of U, $V = U \perp V_{-1}$ and $\dim V_{-1} \leqslant 2$.

Lemma 3.4. *The subspace U has an orthogonal basis consisting of $+$-type vectors and an orthogonal basis consisting of $-$-type vectors.*

Proof. From the definition of U, we have that U is a perpendicular sum of 2-spaces all of type θ. Since $q \equiv \theta \pmod 4$, Lemma 3.2 implies that the 2-spaces in the decomposition of U each have an orthogonal basis consisting of vectors of any fixed non-singular type. The result now follows. $\square$

By [1, 21.2], $V_\theta \perp V_\theta$ has $+$-type independently of the type of θ. Hence U has type $+$-type when m is even and, if m is odd, U has type θ. Notice that when $\dim V_{-1} = 2$, V_{-1} has type $-\theta$ and the type of V is $-\theta^{m+1}$. This multiplicative property of signs does not hold for 1-spaces. Indeed, V_θ is a perpendicular sum of two 1-dimensional $+$-spaces and two $-$-spaces independently of the type of V_θ.

We write U as a perpendicular sum of m copies of V_θ and select an ordered basis $\{e_i, e_{i+1}\}$ for the ith term of the sum where $Q(e_i) = Q(e_{i+1}) = 1$. This yields an orthogonal basis $e_1, \ldots, e_{2m}$ of U consisting of $+$-type vectors with $Q(e_1) = Q(e_2) = \cdots = Q(e_{2m}) = 1$. For $1 \leqslant i \leqslant 2m$, let x_i be the reflection which negates e_i and fixes e_j for $1 \leqslant j \leqslant 2m$ with $j \neq i$. Let $y_i \in \mathrm{GO}(\langle e_i, e_{i+1}\rangle)$ be a reflection of $-$-type which has product of order $(q-\theta)_2$ with x_i. Set $f_i = e_iy_i - e_i$ and $f_{i+1} \in \langle e_iy_i + e_i\rangle$ with $Q(f_{i+1}) = Q(f_i)$. Then f_i and f_{i+1} are $-$-type vectors. Let y_{i+1} be the reflection that negates f_{i+1}. We make these selections so that $Q(f_i)$ is independent of i. Notice that $\langle x_1, \dots, x_m\rangle$ and $\langle y_1, \dots, y_m\rangle$ are elementary abelian 2-groups of order 2^{2m}.

If $\dim V_{-1} = 1$, then we select d_{-1} to be a vector of type $\delta = \theta^m\epsilon$ so that $Q(d_{-1}) = Q(e_1)$ if $\delta = +$ and $Q(d_{-1}) = Q(f_1)$ if $\delta = -$. Similarly, using Lemma 3.2, when $\dim V_{-1} = 2$, we select e_{-1} and f_{-1} such that e_{-1} is perpendicular to f_{-1} and such that $Q(e_{-1}) = Q(e_1)$ and $Q(f_{-1}) = Q(f_1)$. We fix this notation for the remainder of the paper. We denote the reflection which negates d_{-1} by x_{-1} when $\theta^m\epsilon = +$ and otherwise we represent it by y_{-1}. In the event that $\dim V_{-1} = 2$, we let x_{-1} be the reflection which negate e_{-1} and y_{-1} be the reflection which negates f_{-1}.

We can use these bases to write down the centre of G.

Lemma 3.5. *The following hold.*

(i) $Z(G) = \langle -I_n\rangle$ *where* I_n *is the identity of* $G = \mathrm{GO}(V)$.
(ii) $Z(G) \leqslant \Omega(V)$ *if and only if* $V = U$ *(so* $n = 2m$ *and* $\epsilon = \theta^m$*).*
(iii) ${}^{\delta}\mathrm{GO}^{\epsilon}_{2m+1}(q) \cong 2 \times \Omega^{\epsilon}_{2m+1}(q)$ *where* $\delta = \theta^m\epsilon$.
(iv) $\mathrm{SO}^{\epsilon}_{2m+2}(q) \cong 2 \times \Omega^{\epsilon}_{2m+2}(q)$.

Proof. Obviously (i) holds. Let $z \in Z(G)^\#$. Then

$$z = \begin{cases} x_1 \dots x_{2m} & n = 2m \\ x_1 \dots x_{2m}x_{-1} & n = 2m+1 \text{ and } \theta^m\epsilon = + \\ x_1 \dots x_{2m}y_{-1} & n = 2m+1 \text{ and } \theta^m\epsilon = - \\ x_1 \dots x_{2m}x_{-1}y_{-1} & n = 2m+2. \end{cases}$$

In the first case $\mathcal{S}(z)$ is a square and $\det z = 1$ and so $Z(G) \leqslant \Omega(V)$. In the second an third possibilities for z, $\det z = -1$ and $\mathcal{S}(z) = \mathcal{S}(x_{-1})$ or $\mathcal{S}(y_{-1})$. This, respectively, gives $z \in {}^{+}\mathrm{GO}^{\epsilon}_{n}(q)$ and $z \in {}^{-}\mathrm{GO}^{\epsilon}_{n}(q)$. This gives (iii). Finally, if $z = x_1 \dots x_{2m}x_{-1}y_{-1}$, then $\mathcal{S}(z)$ is a non-square and z has determinant 1. This is (iv) and (ii) also now follows. □

Lemma 3.6. *Suppose that* $V = W_1 \perp W_2$, W_1 *has type* ϵ_1 *and* $\dim W_2 \geqslant 2$. *Define* $H_{12} = (\mathrm{GO}(W_1) \times \mathrm{GO}(W_2)) \cap \Omega(V)$.

(i) *If* $\dim W_1 = 2k+1$, *then* H_{12} *contains a subgroup isomorphic to* $\Omega(W_1) \times {}^{\theta^k\epsilon_1}\mathrm{GO}(W_2)$.
(ii) *If* $\dim W_1 = 2k$ *and* $\theta^k = -\epsilon_1$, *then* H_{12} *contains a subgroup isomorphic to* $\Omega(W_1) \times \mathrm{SO}(W_2)$.
(iii) *If* $\dim W_1 = 2k$ *and* $\theta^k = \epsilon_1$ *and* $\dim W_1 \geqslant 4$, *then* $C_{H_{12}}(\Omega(W_1)) \leqslant Z(\Omega(W_1)) \times \Omega(W_2)$.

(iv) *If* $\dim W_1 = 2$ *and* $\theta = \epsilon_1$*, then* $C_{H_{12}}(\Omega(W_1))$ *contains a subgroup isomorphic to* $\mathrm{SO}(W_2)$*.*

Proof. Suppose that $\dim W_1$ is odd. Then $d = \mathrm{diag}(-I_{2k+1}, I_{n-(2k+1)})$ has determinant -1 and $\mathcal{S}(d) = \theta^k\epsilon_1$. Thus letting d_2 be a reflection in $\mathrm{GO}(W_2)$ with spinor norm $\theta^k\epsilon_1$, we see that $\mathrm{diag}(-I_{2k+1}, d_2) \in \Omega(V)$ and

$$\Omega(W_2)\langle \mathrm{diag}(-I_{2k+1}, d_2)\rangle \cong {}^{\theta^k\epsilon_1}\mathrm{GO}(W_2).$$

Hence (i) holds.

Suppose that $\dim W_1 = 2k$ and $\epsilon_1 = -\theta^k$. Then by Lemma 3.5 (iii) the spinor norm of

$$\mathrm{diag}(-I_{2k}, I_{n-2k})$$

is not a square. Pick reflections d_1 and d_2 in $\mathrm{GO}(W_2)$ such that their product has spinor norm a non-square (this is possible as, by hypothesis, $\dim W_2 \geqslant 2$). Then $\mathrm{diag}(-I_{2k}, d_1d_2) \in \Omega(V)$ and this time we see that $\Omega(W_2)\langle \mathrm{diag}(-I_{2k}, d_1d_2)\rangle \cong \mathrm{SO}(W_2)$. This proves (ii).

Now suppose that $\dim W_1 = 2k \geqslant 4$ and $\epsilon_1 = \theta^k$. Then the spinor norm of $\mathrm{diag}(-I_{2k}, I_{n-2k})$ is a square. Suppose that $d \in H_{12}$ centralizes $\Omega(W_1)$ and that $d \notin Z(\Omega(W_1))\Omega(W_2)$. As $\dim W_1 \geqslant 4$, we have

$$C_{\mathrm{GO}(W_1)}(\Omega(W_1)) = \langle \mathrm{diag}(-I_{2k}, I_{n-2k})\rangle.$$

Hence multiplying by $\mathrm{diag}(-I_{2k}, I_{n-2k})$ if necessary we may assume that d is the block diagonal matrix $\mathrm{diag}(I_{2k}, d_2)$ with $d_2 \in \mathrm{GO}(W_2)$. Since $\mathrm{GO}(W_2) \cap \Omega(V) = \Omega(W_2)$, we have (iii).

Finally suppose that $\dim W_1 = 2$ and $\theta = \epsilon_1$. Assume that $d \in H_{12}$ centralizes $\Omega(W_1)$. Then, as $\Omega(W_1)$ is cyclic of order $(q-\theta)/2$ and $\Omega(W_1)$ is centralized by $\mathrm{SO}(W_1)$, we see that d is the block diagonal matrix $\mathrm{diag}(g, d_2)$ for some $g \in \mathrm{SO}(W_1)$ and $d_2 \in \mathrm{GO}(W_2)$. Since g has determinant 1, so does d_2. Thus selecting $g \in \mathrm{SO}(W_1)\backslash\Omega(W_1)$, we see that $\Omega(W_2)\langle \mathrm{diag}(g, d_2)\rangle \cong \mathrm{SO}(W_2)$ and centralizes $\Omega(W_1)$. □

Let

$$\Sigma = \{\{e_1, e_2\}, \{e_3, e_4\}, \ldots, \{e_{2m-1}, e_{2m}\}\}$$

and $X_m \cong \mathrm{Sym}(\Sigma)$ be the subgroup of G which faithfully permutes the ordered parts of the partition of the basis for Σ.

Lemma 3.7. *We have* $X_m \leqslant \Omega_n^\epsilon(q)$*.*

Proof. It suffices to show that the transpositions in X_m lie in $\Omega_n^\epsilon(q)$ and it further suffices to consider the transposition which exchanges $\{e_1, e_2\}$ and $\{e_3, e_4\}$. On the 4-dimensional space $\langle e_1, e_2, e_3, e_4\rangle$, such an element, β say, is a product of two reflections one swapping e_1 and e_3 and one swapping e_2 and e_4. Then β has determinant 1 and, as $Q(e_1 - e_3) = Q(e_2 - e_4)$, β is contained in $\Omega_n^\epsilon(q)$. □

The next lemma illustrates how the bases for the θ-decomposition we have chosen reveal the Sylow 2-structure of G. The subgroups defined in this lemma are stabilizers of θ-decompositions if V.

Lemma 3.8. *One of the following holds.*

(i) $n = 2m$, $V = U$, $\epsilon = \theta^m$ *and*

$$\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(\Sigma) = \mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m)$$

contains a Sylow 2*-subgroup of* G.

(ii) $n = 2m + 1$, codim $U = 1$ *and, setting* $\delta = \theta^m\epsilon$, *we have*

$$(\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(\Sigma)) \times \mathrm{GO}_1^{\delta}(q) = (\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m)) \times \mathrm{GO}_1^{\delta}(q)$$

contains a Sylow 2*-subgroup of* G.

(iii) $n = 2m + 2$, codim $U = 2$, $\epsilon = -\theta^{m+1}$ *and*

$$(\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(\Sigma)) \times \mathrm{GO}_2^{-\theta}(q) = (\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m)) \times \mathrm{GO}_2^{-\theta}(q)$$

contains a Sylow 2*-subgroup of* G.

Proof. If $n = 2$ or $n = 3$, the result is true (where for (iii) we interpret $\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(0)$ as the trivial group). So we proceed by induction to prove the result. We take the orders of the orthogonal groups from [17, pg. 72].

First we suppose that $U = V$. Then $\epsilon = \theta^m$ and we have the following possibilities

$$|\mathrm{GO}_{2m}^{\theta^m}(q) : \mathrm{GO}_{2m-2}^{\theta^{m-1}}(q)| = \begin{cases} (q^{m-1}+1)(q^m-1) & \theta^m = +, \theta^{m-1} = + \\ (q^{m-1}-1)(q^m-1) & \theta^m = +, \theta^{m-1} = - \\ (q^{m-1}+1)(q^m+1) & \theta^m = -, \theta^{m-1} = + \\ (q^{m-1}-1)(q^m+1) & \theta^m = -, \theta^{m-1} = -. \end{cases}$$

Evidently the fourth possibility cannot arise. In the first case, we have $\epsilon = \theta = +$ and $q \equiv 1 \pmod 4$. Therefore

$$((q^{m-1}+1)(q^m-1))_2 = 2(q^m-1)_2 = 2m_2(q-1)_2 = 2m_2(q-\theta)_2.$$

In the second case, $\epsilon = +$, $\theta = -$, $q \equiv -1 \pmod 4$ and m is even. Hence

$$((q^{m-1}-1)(q^m-1))_2 = 2m_2(q-\theta)_2.$$

In the third possibility, $\epsilon = \theta = -$, $q \equiv -1 \pmod 4$ and m is odd. Hence

$$((q^{m-1}+1)(q^m+1))_2 = 2(q^{m-1}+1)_2 = 2m_2(q-\theta)_2.$$

Since the 2-part of the index of $\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m-1)$ in $\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m)$ is exactly $2m_2(q-\theta)_2$, we have proved (i).

Suppose that $\dim V_{-1} = 1$. Then $|\mathrm{GO}_{2m+1}(q) : \mathrm{GO}_{2m}^{\theta^m}(q)| = q^m + \theta^m$. Since $q \equiv \theta \pmod 4$, $q^m \equiv \theta^m \pmod 4$ and so $q^m + \theta^m \equiv 2 \pmod 4$. Hence, using (i) and the definition of type of an orthogonal group of odd degree, we get that (ii) holds.

Assume that $\dim V_{-1} = 2$, and set $n = 2m+2$. Observe that V_{-1} has type $-\theta$. Hence $\epsilon = -\theta^{m+1}$ and we are interested in the index

$$|\mathrm{GO}_{2m+2}^{-\theta^{m+1}}(q) : \mathrm{GO}_{2m}^{\theta^m}(q)| = \begin{cases} (q^m+1)(q^{m+1}-1) & -\theta^{m+1} = +, \theta^m = + \\ (q^m-1)(q^{m+1}-1) & -\theta^{m+1} = +, \theta^m = - \\ (q^m+1)(q^{m+1}+1) & -\theta^{m+1} = -, \theta^m = + \\ (q^m-1)(q^{m+1}+1) & -\theta^{m+1} = -, \theta^m = -. \end{cases}$$

Notice that the second case cannot occur as then $\theta = -$ and m is odd which means that $m+1$ is even and so $\theta^{m+1} = +$. Suppose that the first possibility arises. Then $\theta = -$ and m is even. As $\theta = -$, $q \equiv 3 \pmod 4$, and $m+1$ is odd, $q^{m+1} - 1 \equiv q^m + 1 \equiv 2 \pmod 4$. Hence

$$(q^m+1)(q^{m+1}-1) \equiv 4 \pmod 8.$$

In the third case, we have $\theta = +$ and $q \equiv 1 \pmod 4$ which gives $(q^m+1)(q^{m+1}+1) \equiv 4 \pmod 8$. In the final case, $\theta = -$, $m+1$ is even and $q \equiv 3 \pmod 4$ and so again $(q^m-1)(q^{m+1}+1) \equiv 4 \pmod 8$. Since the Sylow 2-subgroups of $\mathrm{GO}_2^{-\theta}(q)$ have order 4, part (iii) holds. □

We call the subgroups introduced in Lemma 3.8, *θ-decomposition subgroups* of G or more succinctly *θ-subgroups*. Exploiting the descriptions of the θ-subgroups given in Lemma 3.8, we intend to decompose the Sylow 2-subgroup according to the 2-adic decomposition of m. Thus we set

$$m = 2^{m_1} + \cdots + 2^{m_s}$$

with $m_1 > \cdots > m_s \geqslant 0$. Set $J = \{1, \ldots, s\}$. Recalling the definition of Σ from before Lemma 3.7, we write $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_s$ with $|\Sigma_i| = 2^{m_i}$ just as we did for Ω.

Following our standard notation for symmetric groups, T_{m_k} denotes a Sylow 2-subgroup of $\mathrm{Sym}(\Sigma_k) = \mathrm{Sym}(2^{m_k})$ which here we regard as a subgroup of $\mathrm{Sym}(\Sigma_k) \leqslant \mathrm{Sym}(\Sigma)$. Let $\beta_i = \{e_{2i-1}, e_{2i}\} \in \Sigma_j$ and set $D_{\beta_i} = \langle x_{2i-1}, y_{2i-1} \rangle \cong \mathrm{Dih}(2(q-\theta)_2)$. Then we define

$$S_{m_j} = (\prod_{\beta \in \Sigma_j} D_\beta) T_{m_j}$$

and note that S_{m_j} is a Sylow 2-subgroup of $\mathrm{GO}_{2^{m_j}}^{\mu}(q)$ where $\mu = \theta^{2^{m_j}}$ by Lemma 3.8. Of course, for a fixed $\beta \in \Sigma_j$, we could also have described $S_{m_j} = D_\beta \wr T_{m_j}$.

Define $V_{m_j} = [V, S_{m_j}] = \langle\langle\beta\rangle \mid \beta \in \Sigma_j\rangle$ and observe that Lemma 3.8 implies that

$$V = V_{m_1} \perp \cdots \perp V_{m_s} \perp V_{-1}$$

where V_{-1} is either the zero space, one-dimensional of type $\delta = \theta^m\epsilon$ or a 2-dimensional non-degenerate space of type $-\theta$ with orthogonal basis $\{e_{-1}, f_{-1}\}$. In the cases when V_{-1} is zero we take S_{-1} to be the trivial group (acting on a zero-dimensional space). When $\dim V_{-1} = 1$, we set $S_{-1} = \langle x_{-1}\rangle$ if $\theta^m\epsilon = +$ and $\langle y_{-1}\rangle$ if $\theta^m\epsilon = -$. If $\dim V_{-1} = 2$, we write $S_{-1} = \langle x_{-1}, y_{-1}\rangle$. Finally, we put

$$S = \prod_{j=1}^{s} S_{m_j} \times S_{-1}$$

and note that $S \in \mathrm{Syl}_2(G)$ by Lemma 3.8. This gives us a "hands-on" description of S.

We record the following consequence of our notational choices.

Lemma 3.9. *We have V_{m_j} is a perpendicular sum of 2^{m_j} subspaces of θ-type and $\dim V_{m_j} = 2^{m_j+1}$. In particular, if $j \in J$ and $m_j \geqslant 1$, then V_{m_j} is of $+$-type and if $m_j = m_s = 0$, then V_0 is of θ-type.*

Notice that

$$U = V_{m_1} \perp \cdots \perp V_{m_s}$$

and, as we mentioned before, U has type θ^m.

Lemma 3.10. *Suppose that $n \geqslant 4$, $H = \Omega_n^\epsilon(q)$ and $S_H = S \cap H$. Then*

(i) *for $1 \leqslant i \leqslant s$, S_H acts irreducibly on V_{m_i}.*
(ii) *S and S_H act in the same way on V_{-1}.*

Furthermore, if $W \leqslant V$ is S_H-invariant and non-zero, then W is S-invariant and non-degenerate.

Proof. Suppose first that $\dim V_{m_i} \geqslant 4$. To simplify notation we may as well suppose that $V = V_{m_i}$ and so has basis $e_1, \ldots, e_{2m}$. Recall x_j is the reflection that negates e_j and $x_1, \ldots, x_{2m} \in S$. For $1 \leqslant j < k \leqslant 2m$, x_jx_k has spinor norm a square and determinant 1 and so $x_jx_k \in S_H$. Assume that W is a non-zero S_H-invariant subspace of V. We know $T_m \leqslant X_m \leqslant H$ by Lemma 3.7, and so T_m leaves W invariant. Choose a non-zero $w = \sum_{i=1}^{2m} \lambda_i e_i \in W$ such that the number $d(w)$ of terms with $\lambda_i \neq 0$ is minimal. If $d(w) = n$, then $d(w + wx_{n-1}x_n) = d(w) - 2$, a contradiction as $\dim V_{m_i} \geqslant 4$. Hence we can suppose $\lambda_j = 0$ and $\lambda_k \neq 0$ for some $j, k \in \{1, \ldots, 2m\}$. Then $d(w + wx_jx_k) = d(w) - 1$ which is contradiction unless $w + wx_jx_k = 0$. Because of this we may suppose that $e_k \in W$ for some $k \in \{1, \ldots, 2m\}$. Applying elements from T_m, we have $e_\ell \in W$ for all $\ell \equiv k \pmod 2$. We know $y_1y_3 \in S_H$.

Since $e_1y_1y_3 = e_1y_1 \in \langle e_1, e_2\rangle \backslash \langle e_1\rangle$ and $e_2y_1y_3 = e_2y_1 \in \langle e_1, e_2\rangle \backslash \langle e_2\rangle$, we deduce that $\langle e_1, e_2\rangle \leqslant W$ and, as W is T_m-invariant, we have $V = W$. This proves (i) when $\dim V_{m_i} \geqslant 4$. Suppose that $\dim V_{m_i} < 4$. Then $m_i = m_s = 0$ and $V_{m_s} = V_0$ is isometric to V_θ. Since $n \geqslant 4$, either $n = 4$, $m_1 = 0$ and $V = V_0 + V_{-1}$ with $V_{-1} = \langle e_{-1}, f_{-1}\rangle$ or $m_1 \geqslant 1$.

Taking $\Sigma_0 = \{\lambda\}$ with $\lambda = \{e_a, e_{a+1}\}$, we have $S_0 = D_\lambda$ is non-abelian. Hence S_0 acts irreducibly on V_0 as $\dim V_0 = 2$. Now set $x = x_1x_a$ and $y = y_2y_{a+1}$ if $m_1 \geqslant 1$ or $x = x_ax_{-1}$ and $y = y_{a+1}y_{-1}$ if $m_1 = 0$. Then $\langle x, y\rangle \leqslant S_H$ and acts on V_0 as S_0 and thus V_0 is irreducible as an S_H-module as claimed. This confirms that (i) holds.

Part (ii) is demonstrated by considering the one of or both of elements x_1x_{-1} and y_1y_{-1}.

By parts (i) and (ii), as an S_H-module, V is a direct sum of the pairwise non-isomorphic irreducible submodules. Hence every irreducible S_H-submodule of V is S-invariant and so it follows from Maschke's Theorem that every S_H-submodule is an S-submodule. □

Lemma 3.11. *The following hold.*

(i) *If $V = U$ (so $n = 2m$ and $\epsilon = \theta^m$), then G contains $\mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(n)$. In particular, G contains two conjugacy classes of subgroups isomorphic to $2 \wr \mathrm{Sym}(n)$.*

(ii) *If $\operatorname{codim} U = 1$ (so $n = 2m+1$ and $\delta = \theta^m\epsilon$), then G contains $\mathrm{GO}_1^{\delta}(q) \wr \mathrm{Sym}(n)$.*

(iii) *If $\operatorname{codim} U = 2$ (so $n = 2m+2$ and $\epsilon = -\theta^{m+1}$), then G contains the two subgroups $\mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(n-1) \times \mathrm{GO}_1^{\mp}(q)$.* □

Proof. This follows from Witt's Lemma. □

We consider the subgroups appearing in Lemma 3.11 when $q \equiv 3, 5 \pmod 8$ and investigate how they intersect with our various candidates for H. Recall that we have fixed $e_1, \ldots, e_{2m}$ to be an orthogonal bases for U with e_i of $+$-type just after Lemma 3.11. In this case a Sylow 2-subgroup of $\mathrm{GO}_2^{\theta}(q)$ is generated by two reflections $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (with respect to $\{e_1, e_2\}$) and so in this case, as 2 is not a quadratic residue mod p because $q \equiv 3, 5 \pmod 8$, we can manufacture the basis $f_1, \ldots, f_{2m}$ as follows: $f_{2j+1} = e_{2j+1} + e_{2j+2}$ and $f_{2j+2} = e_{2j+1} - e_{2j+2}$ for $0 \leqslant j \leqslant m-1$. Then the type of f_i is $-$ and $f_1, \ldots, f_{2m}$ is an orthogonal basis for U. In particular, in this instance we see that S preserves both these decompositions and that the groups described in Lemma 3.11 all contain S.

Lemma 3.12. *Suppose that $q \equiv 3, 5 \pmod 8$.*

(i) *If* $V = U$ *(*$n = 2m$ *and* $\epsilon = \theta^m$*), then*

$$(\mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(n)) \cap H \sim \begin{cases} 2^{n-1}.\mathrm{Sym}(n) & H = {}^{\mp}\mathrm{GO}_n^{\epsilon}(q) \\ \mathrm{GO}_1^{\pm}(q) \wr \mathrm{Alt}(n) & H = {}^{\pm}\mathrm{GO}_n^{\epsilon}(q) \\ 2^{n-1}.\mathrm{Sym}(n) & H = \mathrm{SO}_n^{\epsilon}(q) \\ 2^{n-1}.\mathrm{Alt}(n) & H = \Omega_n^{\epsilon}(q) \end{cases}.$$

(ii) *If* codim $U = 1$ *(*$n = 2m+1$ *and* $\delta = \theta^m\epsilon$*), then*

$$(\mathrm{GO}_1^{\delta}(q) \wr \mathrm{Sym}(n)) \cap H \sim \begin{cases} 2^{n-1}.\mathrm{Sym}(n) & H = {}^{-\delta}\mathrm{GO}_n^{\epsilon}(q) \\ \mathrm{GO}_1^{\delta}(q) \wr \mathrm{Alt}(n) & H = {}^{\delta}\mathrm{GO}_n^{\epsilon}(q) \\ 2^{n-1}.\mathrm{Sym}(n) & H = \mathrm{SO}_n^{\epsilon}(q) \\ 2^{n-1}.\mathrm{Alt}(n) & H = \Omega_n^{\epsilon}(q) \end{cases}.$$

(iii) *If* codim $U = 2$ *(*$n = 2m+2$ *and* $\epsilon = -\theta^{m+1}$*), then*

$$(\mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(n-1) \times \mathrm{GO}_1^{\mp}(q)) \cap H \sim \begin{cases} 2^{n-2}.\mathrm{Sym}(n-1) \times 2 & H = {}^{\mp}\mathrm{GO}_n^{\epsilon}(q) \\ 2^{n-1}.\mathrm{Sym}(n-1) & H = {}^{\pm}\mathrm{GO}_n^{\epsilon}(q) \\ 2^{n-1}.\mathrm{Sym}(n-1) & H = \mathrm{SO}_n^{\epsilon}(q) \\ 2^{n-2}.\mathrm{Sym}(n-1) & H = \Omega_n^{\epsilon}(q) \end{cases}.$$

(Notice that the subgroups of the same shape in lines one and three in the above statements (i), (ii) and (iii) are not equal.) In all cases $S \cap H$ *is a Sylow* 2*-subgroup of these groups.*

Proof. This follows from the discussion prior to the lemma. □

Extending our notation to cover all possibilities for $H \leqslant G$ we set $S_H = S \cap H \in \mathrm{Syl}_2(H)$, $B_H = N_H(S \cap H)$ and $B = B_G$. As we now see, if $n \geqslant 6$, $B_H = B \cap H = S_H$.

Lemma 3.13. *We have that* $B \cap H = S_H$ *is a Sylow* 2*-subgroup of* H*. Furthermore* $B \cap H = S_H = B_H$ *unless* $H < G$*,* $q \equiv 3, 5 \pmod 8$ *and one of the following holds:*

(i) $G = \mathrm{GO}_3^{\epsilon}(q)$ *and* $H \leqslant {}^{\epsilon}\mathrm{GO}_3^{\epsilon}(q) \cong 2 \times \mathrm{PSL}_2(q)$ *when* $q \equiv 5 \pmod 8$ *and* $H \leqslant {}^{-\epsilon}\mathrm{GO}_3^{\epsilon}(q) \cong 2 \times \mathrm{PSL}_2(q)$ *when* $q \equiv 3 \pmod 8$*.*

(ii) $G = \mathrm{GO}_4^{+}(q)$ *and* $H \leqslant {}^{\pm}\mathrm{GO}_4^{+}(q)$*.*

(iii) $G = \mathrm{GO}_5^{\epsilon}(q)$ *and* $H \leqslant {}^{\epsilon}\mathrm{GO}_5^{\epsilon}(q) \cong 2 \times \mathrm{PSp}_4(q)$*.*

Proof. That S_H is a Sylow 2-subgroup of H follows from Lemma 3.8 and our construction of S. The fact that $B = S = N_G(S)$ is stated as Theorem 5 of [2]. For $H < G$, we refer to [7, Theorem 1] to see that typically $B \cap H = S_H = B_H$ and to locate the exceptions which occur when $n = 3, 4$ and 5 and $q \equiv 3, 5 \pmod 8$.

For $n = 3$, we have $K = Z(G)G' \cong 2 \times \Omega_3^\epsilon(q) \cong 2 \times \mathrm{PSL}_2(q)$ is ${}^{\theta\epsilon}\mathrm{GO}_3^\epsilon(q)$ by Lemma 3.5 (ii) and this leads to the case distinction in (i).

When $n = 4$, we additionally know that V has $+$-type. Since $q \equiv 3, 5 \pmod 8$, $B/O_2(B) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ and we find $B_H/O_2(B) \cong 3 \times \mathrm{Sym}(3)$ when $H = {}^{\pm}\mathrm{GO}_4^+(q)$ and $B_H/O_2(B) = 3 \times 3$ when $H = \Omega_4^+(q)$. When $H = \mathrm{SO}_4^+(q)$, the Sylow 3-subgroup is inverted. Hence we have the three possibilities as listed in (ii).

When $n = 5$, Lemma 3.5 (ii) gives $K = Z(G)G' \cong 2 \times \Omega_5^\epsilon(q)$ is ${}^{\epsilon}\mathrm{GO}_5^\epsilon(q)$ as $m = 2$. This is (iii).

□

Our general strategy for locating the 2-minimal subgroups of H is guided by the Proposition 3.14. As mentioned in the introduction the existence of the exceptional Coxeter groups E_7 and E_8 play a role in the subgroup structure of the orthogonal groups in dimensions 7 and 8. The Coxeter group E_6 doesn't get invited to the party as its Sylow 2-subgroup is a factor of 2 too small to contain a Sylow 2-subgroup of $\Omega_6^\pm(q)$. This can lead to some confusion as $H = \Omega_6^+(3)$ contains subgroups $\mathrm{GO}_5^\pm(3) \cap H \cong \mathrm{PSU}_4(2).2 \cong \mathrm{E}_6$ but the latter group contains reflections whereas $\Omega_6^+(3)$ does not. In dimension 8, the $+$-type reflections and $-$-type reflections become conjugate in the conformal group. Hence in $\mathrm{GO}_8^+(q)$ there are two conjugacy classes of subgroups isomorphic to E_8 one generated by $+$-type reflections and the other by $-$-type reflections. These classes decompose into two classes in ${}^{\pm}\mathrm{GO}_8^+(q)$ and eventually, when $q \equiv 3, 5 \pmod 8$, we obtain four conjugacy classes of subgroups containing the Sylow 2-subgroup of $\Omega_8^+(q)$ each being isomorphic to $\mathrm{E}_8' \cong 2{\cdot}\Omega_8^+(2)$. The situation is similar in dimension 7 except that there is only one conjugacy class of subgroups isomorphic to E_7 and this subgroup is generated by reflections of type $\theta^3\epsilon$ and gives rise to just two conjugacy classes of subgroups isomorphic to $\mathrm{E}_7' \cong \mathrm{Sp}_6(2)$ in $\Omega_7^\epsilon(q)$.

Proposition 3.14. *Suppose that $G = \mathrm{GO}_n^\epsilon(q)$, $n \geqslant 6$ and $P < H$ has odd index in H. Then either*

(i) *P normalizes a subfield subgroup of G;*
(ii) *P leaves a proper subspace of V invariant;*
(iii) *P leaves invariant a decomposition of V into an orthogonal sum of non-degenerate subspaces of dimension 2^k for some $k \geqslant 0$;*
(iv) *$n = 7$, $q \equiv 3, 5 \pmod 8$ and $H = {}^{\theta^3\epsilon}\mathrm{GO}_7^\epsilon(q)$ with P contained in a subgroup isomorphic to E_7;*

(v) $n = 7$, $q \equiv 3, 5 \pmod 8$ *and* $H = \Omega_7^\epsilon(q)$ *with* P *contained in a subgroup isomorphic to* $\mathrm{E}_7' \cong \mathrm{Sp}_6(2)$. *There are two* H*-conjugacy classes of such subgroups and they are conjugate in* G*;*

(vi) $n = 8$, $q \equiv 3, 5 \pmod 8$ *and* $H = {}^{\pm}\mathrm{GO}_8^+(q)$ *with* P *contained in a subgroup isomorphic to* E_8. *There are two* H*-conjugacy classes of such subgroups and they are conjugate in* G*; or*

(vii) $n = 8$, $q \equiv 3, 5 \pmod 8$ *and* $H = \Omega_8^+(q)$ *with* P *contained in a subgroup isomorphic to* $\mathrm{E}_8' \cong 2{}^{\cdot}\Omega_8^+(2)$. *There are four* H*-conjugacy classes of such subgroups and two conjugacy classes in* G.

Proof. By Lemma 3.13, as $n \geqslant 6$, B_H is both a Sylow 2-subgroup of H and its own normalizer in H. Suppose that (i), (ii) and (iii) do not hold. Then by [9, Theorem] or [11, 12], $n = 7$ or $(n, \epsilon) = (8, +)$ and P is contained in a subgroup of G isomorphic to $\mathrm{E}_7 \cong 2 \times \mathrm{Sp}_6(2)$ or E_8 (which has shape $2{}^{\cdot}\Omega_8^+(2).2$) respectively. Since both are generated by a single conjugacy class of reflections, they do not contain a Sylow 2-subgroup of G. However, they do contain Sylow 2-subgroups of some of the subgroups of G of index 2 . The exact description of these groups is given in the statement. For this we note that in odd dimensional orthogonal groups $\mathrm{GO}_{2m+1}^\epsilon(q)$ the subgroup isomorphic to $2 \times \Omega_{2m+1}^\epsilon(q)$ is ${}^{\theta^m\epsilon}\mathrm{GO}_{2m+1}^\epsilon(q)$ by Lemma 3.5 (ii). □

With all our notation established, we now gather it together in one place.

Notation 3.15. *We take* $n \geqslant 2$ *a natural number,* $q = p^a$ *with* p *an odd prime,* $G = \mathrm{GO}_n^\epsilon(q)$, $\theta \equiv q \pmod 4$ *and fix a* θ*-decomposition of* V.

(i) H *is one of the five groups* $\mathrm{GO}_n^\epsilon(q)$, ${}^+\mathrm{GO}_n^\epsilon(q)$, ${}^-\mathrm{GO}_n^\epsilon(q)$, $\mathrm{SO}_n^\epsilon(q)$ *and* $\Omega_n^\epsilon(q)$.

(ii) $V = U \perp V_{-1}$ *where* $\dim U = 2m$, $\dim V_{-1} \leqslant 2$ *and* $n = 2m + \dim V_{-1}$.

(iii) $m = 2^{m_1} + \cdots + 2^{m_s}$ *with* $m_1 > \cdots > m_s \geqslant 0$ *and* $J = \{1, \ldots, s\}$.

(iv) U *has an orthogonal bases* $e_1, \ldots, e_{2m}$ *and* $f_1, \ldots, f_{2m}$ *with* $Q(e_1) = Q(e_2) = \cdots = Q(e_{2m})$ *a square and* $Q(f_1) = Q(f_2) = \cdots = Q(f_{2m})$ *a non-square. Either* $V_{-1} = 0$, $V_{-1} = \langle d_{-1}\rangle$ *with* d_{-1} *of type* $\theta^m\epsilon$ *and* $Q(d_{-1}) \in \{Q(e_1), Q(f_1)\}$ *or* $V_{-1} = \langle e_{-1}, f_{-1}\rangle$, $Q(e_{-1}) = Q(e_1)$ *and* $Q(f_{-1}) = Q(f_1)$.

(v) *for* $1 \leqslant i \leqslant 2m$, x_i *is the reflection negating* e_i *and* y_i *is the reflection negating* f_i. *We have* $\langle x_i, y_i\rangle \cong \mathrm{Dih}(2(q - \theta)_2)$.

(vi) $S = \prod_{j\in J} S_{m_j} \times S_{-1}$ *is a Sylow 2-subgroup of* G *with* $V_{m_j} = [V, S_{m_j}]$ *and* $[V, S_{-1}] = V_{-1}$ *as described before Lemma 3.9.*
(vii) $S_H = S \cap H \in \mathrm{Syl}_2(H)$, $B_H = N_H(S \cap H)$ *and* $B = N_G(S)$.

4. 2-minimal subgroups of orthogonal groups in dimension at most 5

In this section we continue to use the notation as in 3.15 and capture the 2-minimal subgroups when $n \leqslant 5$. We begin this section by itemizing the 2-minimal subgroups of $\Omega_3^{\pm}(q) \cong \mathrm{PSL}_2(q)$. In Theorem 4.1, the superscript [2] indicates that there are two conjugacy classes of the given group.

Theorem 4.1. *Suppose that* $H = \Omega_3^{\epsilon}(q) \cong \mathrm{PSL}_2(q)$ *with* $q = p^a$ *odd.*

(i) *If* $q \equiv 3, 5 \pmod 8$ *and* $p \neq 3, 5$, *then one of the following holds:*
  (a) $q \equiv \pm 11, \pm 19 \pmod{40}$ *and*
  $$\mathcal{M}(H, B_H) = \{\mathrm{Alt}(5)^{[2]}, \Omega_3^{\epsilon}(p^{s^t}) \mid s^t \in \Pi(a)\}; \text{ or}$$
  (b) $q \not\equiv \pm 11, \pm 19 \pmod{40}$ *and*
  $$\mathcal{M}(H, B_H) = \{\Omega_3^{\epsilon}(p^{s^t}) \mid s^t \in \Pi(a) \cup \{1\}\}.$$
(ii) *If* $q \equiv 3, 5 \pmod 8$ *and* $p = 3$, *then*
  $$\mathcal{M}(H, B_H) = \{\Omega_3^{\epsilon}(3^{s^t}) \mid s^t \in \Pi(a)\}.$$
(iii) *If* $q \equiv 3, 5 \pmod 8$ *and* $p = 5$, *then*
  $$\mathcal{M}(H, B_H) = \{\Omega_3^{\epsilon}(5^{s^t}) \mid s^t \in \Pi(a) \cup \{1\}\}.$$
(iv) *If* $q \equiv 1 \pmod 8$, *then one of the following holds:*
  (a) $a_2 > 2$ *or* $a_2 = 2$ *and* $q \equiv 1 \pmod{16}$,
  $$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q-1), B_H) \cup \{\mathrm{SO}_3^{\pm}(p^{a_2/2})\};$$
  (b) $p = 5$, $a_2 = 2$ *and*
  $$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q-1), B_H) \cup \{\mathrm{SO}_3^{\pm}(5)\} \cup \{\mathrm{Sym}(4)^{[2]}\};$$
  (c) $p = 3$, $a_2 = 2$ *and*
  $$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q-1), B_H) \cup \{\mathrm{SO}_3^{\pm}(3)\};$$
  (d) $a_2 = 2$ *and* $q \equiv 9 \pmod{16}$ *with* $p > 5$,
  $$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q-1), B_H) \cup \{\mathrm{Sym}(4)^{[2]}\};$$
  (e) $q \equiv 1 \pmod{16}$, $a_2 = 1$,
  $$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q-1), B_H) \cup \{\Omega_3^{\epsilon}(p)\}; \text{ or}$$

(f) $q \equiv 9 \pmod{16}$, $a_2 = 1$,

$$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q-1), B_H) \cup \{\mathrm{Sym}(4)^{[2]}\}.$$

(v) *If* $q \equiv 7 \pmod 8$, *then one of the following holds:*

(a) $q \equiv 7 \pmod{16}$,

$$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q+1), B_H) \cup \{\mathrm{Sym}(4)^{[2]}\};\ or$$

(b) $q \equiv 15 \pmod{16}$,

$$\mathcal{M}(H, B_H) = \mathcal{M}(\mathrm{Dih}(q+1), B_H) \cup \{\mathrm{SO}_3^{\epsilon}(p)\}.$$

Proof. See Theorem 13.2 of [13]. □

We make some remarks about the subgroups appearing in Theorem 4.1. The subgroups $\mathrm{Dih}(q-\theta)$ are $\mathrm{GO}_2^{\theta}(q) \times \mathrm{GO}_1(q)$. Just as in Proposition 3.14 we see the shadow of Coxeter groups in the picture. In this case the projection of $\mathrm{H}_3 \cong 2 \times \mathrm{Alt}(5)$ appears in (i) and $\mathrm{B}_3 \cong 2 \wr \mathrm{Sym}(3)$ appears in various other parts.

In the next theorem notice that $\mathrm{SO}_3(q) \cong {}^{-\epsilon\theta}\mathrm{GO}_3^{\epsilon}(q) \cong \mathrm{PGL}_2(q)$.

Theorem 4.2. *Assume that* $H = \mathrm{SO}_3(q)$ *or* $H = {}^{-\epsilon\theta}\mathrm{GO}_3^{\epsilon}(q)$ *with* $q = p^a$ *odd. Then under the given conditions* $\mathcal{M}(H, B_H)$ *is as follows.*

(i) $q \equiv \theta \pmod 8$ *and*

$$\mathcal{M}(\mathrm{Dih}(2(q-\theta)), B_H) \cup \{{}^{-\epsilon\theta}\mathrm{GO}_3^{\epsilon}(p^{a_2})\}.$$

(ii) $q \equiv 4 + \theta \pmod 8$, $p \neq 5$, *and*

$$\mathcal{M}(\mathrm{Dih}(2(q-\theta)), B_H) \cup \{\mathrm{Sym}(4)\}.$$

(iii) $q = 5^a$ *with* a *odd and*

$$\mathcal{M}(\mathrm{Dih}(2(q-1)), B_H) \cup \{{}^{-\epsilon\theta}\mathrm{GO}_3^{\epsilon}(5)\} \cup \{\mathrm{Sym}(4)\}.$$

Proof. This can be written down using [13, Proposition 9.3]. □

Since $\mathrm{GO}_3^{\epsilon}(q) \cong 2 \times \mathrm{PGL}_2(q)$, and ${}^{\epsilon\theta}\mathrm{GO}_3^{\epsilon}(q) \cong 2 \times \Omega_3^{\epsilon}(q)$, we may now use Theorems 4.1 and 4.2 to write down the 2-minimal subgroups for all H in dimension 3.

For application in the proof of our main theorem, Theorem A, we present the following corollary.

Corollary 4.3. *Suppose that* $G = \mathrm{GO}_3^{\epsilon}(q)$ *and* $P \in \mathcal{M}(G, B) \backslash \mathcal{M}(\mathrm{GO}_2^{\theta}(q) \times \mathrm{GO}_1^{\theta\epsilon}(q))$. *Then one of the following holds:*

(i) $q \equiv 1, 7 \mod 8$ *and* $P = \mathrm{GO}_3^{\epsilon}(p^{a_2})$;

(ii) $q \equiv 3, 5 \pmod 8$ *and* $P \cong \mathrm{GO}_1^{\epsilon\theta}(q) \wr \mathrm{Sym}(3)$;*or*

(iii) $p = 5$, a *is odd and* $P = \mathrm{GO}_3^{\epsilon}(5)$.

In particular, $G \in \mathcal{M}(G, B)$, *if and only if* $G = \mathrm{GO}_3^{\epsilon}(p^{a_2})$ *with* $p^{a_2} \equiv 1, 7 \pmod 8$, $G = \mathrm{GO}_3^{\epsilon}(5)$ *or* $G \cong \mathrm{GO}_3^{\epsilon}(3) \cong \mathrm{GO}_1^{-\epsilon}(3) \wr \mathrm{Sym}(3)$. □

If $G = \mathrm{GO}_4^+(3)$, then $G/O_2(G) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ and so this group has two 2-minimal subgroups each of index 3 in G. Since $\mathrm{GO}_4^+(3)$ is isomorphic to the Coxeter group of type F_4, these novelty 2-minimal subgroups perpetuate in $\mathrm{GO}_4^+(q)$ whenever $q \equiv 3, 5 \pmod 8$. Whenever $q \equiv 3, 5 \pmod 8$ and $H = \Omega_4^+(q)$, then F_4 is realized as $N_G(S_H)$ where S_H is the central product of two quaternion groups of order 8. Notice that F_4 contains $\mathrm{GO}_1^+(q) \wr \mathrm{Sym}(4) = \mathrm{B}_4$ and $\mathrm{GO}_1^-(q) \wr \mathrm{Sym}(4) = \mathrm{B}_4$ as maximal subgroups.

Lemma 4.4. *Suppose that $G = \mathrm{GO}_4^+(q)$ and $P \in \mathcal{M}(H, B_H)$. Let K_1 and K_2 be subnormal subgroups of G isomorphic to $\mathrm{SL}_2(q)$, $K = K_1K_2$ and $B_0 = N_{B_H}(K_1) = N_{B_H}(K_2)$. Then either*

(i) *there exists*

$$X \in \mathcal{M}(K_1B_0, N_{K_1B_0}(S \cap B_0)) \cup \mathcal{M}(K_2B_0, N_{K_2B_0}(S \cap B_0))$$

such that $P = \langle X, B_H \rangle$; or

(ii) *$q \equiv 3, 5 \pmod 8$ and $P \in \mathcal{M}(N_H(S \cap K), B_H)$.*

Proof. Suppose first that $H \in \{\mathrm{SO}_4^+(q), \Omega_4^+(q)\}$. Then $B_H = N_H(S \cap H)$ normalizes K_1 and K_2 and so Lemma 2.8 applies to give possibility (i) or (ii) holds.

So we may suppose that $H = \mathrm{GO}_4^+(q)$ or $H = {}^{\pm}\mathrm{GO}_4^+(q)$.

Assume that (ii) does not hold. Then $G \neq \mathrm{GO}_4^+(3)$ and so K_1 and K_2 are components of G and there exists a reflection $b \in B_H$ such that $K_1^b = K_2$. Set $S_0 = S \cap B_0$.

We have $P = B_H(P \cap K)$, $(P \cap K_1)^b = P \cap K_2$ and both $P \cap K_1$ and $P \cap K_2$ are normalized by $P \cap K$. If P normalizes $S \cap K$, then $N_H(S \cap K)$ is not a 2-group by Lemma 3.13 and so $q \equiv 3, 5 \pmod 8$ and (ii) holds, a contradiction. Hence $P \cap K$ does not normalize $S \cap K$ and so $S \cap K_1$ is not normalized by $P \cap K$. Set $R = \langle (S \cap K_1)^{P \cap K} \rangle$. Then R is normalized by B_0.

Set $X = RB_0$. We claim that $N_{K_1B_0}(S_0) \leqslant X$. This plainly holds if $B_0 = N_{K_1B_0}(S_0)$. Assume that $B_0 < N_{K_1B_0}(S_0)$. In this case Lemma 3.13 implies $q \equiv 3, 5 \pmod 8$, $S_0 = S \cap K$ and we know $H = {}^{\pm}\mathrm{GO}_4^+(q)$. In addition, we have $B_0 \leqslant K$, $|B_H : S \cap H| = 3 = |N_K(S \cap K) : B_0|$ and $|B_H : B_0| = 2$. Since B_0 normalizes R and K_2 centralizes R, R is normalized by

$$B_0N_{K_2}(S \cap K_2) = N_K(S \cap K) \geqslant N_{K_1}(S \cap K).$$

Using Theorem 4.1 or more directly [4, II.8.27] we observe that any subgroup of K_1 which is normalized by $N_{K_1}(S \cap K_1)$ contains $N_{K_1}(S \cap K_1)$. Hence $N_{K_1}(S \cap K_1) \leqslant R$ and $N_{K_1}(S \cap K_1)B_0 = N_{K_1B_0}(S_0) \leqslant X$ as claimed.

Since $P \cap K_1$ normalizes R, we have $P \cap K_1 = RN_{P\cap K_1}(S \cap K_1) = R$. Hence $P = \langle X, B_H\rangle$. Assume that X is not 2-minimal. Then X contains maximal subgroups U_1 and U_2 both containing $N_X(S_0)$ with $U_1 \neq U_2$. We have $BU_1U_1^b$ is a proper over-group of B in P containing B. Similarly $BU_2U_2^b$ is a proper over-group of B contained in P. Since $(P \cap K)B_0 = \langle U_1, U_2\rangle$, we have a contradiction to the 2-minimality of P. Hence X is a 2-minimal subgroup of K_1B_0 and $P = \langle X, B_H\rangle$ as claimed in (i). □

Lemma 4.5. *Suppose that $G = \mathrm{GO}_4^+(q)$. Let K_1 and K_2 be subnormal subgroups of G isomorphic to $\mathrm{SL}_2(q)$, $K = K_1K_2$ and select reflections x_1 and y_3 such that $[V, x_1] = \langle e_1\rangle$ and $[V, y_3] = \langle f_3\rangle$. Put $B_0 = N_{B_H}(K_1) = N_{B_H}(K_2)$. Assume that $P \in \mathcal{M}(H, B_H)$. Then*

(i) *If $X \in \mathcal{M}(K_1B_0, N_{K_1B_0}(S \cap B_0)) \cup \mathcal{M}(K_2B_0, N_{K_1B_0}(S \cap B_0))$ is such that $P = \langle X, B_H\rangle$, then either*

(a) *$H = G = \mathrm{GO}_4^+(q)$, $K_1\langle x_1y_3\rangle/Z(K_1) \cong \mathrm{PGL}_2(q)$ and*

$$\mathcal{M}(H, B) = \{RR^{x_1}B \mid R \in \mathcal{M}(K_1\langle x_1y_3\rangle, B \cap K_1\langle x_1y_3\rangle)\}$$

(see Theorem 2.6 for the candidates for R).

(b) *$H = {}^{\pm}\mathrm{GO}_4^+(q)$,*

$$\mathcal{M}(H, B_H) = \{RR^{x_1}B_H \mid R \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))\}$$

(see Theorem 4.1 for the candidates for R).

(c) *$H = \mathrm{SO}_4^+(q)$ and*

$$\mathcal{M}(H, B_H) = \{RB_H, R^{x_1}B_H \mid R \in \mathcal{M}(K_1\langle x_1y_3\rangle, B_H \cap K_1\langle x_1y_3\rangle)\}$$

(see Theorem 2.6 for the candidates for R).

(d) *$H = \Omega_4^+(q)$ and*

$$\mathcal{M}(H, B_H) = \{RB_H, R^{x_1}B_H \mid R \in \mathcal{M}(K_1, B_H \cap K_1)\}.$$

(see Theorem 4.1 for the candidates for R).

(ii) *If $q \equiv 3, 5 \pmod 8$ and $P \in \mathcal{M}(N_H(S \cap K), B_H)$, then one of the following holds:*

(a) *$H = G$, and $P = [N_K(S \cap K), x_1]B = \mathrm{GO}_1^+(q) \wr \mathrm{Sym}(4)$ or $P = [N_K(S \cap K), y_3]B = \mathrm{GO}_1^-(q) \wr \mathrm{Sym}(4)$.*

(b) *$H = {}^{+}\mathrm{GO}_4^+(q)$, and $P = [N_K(S \cap K), x_1]B_H$.*

(c) *$H = {}^{-}\mathrm{GO}_4^+(q)$, and $P = [N_K(S \cap K), y_3]B_H$.*

(d) *$H = \mathrm{SO}_4^+(q)$, and $P \in \{N_{K_1}(S\cap K)B_H, N_{K_2}(S\cap K)B_H, [N_K(S\cap K), x_1]B_H, [N_K(S \cap K), y_3]B_H\}$.*

Proof. Suppose that $X \in \mathcal{M}(K_1B_0, N_{K_1B_0}(S \cap B_0))$ and $P = \langle X, B_H\rangle$ be as in Lemma 4.4 (i). Set $D = \langle x_1y_3\rangle \cap H$. Then $B_0 = (B_H \cap K)D$ and $|D| \leqslant 2$. Since $X \geqslant N_{K_1B_0}(S \cap B_0) \geqslant B_0$, we have

$$X = (X \cap K_1)B_0 = (X \cap K_1)D(B_0 \cap K_2).$$

Set $R = (X \cap K_1)D$. Now

$$\begin{aligned} N_{K_1D}(S \cap K_1D) &= N_{K_1D}((S \cap K_1)D) \\ &= N_{K_1B_0}((S \cap K)D) \cap K_1D \\ &\leqslant X \cap K_1D = R. \end{aligned}$$

Let U^* be the unique maximal subgroup of X containing $N_{K_1B_0}(S \cap B_0)$ and assume that $N_{K_1D}(S \cap K_1D) \leqslant T < R$. Then $T = (T \cap K_1)D$ and so $T_0 = TN_{K_1B_0}(S \cap B_0) = (U^* \cap K_1)D(B_0 \cap K_2)$ and $T_0 \cap K_1D = T$. Hence $N_{K_1B_0}(S \cap B_0) < T_0 < X$ which means that $T_0 \leqslant U^*$. It follows that $T \leqslant U^* \cap R$ and so $R \in M(K_1D, N_{K_1D}(S \cap K_1D))$. Now $P = \langle X, B_H \rangle = RB_H$ if $B_H = B_0$ and otherwise $P = RR^{x_1}B_H$. This proves part (i).

If Lemma 4.4 (ii) holds, then part (ii) follows as $N_G(S \cap K) = \mathrm{F}_4$ and $\mathrm{F}_4/O_2(\mathrm{F}_4) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. □

Corollary 4.6. *Suppose that* $G = \mathrm{GO}_4^+(q)$ *and* $P \in \mathcal{M}(G,B) \backslash \mathcal{M}(\mathrm{GO}_2^\theta(q) \wr T_1, B)$*. Then one of the following holds*

(i) $q \equiv 1, 7 \pmod 8$ *and* $P = \mathrm{GO}_4^+(p^{a_2})$*;*
(ii) $q \equiv 3, 5 \pmod 8$*,* $P \in \mathcal{M}(\mathrm{F}_4, B) \in \{\mathrm{GO}_1^+(q) \wr \mathrm{Sym}(4), \mathrm{GO}_1^-(q) \wr \mathrm{Sym}(4)\}$*;*
(iii) $p = 5$*, and* $P = \mathrm{GO}_4^+(5)$*.*

In particular, $G \in \mathcal{M}(G,B)$ *if and only if* $G = \mathrm{GO}_4^+(p^{a_2})$ *with* $p^{a_2} \equiv 1, 7 \pmod 8$*, or* $q = 5$ *and* $G = \mathrm{GO}_4^+(5)$*.*

Proof. Parts (i), (ii) and (iii) follow from Lemma 4.5.

Suppose that $G \in \mathcal{M}(G,B)$. If $q \equiv 1, 7 \pmod 8$, then $p^{a_2} \equiv 1, 7 \pmod 8$. If $p^{a_2} < q$, then $\mathrm{GO}_2^\theta(q) \wr T_1$ is not contained in any proper subfield subgroup and this leads to a contradiction. Hence if G is 2-minimal, then $q = p^{a_2}$.

Suppose that $q \equiv 3, 5 \pmod 8$ with $q > 3$. Then the subgroup F_4 is contained in $\mathrm{GO}_4^+(p)$ and this subgroup with $\mathrm{GO}_2^\theta(q) \wr T_1$ generates G unless $\mathrm{GO}_2^\theta(q) \wr T_1 \leqslant \mathrm{F}_4$. This latter possibility holds only if $q - \theta$ is a power of 2. Since $q \equiv 3, 5 \pmod 8$, this is only if $q \in 5$ as $q \neq 3$. If $q = 3$, then $\mathrm{GO}_4^+(3) = \mathrm{F}_4$, and this group is not 2-minimal. □

Before we study the 2-minimal subgroups of $\mathrm{GO}_4^-(q)$, we draw attention to a perplexing consequence of the definition of the type of a quadratic space of odd dimension. The situation of interest arises when m is odd, $n = 2m + 2$ and $q \equiv -1 \pmod 4$ so that $\theta = -$. Now in this case U has type $\theta^m = -$ and V_{-1} has type $-\theta = +$. In particular, V has $-$-type. We take our standard basis $e_1, \ldots, e_{2m}$ for U and let e_{-1}, f_{-1} be our usual basis for V_{-1}. Then $W = f_{-1}^\perp = \langle e_1, \ldots, e_{-1} \rangle$ and

$e_{-1}^{\perp} \cap W = U$ which has $-$-type. Since e_{-1} is a $+$-vector, it follows from the definition of type of an odd dimensional orthogonal space that W has $-$-type. Thus the stabilizer of the decomposition $W \perp W^{\perp}$ has type $\mathrm{GO}_{2m+1}^{-}(q) \times \mathrm{GO}_1^{-}(q)$ whereas we may have expected the superscripts to product to a $-$. This leads to the matching of signs being rather bizarre in part (iii) of the next lemma.

Lemma 4.7. *Suppose that* $G = \mathrm{GO}_4^{-}(q)$ *and* $P \in \mathcal{M}(H, B_H)$. *Then* $P = \widehat{P} \cap H$ *where* $\widehat{P} \in \mathcal{M}(G, B)$. *Furthermore, if* $H = G$ *and* $P \in \mathcal{M}(G, B)$*, then one of the following occurs:*

(i) P *is contained in* $\mathrm{GO}_2^{+}(q) \times \mathrm{GO}_2^{-}(q)$*;*
(ii) P *is isomorphic to one of the two subgroups* $\mathrm{GO}_3^{\pm}(p^{a_2}) \times \mathrm{GO}_1^{\mp}(q)$ *if* $q \equiv 1 \pmod 8$*;*
(iii) P *is isomorphic to one of the two subgroups* $\mathrm{GO}_3^{\pm}(p) \times \mathrm{GO}_1^{\pm}(q)$ *if* $q \equiv 7 \pmod 8$*;*
(iv) P *is isomorphic to one of the two subgroups* $\mathrm{GO}_1(q)^{\pm} \wr \mathrm{Sym}(3) \times \mathrm{GO}_1(q)^{\mp}$ *if* $q \equiv 3, 5 \pmod 8$*; or*
(v) $q = 5^a$*,* a *odd,* $P \cong \mathrm{GO}_3^{\pm}(5) \times \mathrm{GO}_1^{\mp}(5)$.

Proof. The subgroups presented in parts (ii) to (iv) are 2-minimal. To prove the lemma, we show that there are no more by determining all the 2-minimal subgroups of of $\Omega_4^{-}(q) \cong \mathrm{PSL}_2(q^2)$ using Theorem 4.1 and noting that these are the intersection of the groups listed in (i) to (iv) with $\Omega_4^{-}(q)$ and are consequently normalized by B. To use Theorem 4.1 in a transparent way, we temporarily take $\pi = q^2 = p^{2a}$ and let $\alpha_2 = 2a_2$. Thus we see that $\alpha_2 > 1$ and $\pi \equiv 1 \pmod 8$. Thus we are immediately in case (iv) of Theorem 4.1. Hence we have the following possibilities for the 2-minimal subgroups of $\Omega_4^{-}(q)$:

(i) $\alpha_2 > 2$ or $\alpha_2 = 2$ and $\pi \equiv 1 \pmod{16}$,
$$\mathcal{M}(G, B) = \mathcal{M}(\mathrm{Dih}(\pi - 1), B) \cup \{\mathrm{PGL}_2(p^{\alpha_2/2})^{[2]}\};$$
(ii) $p = 5$, $\alpha_2 = 2$ and
$$\mathcal{M}(G, B) = \mathcal{M}(\mathrm{Dih}(\pi - 1), B) \cup \{\mathrm{PGL}_2(5)^{[2]}\} \cup \{\mathrm{Sym}(4)^{[2]}\};$$
(iii) $p = 3$, $\alpha_2 = 2$ and
$$\mathcal{M}(G, B) = \mathcal{M}(\mathrm{Dih}(\pi - 1), B) \cup \{\mathrm{PGL}_2(3)^{[2]}\};$$
(iv) $p > 5$, $\alpha_2 = 2$ and $\pi \equiv 9 \pmod{16}$,
$$\mathcal{M}(G, B) = \mathcal{M}(\mathrm{Dih}(\pi - 1), B) \cup \{\mathrm{Sym}(4)^{[2]}\}.$$

Now we note that $\mathrm{GO}_2^{+}(q) \times \mathrm{GO}_2^{-}(q)$ intersects H in a dihedral group $\mathrm{Dih}(\pi - 1)$, the groups $\mathrm{PGL}_2(p^{\alpha_2/2})$ are just the groups $\mathrm{SO}_3(p^{\alpha_2/2})$ and the two classes of subgroups isomorphic to $\mathrm{Sym}(4)$, which arise when $\pi \equiv 9 \pmod{16}$ or $p \in \{3,5\}$ with $\alpha_2 = 2$, are just $\mathrm{GO}_1^{\mp}(q) \times \mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(3)$ intersected with $\Omega_4^-(q)$. These considerations validate the lemma. □

Lemma 4.8. *Suppose that* $G = \mathrm{GO}_5^{\epsilon}(q)$ *and* $P \in \mathcal{M}(H, B_H)$. *Then* $P = \widehat{P} \cap H$ *where* $\widehat{P} \in \mathcal{M}(G, B)$. *In addition, if* $H = G$ *and* $P \in \mathcal{M}(G, B)$, *then one of the following holds:*

(i) $P \in \mathcal{M}(\mathrm{GO}_4^+(q) \times \mathrm{GO}_1^{\epsilon}(q), B)$;
(ii) $q \equiv 3, 5 \pmod 8$, $P \cong \mathrm{GO}_1^{\epsilon}(q) \wr \mathrm{Sym}(5)$; *or*
(iii) $q \equiv 1, 7 \pmod 8$, $P \cong \mathrm{GO}_5^{\epsilon}(p^{a_2})$.

In particular, $H \in \mathcal{M}(H, B_H)$ *if and only if* $q \equiv 1, 7 \pmod 8$ *and* $a = a_2$.

Proof. Since $\mathrm{GO}_5^{\epsilon}(q) \cong 2 \times \mathrm{PGSp}_4(q)$ and $\Omega_5^{\epsilon}(q) \cong \mathrm{PSp}_4(q)$, we may read the 2-minimal subgroups for $\Omega_5^{\epsilon}(q)$ from Lemma 3.1 of [14]. Given this list we then argue as in Lemma 4.7 that the statement of Lemma 4.8 is correct. □

We close this section with a lemma which applies when $n = 7$ and when $n = 8$.

Lemma 4.9. *Suppose that* $q \equiv 3, 5 \pmod 8$ *and that* $n \in \{7, 8\}$. *Assume that* $K \leqslant H$ *is one of the exceptional configurations given in Proposition 3.14 (iv) –(vii) (so* K *is isomorphic to one of* E_7, E_7', E_8 *or* E_8'*). If* $P \in \mathcal{M}(K, B \cap K)$, *then either* P *leaves a proper subspace of* V *invariant or* P *leaves a decomposition of* V *into an orthogonal sum of equal dimensional subspaces invariant.*

Proof. Suppose first that $n = 7$. Then $K \cong \mathrm{E}_7 \cong 2 \times \mathrm{Sp}_6(2)$ or $K \cong \mathrm{Sp}_6(2)$. Let W be the natural $\mathrm{Sp}_6(2)$ symplectic module. The 2-minimal subgroups of K' are the minimal parabolic subgroups of K' and are contained in the stabilizer of an isotropic 1-space of W or in the stabilizer of an isotropic 2-space of W. In particular, every 2-minimal subgroup of K containing B centralizes a non-central involution of K in the centre of $B \cap K'$. As the centralizer in V (the orthogonal module for G) of such an involution is a proper subspace of V, we have our conclusion in this case.

Suppose that $n = 8$ and, so, this time K is either E_8, which has shape $2{\cdot}\Omega_8^+(2).2$, or K has index 2 in E_8. The centralizer X of a 2-central involution in $\Omega_8^+(2)$ has shape $2_+^{1+8}{:}(\mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3))$ and this class of involutions lift to elements of order 2 in K [3]. Let F be the fours group such that F maps to the central subgroup of X, then, letting z be the central involution of G and $f \in F \backslash \langle z \rangle$, we have $V = [V, f] \perp [V, fz]$ is a non-trivial decomposition of V preserved by

X and hence by all the 2-minimal subgroups contained in X. The only member of $P \in \mathcal{M}(K', B \cap K')$ which does not normalize F is the one which corresponds to the middle node of the Dynkin diagram for D_4. This subgroup P^* is normalized by B_H and the product $P = B_H P^*$ is a 2-minimal subgroup of H. Viewing P as contained in E_8, we see P is contained in the subgroup which corresponds to the Weyl group of type D_8 which itself has shape $2^7 : \mathrm{Sym}(8)$. Now, in K the normal subgroup R of order 2^7 in $\mathrm{D}_8 \sim 2^7 : \mathrm{Sym}(8)$ remains normal in $\mathrm{D}_8 \cap K$. We may write $V = C_V(R) \perp \bigoplus_{|R:R_1|=2} C_V(R_1)$ and as $\mathrm{Alt}(8)$ has no subgroups of index less than 8, we get that $V = \bigoplus_{|R:R_1|=2} C_V(R_1)$ and each non-zero summand has dimension 1; furthermore, the sum is orthogonal. Hence $P \leqslant \mathrm{GO}_1^{\tau}(q) \wr \mathrm{Sym}(8)$ for some τ where τ is the type of reflections used to generate E_8. This completes the proof of the lemma. □

5. The 2-minimal subgroups of θ-decomposition subgroups

In this section we maintain the notation and assumptions presented in 3.15 and in addition we assume that $n \geqslant 6$. Our intention is to start the study of 2-minimal subgroups of the θ-subgroup in each of the different candidates for H. Of course, Lemma 3.13 tells us that $B_H = S_H$ and that $B = N_G(S) = S$. Since B is so ingrained as the normalizer of S in G, when we wish to emphasize this role we still use B in place of S.

Our *modus operandi* for pinning down the 2-minimal subgroups of orthogonal groups is to find a smaller collection of subgroups which corral the 2-minimal subgroups. To this end we let N be one of the θ-subgroups which appear in Lemma 3.8. Recall that

$$\Sigma = \{\{e_1, e_2\}, \{e_3, e_4\}, \ldots, \{e_{2m-1}, e_{2m}\}\}.$$

For $\beta_i = \{e_{2i-1}, e_{2i}\} \in \Sigma$, set $F_i = \mathrm{GO}(\langle e_{2i-1}, e_{2i}\rangle) = \mathrm{GO}_2(\langle \beta_i \rangle) \cong \mathrm{GO}_2^{\theta}(q)$ (understanding as usual that it acts trivially on $\langle e_{2i-1}, e_{2i}\rangle^{\perp}$). Considering the three possibilities for N portrayed in Lemma 3.8, we let $F^* = F_1 \times \cdots \times F_m$ be the base group of the wreath product subgroup of N. Define $F_{-1} = \mathrm{GO}(V_{-1})$ and $F = F^* \times F_{-1}$. This means

$$F = \begin{cases} F^* & V = U \\ F^* \times \mathrm{GO}_1^{\epsilon\theta^m}(q) & \operatorname{codim} U = 1 \\ F^* \times \mathrm{GO}_2^{-\theta}(q) & \operatorname{codim} U = 2 \end{cases}$$

and

$$N = \begin{cases} F_1 \wr \mathrm{Sym}(m) & V = U \\ F_1 \wr \mathrm{Sym}(m) \times \mathrm{GO}_1^{\epsilon\theta^m}(q) & \operatorname{codim} U = 1 \\ F_1 \wr \mathrm{Sym}(m) \times \mathrm{GO}_2^{-\theta}(q) & \operatorname{codim} U = 2. \end{cases}$$

We further define C^* to be the direct product of the cyclic 2-subgroups of maximal order in F_1 to F_m and put $C = C^* \times S_{-1}$. Obviously C is abelian and N normalizes C if $n = 2m$, or $n = 2m+1$.

Lemma 5.1. *We have $N = N_G(C^*) = N_G(\Omega_1(C^*)) \geqslant N_G(C)$ and C is weakly closed in S with respect to G.*

Proof. We write $C = C_1 \times \cdots \times C_m \times S_{-1}$ where, for $1 \leqslant i \leqslant m$, C_i is the maximal cyclic subgroup of F_i. Let c_i be the involution in C_i. We first show that $N_G(C^*) = N_G(\Omega_1(C^*)) = N$. Since $N \leqslant N_G(C^*) \leqslant N_G(\Omega_1(C^*))$, it suffices to show that $N = N_G(\Omega_1(C^*))$. Because $[V, c_i]$ is a 2-dimensional space of type θ, for $d \in S_{-1}$, we have $[V, d]$ has dimension at most 1 or is of type $-\theta$ and all the other elements of $\Omega_1(C)$ have commutator subspace of dimension at least 3, we infer that $N_G(\Omega_1(C))$ permutes the set $\{c_1, \dots, c_m\}$. Since N permutes this set as $\mathrm{Sym}(m)$, we get

$$N_G(\Omega_1(C^*)) = N\left(\bigcap_{i=1}^{m} C_G(c_i)\right) = NF = N.$$

Because $\Omega_1(C^*) = \Omega_1(C)$, we have $N_G(C) \leqslant N_G(C^*) = N$.

Now suppose that $C^g \leqslant S$. If $N \cong \mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m)$, then S is a wreath product of a dihedral group of order at least 8 with a Sylow 2-subgroup of $\mathrm{Sym}(m)$, and the lemma follows from Lemma 2.5. If $N = \mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m) \times \mathrm{GO}_2^{-\theta}(q)$ or $N = \mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m) \times \mathrm{GO}_1(q)$, then S is isomorphic to the direct product of a wreath product of a dihedral group of order at least 8 with a Sylow 2-subgroup of $\mathrm{Sym}(m)$ and an elementary abelian group S_{-1} of order 2 or 4. Applying Lemma 2.5 to S/S_{-1}, we see that $C^g S_{-1}/S_{-1}$ is contained in the base group of S/S_{-1} and then that $C^g \geqslant S_{-1}$ by comparing the order of C and C^g. But then $C^g = C$, as claimed. □

Lemma 5.2. *We have $(S_H \cap F)' = (S \cap F)'$ and*

$$\Omega_1(C^*) = \Omega_1((S_H \cap F)') = \Omega_1((S \cap F)')$$

is a characteristic subgroup of $S_H \cap F$.

Proof. Since $n \geqslant 6$, $m \geqslant 2$. For $1 \leqslant i \leqslant m$, recall that $D_i = S \cap F_i$ is a dihedral group of order $2(q-\theta)_2$ at least 8. We know $D_i = \langle x_{2i-1}, y_{2i-1} \rangle$. Now $S \cap F = D_1 \dots D_m S_{-1}$ and so

$$(S \cap F)' = D_1' \dots D_m' = \langle [x_{2i-1}, y_{2i-1}] \mid 1 \leqslant i \leqslant m \rangle$$

and $\Omega_1(C^*) = \Omega_1((S \cap F)')$.

To prove the lemma, it remains to show that $(S_H \cap F)' = (S \cap F)'$ and to do this it is sufficient to assume that $H = \Omega(V)$. We only need to show that $[x_{2i-1}, y_{2i-1}] \in (S_H \cap F)'$ for $1 \leqslant i \leqslant m$. Since $m \geqslant 2$, there exists $1 \leqslant j \leqslant m$ with $i \neq j$ and, in addition, as $n \geqslant 5$, there exists a reflection $t \in F$ such that t commutes with D_iD_j. We assume notation is chosen so that t is conjugate to x_{2i-1}. Then $y_{2i-1}y_{2j-1}$ and $x_{2i-1}t$ have determinant 1 and spinor norm a square. Therefore $y_{2i-1}y_{2j-1}, x_{2i-1}t \in H$ and so

$$[x_{2i-1}, y_{2i-1}] = [x_{2i-1}t, y_{2i-1}y_{2j-1}] \in (S_H \cap F)'.$$

It follows that $(S_H \cap F)' = (S \cap F)'$, as claimed. □

Recall that $X_m = \mathrm{Sym}(\Sigma) \cong \mathrm{Sym}(m)$ is the subgroup of N which faithfully permutes Σ and $N = FX_m$.

Lemma 5.3. *We have $N_G(S \cap F) = N_G(S_H \cap F) = (S \cap F)X_m$.*

Proof. Obviously, $N_G(S \cap F) \leqslant N_G(S_H \cap F)$. Since $\Omega_1(C^*)$ is a characteristic subgroup of $S \cap F_H$ by Lemma 5.2, Lemma 5.1 implies that $N_G(S_H \cap F) = N_N(S_H \cap F)$. Furthermore, $X_m \leqslant N_G(S_H \cap F)$. Since $N = FX_m$, we only need to show $N_F(S_H \cap F) = S \cap F$. This follows as F is a direct product of dihedral groups and $S_H \cap F$ does not centralize any elements of odd order in F. This proves the lemma. □

Finally we can establish a result on 2-minimal subgroups of N.

Lemma 5.4. *Suppose that $P \in \mathcal{M}(N \cap H, B_H)$. Then $PB \in \mathcal{M}(N, B)$.*

Proof. We must show that the reflections in B normalize P. Let $F_H = F \cap H$. As $P \leqslant N$, P normalizes F_H. Hence by Lemma 2.6, either $P \leqslant B_HF_H$ or $P \leqslant N_G(S \cap F_H)$. Suppose that the latter possibility holds. Then noting that $S \cap F_H = S_H \cap F$, Lemma 5.3 yields $N_G(S \cap F_H) = N_G(S \cap F)$. Consequently

$$[S \cap F, P] \leqslant S \cap F \cap H = S \cap F_H \leqslant P.$$

As $B = (S \cap F)B_H$, we are done in this case. So we now suppose that $P \leqslant B_HF_H$. Then $P = B_H(P \cap F_H)$ and, as F_H has a normal 2-complement, $P = B_HO_{2'}(P)$. Since B_H is normalized by B and $O_{2'}(P)$ is normalized by B_HC, we just need to show that $O_{2'}(P)$ is normalized by reflections in B.

Note that, as $n \geqslant 6$, x_1x_3 and $x_3x_5^* \in \Omega_n^\epsilon(q)$ where, if $n = 6$ and $2m = 4$, $x_5^* = x_{-1}$ and otherwise $x_5^* = x_5$. Hence these elements are in P. Now let $g \in O_{2'}(P)$. Then $[g, x_1x_3] \in O_{2'}(P)$ and

$$[g, x_1x_3] = [g, x_1]^{x_3}[g, x_3] = [g, x_1][g, x_3]$$

as $[g, x_1] \in F_1$, $x_3 \in F_2$ and $[F_1, F_2] = 1$. Therefore

$$([g, x_1][g, x_3])^{x_3x_5^*} = [g, x_1][g, x_3]^{-1} \in O_{2'}(P).$$

Because $O_{2'}(F)$ is abelian, it follows that

$$[g, x_1]^2 = [g, x_1][g, x_3][g, x_1][g, x_3]^{-1} \in O_{2'}(P)$$

which as $O_{2'}(F)$ has odd order means that $[g, x_1] \in O_{2'}(P)$. Thus $[O_{2'}(P), x_1] \leqslant O_{2'}(P)$ and a similar argument demonstrates that y_1 normalizes $O_{2'}(P)$. We conclude that B normalizes P and $PB \in \mathcal{M}(G, B)$. □

We now enumerate the 2-minimal subgroups in $\mathcal{M}(N, B)$. We first consider the case when $n = 2m$. In this case $N = F_1 \wr \mathrm{Sym}(\Sigma) = F_1 \wr \mathrm{Sym}(m)$. As normal 2-subgroups are contained in every 2-minimal subgroup, the 2-minimal subgroups of N are described in Theorem 2.4. Hence the 2-minimal subgroups of $\mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m)$ are

$$\begin{aligned} \mathcal{X} \;=\; & \{O_2(F)X(n_i + n_j) \mid i, j \in I, i < j\} \\ & \cup \{O_2(F)X(n_i; j) \mid i \in I, j \in \{2, \ldots, n_i - 1\}\} \\ & \cup \{U(n_i; f^c; n_i)S \mid i \in I, \text{and } f^c \in \Pi(q - \theta)\}, \end{aligned}$$

where these subgroups are as described in Definition 2.3 modulo $O_2(F)$.

Recall that

$$U(n_i; f^c; n_i)S \cong \mathrm{Dih}(2(q-\theta)_2 f^c) \wr T_{n_i - 1} \times \prod_{j=1, j\neq i}^{s} S_{m_j}.$$

Because of Lemma 3.8, we obtain a collection of 2-minimal subgroups of G contained in N as follows.

Definition 5.5.

$$\mathcal{N}(G, B) = \{Y \times S_{-1} \mid Y \in \mathcal{X}\}.$$

Lemma 3.8 also provides the following set of 2-minimal subgroups.

Definition 5.6. *When $n = 2m + 2$ with $\epsilon = -\theta^{m+1}$ define*

$$\mathcal{D}_2(G, B) = \{\prod_{k \in J} S_{m_k} \times \mathrm{Dih}(4 f^s)) \mid f^s \in \Pi(q + \theta)\}.$$

Obviously the subgroups in $\mathcal{D}_2(G, B)$ and $\mathcal{N}(G, B)$ are 2-minimal.

We record the following observation:

Lemma 5.7. *$\mathcal{M}(N, B) = \mathcal{N}(G, B) \cup \mathcal{D}_2(G, B)$.*

Proof. This comes from Lemma 2.7 and Theorem 2.4. □

Proposition 5.8. *Suppose that $n \geqslant 6$ and $P \in \mathcal{M}(N \cap H, B_H)$. Then $P = \widehat{P} \cap H$ where $\widehat{P} \in \mathcal{M}(N, B) = \mathcal{N}(G, B) \cup \mathcal{D}_2(G, B)$.*

Proof. For $P \in \mathcal{M}(N \cap H, B_H)$, we have shown in Lemma 5.4 that $PB \in \mathcal{M}(G, B)$. Thus there is a 2-minimal subgroup $\widehat{P}$ of G such that $PB = \widehat{P}$ and $\widehat{P} \cap H = PB_H = P$. Conversely, suppose that $\widehat{P} \in \mathcal{M}(G, B)$. Then $\widehat{P} \cap H$ is generated by 2-minimal subgroups in $\mathcal{M}(\widehat{P} \cap H, B_H)$. Since each of these is normalized by B, we obtain $\widehat{P} = BR$ for some $R \in \mathcal{M}(H, B_H)$. Using Lemma 5.7 finishes the explanation. □

6. The 2-minimal subgroups of the orthogonal groups

We continue with our standard assumptions as listed in 3.15.

Lemma 6.1. *Suppose that $n \geqslant 6$. Then $H \in \mathcal{M}(H, B_H)$ if and only if $n = 2^{m_1+1} + 1$, $q \equiv 1, 7 \pmod 8$ and $a = a_2$.*

Proof. Lemma 3.13 indicates that $B_H = S_H$. Assume that $H \in \mathcal{M}(H, B_H)$. We consider the various possibilities for the over-groups of B_H. Recall that $m = 2^{m_1} + \cdots + 2^{m_s}$ with $m_1 > \cdots > m_s \geqslant 0$. Assume that $s > 1$. Then $Y_H = (\Omega(V_{m_1}) \times \Omega(V_{m_1}^{\perp}))B_H$ is an over-group of B_H contained in H. Since Y_H is a maximal subgroup of H by [9] and, as $s > 1$, $N \cap H$ is not contained in Y_H, H is not 2-minimal, a contradiction. Hence we have $s = 1$ and so $2m = 2^{m_1+1} \geqslant 4$.

If $n = 2m = 2^{m_1+1}$, then $n \geqslant 8$ and $\epsilon = \theta^m = \theta^{2^{m_1}} = +$. Thus H contains a maximal subgroup isomorphic to $(\Omega_4^+(q) \wr \mathrm{Sym}(m/2))B_H$ and this subgroup together with $N \cap H$, where $N \cong \mathrm{GO}_2^{\theta}(q) \wr \mathrm{Sym}(m)$, generate H. Hence H is not 2-minimal in this case.

If $n = 2m+2$, then $\epsilon = -\theta^{m+1}$ and G contains subgroups $\mathrm{GO}_{2m+1}^+(q) \times \mathrm{GO}_1^-(q)$ and $\mathrm{GO}_{2m+1}^-(q) \times \mathrm{GO}_1^+(q)$ both containing S and which together generate G. Intersecting these subgroups with H, we have H is not 2-minimal in this case either.

So we are left with the case $n = 2m + 1 = 2^{m_1+1} + 1$. Note that as m is a power of 2, $\theta^m = +$ and correspondingly $\epsilon\theta^m = \epsilon$.

Let
$$M = \Omega(V_{m_1})B_H.$$
If $q \equiv 3, 5 \pmod 8$, then, in addition to M, G also contains $\mathrm{GO}_1^{\epsilon}(q) \wr \mathrm{Sym}(n) \cap H$ as an over-group of B_H by Lemma 3.11 and so H is not 2-minimal in this situation. Therefore we must have $q \equiv 1, 7 \pmod 8$. If $a_2 < a$, then $\Omega_n^{\epsilon}(p^{a_2})B_H$ also contains B_H and this subgroup together with M generates H which again means that H is not 2-minimal. Hence we have $a = a_2$ and this is our example. Conversely, by Proposition 3.14 these groups are 2-minimal.

□

Lemma 6.2. *Suppose that $n \geqslant 5$, $q \equiv 3, 5 \pmod 8$ and $P \in \mathcal{M}(H, B_H)$. Assume that P is contained in one of the subgroups listed in Lemma 3.11 (i) or (ii). Then one of the following holds.*

(i) *P leaves a proper subspace of V invariant; or*
(ii) *P leaves invariant a decomposition of V into an orthogonal sum of subspaces of dimension $2^k > 1$; or*
(iii) *$n = 2^{m_1+1} + 1$, and $P = \mathrm{GO}_1^{\epsilon}(q) \wr \mathrm{Sym}(n) \cap H$.*

Proof. This follows from Lemma 3.12 by using the description of 2-minimal subgroups of $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ given in Theorem 2.2. □

Inspired by Corollary 4.3 and Lemmas 4.8, 6.1 and 6.2, we produce the following sets of 2-minimal subgroups of G which depend on n and the congruence of q mod 8. We recall the 2-adic decomposition of m is $2^{m_1} + \cdots + 2^{m_s}$ and $J = \{1, \ldots, s\}$.

Definition 6.3. *Assume that $n \geqslant 6$.*

(i) *Suppose that n is odd.*
(a) *If $q \equiv 1, 7 \pmod 8$, then define*

$$\mathcal{O}_{1,7}(G, B) = \left\{ \mathrm{GO}^{\delta_j}_{2^{m_j+1}+1}(p^{a_2})B \mid \delta_j = \begin{cases} \epsilon\theta^m & m_j > 0 \\ \epsilon & m_j = 0 \end{cases}, j \in J \right\}.$$

(b) *If $q \equiv 3, 5 \pmod 8$, then*

$$\mathcal{O}_{3,5}(G, B) = \{(\mathrm{GO}_1^{\epsilon}(q) \wr \mathrm{Sym}(2^{m_j+1} + 1)B \mid j \in J\}.$$

(ii) *Suppose that $n = 2m + 2$ and $\epsilon = -\theta^{m+1}$.*
(a) *If $q \equiv 1, 7 \pmod 8$, then*

$$\mathcal{O}_{1,7}(G, B) = \{((\mathrm{GO}^{\tau}_{2^{m_j+1}+1}(p^{a_2}) \times \mathrm{GO}_1^{-\tau}(q))B \mid j \in J, \tau = \pm\}.$$

(b) *If $q \equiv 3, 5 \pmod 8$, then*

$$\mathcal{O}_{3,5}(G, B) = \{(\mathrm{GO}_1^{\tau}(q) \wr \mathrm{Sym}(2^{m_j+1} + 1) \times \mathrm{GO}_1^{-\tau}(q))B \mid j \in J, \tau = \pm\}.$$

Remark 6.4. (i) *Note that when n is odd, the sets $\mathcal{O}_{3,5}(G, B)$ and $\mathcal{O}_{1,7}(G, B)$ have sizes $|J| = s$ whereas when $n = 2m + 2$ they have size $2|J| = 2s$.*
(ii) *When $n = 2m + 2$, we could suppress mention of the subgroup $\mathrm{GO}_1^{-\tau}(q)$ as it is contained in B, however we think that its inclusion adds meaning to the description.*
(iii) *In Definition 6.3 (i)(a), δ_j does not depend on j when $m_j > 1$.*
(iv) *This definition is only for $n > 2m$.*

Suppose that $n = 2m = 2^k\ell$ with $k \geqslant 1$ and let N_k be the subgroup of G which preserves the decomposition

$$V = \langle e_1, \ldots, e_{2^k} \rangle \perp \cdots \perp \langle e_{2m-2^k+1}, \ldots, e_{2m} \rangle$$

of V into a perpendicular sum of ℓ subspaces of V of dimension 2^k each preserved by B. We have $N = N_1$ is the θ-subgroup studied in Section 5. For $k \geqslant 2$, $N_k = \mathrm{GO}^+_{2^k}(q) \wr \mathrm{Sym}(\ell)$. Let L_k be the base group of N_k and note that $L_1 = F$ as in Section 5.

Lemma 6.5. *Suppose that $n = 2^k\ell$ with $k \geqslant 1$ and F_k is the base group of N_k. If P acts irreducibly on V and $P \in \mathcal{M}(N_k \cap H, B_H)$, then either*

(i) *$\ell = 2^b$ and $P \in \mathcal{M}((L_k \cap H)B_H, B_H)$; or*

(ii) *$P \in \mathcal{M}(N \cap H, B_H)$.*

Proof. By Lemma 2.6, either $P \in \mathcal{M}((L_k \cap H)B_H, B_H)$ or $P \leqslant N_G(S_H \cap L_k)$. If $P \in \mathcal{M}(L_kB, B)$, then as P acts irreducibly on V, we must have $\ell = 2^b$ and so (i) holds. On the other hand, if $P \leqslant N_G(S_H \cap L_k)$, then, as $C \leqslant S \cap F \leqslant S \cap L_k$ and C is weakly closed in S with respect to G by Lemma 5.1, we obtain $P \leqslant N_G(C) = N$ which yields possibility (ii). □

Set

$$\mathcal{X}_4 = \begin{cases} \{\mathrm{GO}^+_4(p^{a_2})\} & q \equiv 1,7 \pmod 8 \\ \{\mathrm{GO}^{\pm}_1(q) \wr \mathrm{Sym}(4)\} & q \equiv 3,5 \pmod 8, q \neq 5^a, a \text{ odd} \\ \{\mathrm{GO}^{\pm}_1(q) \wr \mathrm{Sym}(4), \mathrm{GO}^+_4(5)\} & q = 5^a, a \text{ odd} \end{cases}.$$

By Corollary 4.6, the members of $\mathcal{X}_4$ are 2-minimal and are the candidates for 2-minimal subgroups of $\mathrm{GO}^+_4(q)$ which are not contained in $\mathrm{GO}^{\theta}_2(q) \wr T_1$. We define two further collections of 2-minimal subgroups.

Definition 6.6. (i)

$$\mathcal{G}^+_4(G, B) = \{X \wr T_{m_i-1} \times \prod_{j \in J\setminus\{i\}} S_{m_j} \mid 1 \leqslant i \leqslant s, m_i \geqslant 1 \text{ and } X \in \mathcal{X}_4\}.$$

(ii) *When m is odd, $q = 5^a$ with a odd and $n \neq 2m$*

$$\mathcal{G}_3(G, B) = \begin{cases} \{\mathrm{GO}^{\delta}_3(5) \times \prod_{j=1}^{s-1} S_{m_j}\} & n = 2m+1 \\ \{\mathrm{GO}^{\tau}_3(5) \times \mathrm{GO}^{-\tau}_1(5) \times \prod_{j=1}^{s-1} S_{m_j} \mid \tau \in \pm\} & n = 2m+2 \end{cases}.$$

7. A proof of Theorem A

At last we prove the main theorem of this paper. For convenience we repeat its statement here.

Theorem A. *Suppose that $n \geqslant 5$ and $G = \mathrm{GO}^{\epsilon}_n(q)$ where $q = p^a$ is odd. Let $S \in \mathrm{Syl}_2(G)$ and $B = N_G(S)$. Assume that H is a subgroup of G which contains $\Omega^{\epsilon}_n(q)$. If $P \in \mathcal{M}(H, N_H(S \cap H))$, then there exists*

$$\widehat{P} \in \mathcal{O}_{1,7}(G,B) \cup \mathcal{O}_{3,5}(G,B) \cup \mathcal{N}(G,B) \cup \mathcal{D}_2(G,B) \cup \mathcal{G}^+_4(G,B) \cup \mathcal{G}_3(G,B)$$

such that $P = \widehat{P} \cap H$ or $n \in \{7,8,9\}$, $q \equiv 3,5 \pmod 8$ and either

(i) $H = \Omega_9^\epsilon(q)$ *or* ${}^\epsilon\mathrm{GO}_9^\epsilon(q) \cong 2 \times \Omega_9^\epsilon(q)$ *and* $\langle P, B\rangle = \mathrm{GO}_1^\pm(q) \wr \mathrm{Sym}(4) \wr 2 \times S_{-1} \in \mathcal{G}_4^+(G, B)$ *and*

$$\mathcal{M}(\langle P, B\rangle \cap H, N_H(S \cap H)) = \{P, P^x\}$$

where $x \in B$ *is a reflection of type* $-\epsilon$*.*

(ii) $n = 8$*,* P *acts irreducibly on* V *and either*

(a) $H = \Omega_8^+(q)$ *and* $\langle P, B\rangle = \mathrm{GO}_1^\pm(q) \wr \mathrm{Sym}(4) \wr 2 \in \mathcal{G}_4^+(G, B)$ *and*

$$\mathcal{M}(\langle P, B\rangle \cap H, N_H(S \cap H)) = \{P, P^x\}$$

where $x \in B$ *is a reflection.*

(b) $H = {}^\tau\mathrm{GO}_8^+(q)$*,* $\tau = \pm$ *and* $\langle P, B\rangle = \mathrm{GO}_1^\tau(q) \wr \mathrm{Sym}(4) \wr 2 \in \mathcal{G}_4^+(G, B)$ *and*

$$\mathcal{M}(\langle P, B\rangle \cap H, N_H(S \cap H)) = \{P, P^x\}$$

where $x \in B$ *is a reflection of type* $-\tau$*.*

(iii) $H \cong \Omega_7^\epsilon(q)$ *or* ${}^\epsilon\mathrm{GO}_7^\epsilon(q) \cong 2 \times \Omega_7^\epsilon(q)$*,* $\langle P, B\rangle = \langle \widehat{P}_1, \widehat{P}_2\rangle$ *where* $\widehat{P}_1 = \mathrm{GO}_1^{\theta^3\epsilon}(q) \wr \mathrm{Sym}(4) \times S_0 \times S_{-1} \in \mathcal{G}_4^+(G, B)$ *and* $\widehat{P}_2 = S_1 \times \mathrm{GO}_1^{\theta^3\epsilon}(q) \wr \mathrm{Sym}(3) \in \mathcal{O}_{3,5}(G, B)$*. Furthermore,*

$$\mathcal{M}(\langle P, B\rangle \cap H, N_H(S \cap H)) = \{\widehat{P}_1 \cap H, \widehat{P}_2 \cap H, P, P^x\}$$

where x *is a reflection in* B *of type* $-\epsilon$*.*

Proof. Let $\mathcal{W}(G, B)$ be the set of 2-minimal subgroups

$$\mathcal{O}_{1,7}(G, B) \cup \mathcal{O}_{3,5}(G, B) \cup \mathcal{N}(G, B) \cup \mathcal{D}_2(G, B) \cup \mathcal{G}_4^+(G, B) \cup \mathcal{G}_3(G, B).$$

For inductive reasons, we observe that, if $n < 6$, then $\mathcal{W}(G, B)$ in fact contains all the 2-minimal subgroups of $\mathrm{GO}_n^\epsilon(q)$ by Corollaries 4.3 and 4.6 and Lemmas 4.7 and 4.8 (we don't consider the proper subgroups H in these cases).

Assume that $n \geqslant 6$ and that the theorem is false. Accordingly, let $P \in \mathcal{M}(H, B_H)$ be a counter example chosen with n is as small as possible. In particular, P is not equal to $\widehat{P} \cap H$ for any $\widehat{P} \in \mathcal{W}(G, B)$. By Lemma 6.1, we may assume that $P < H$ as otherwise $P = \widehat{P} \cap H$ with $\widehat{P} \in \mathcal{O}_{1,7}(G, B) \subset \mathcal{W}(G, B)$.

We analyze the action of P on V. Suppose P is in a maximal subgroup which preserves a proper subspace of V. Then P acts reducibly on V and there exists a P-invariant proper subspace W of V which we select to have maximal dimension. Since W is B_H-invariant, it is B-invariant and non-degenerate by Lemma 3.10. Hence $V = W \perp W^\perp$ is a $\langle P, B\rangle$-invariant decomposition. The maximal choice of W implies that $\dim W \geqslant \dim W^\perp$. Set $\widehat{M_1} = \mathrm{GO}(W)$, $\widehat{M_2} = \mathrm{GO}(W^\perp)$, $M_1 = \Omega(W)$

and $M_2 = \Omega(W^\perp)$. Suppose that PB is a group. Then $PB \in \mathcal{M}(G, B)$ and $PB \leqslant \widehat{M_1} \times \widehat{M_2}$. Thus Lemma 2.7 implies that

$$PB \in \mathcal{M}(\widehat{M_1} \times (B \cap \widehat{M_2}), B) \cup \mathcal{M}((B \cap \widehat{M_1}) \times \widehat{M_2}, B).$$

Hence $PB = RB$ where, by induction,

$$\begin{aligned} R &\in \mathcal{M}(\widehat{M_1}, B \cap \widehat{M_1}) \cup \mathcal{M}(\widehat{M_2}, B \cap \widehat{M_2}) \\ &= \mathcal{W}(\widehat{M_1}, B \cap \widehat{M_1}) \cup \mathcal{W}(\widehat{M_2}, B \cap \widehat{M_2}). \end{aligned}$$

By construction, this means that $PB = RB \in \mathcal{W}(G, B)$, a contradiction. In particular, we have shown that P is not normalized by B and so we must have $P \leqslant H < G$.

Suppose that

$$P \in \mathcal{M}(M_1 B_H, B_H) \cup \mathcal{M}(M_2 B_H, B_H)$$

and $\dim W \geqslant \dim W^\perp \geqslant 2$. Assume that $P \in \mathcal{M}(M_i B_H, B_H)$ for some $i \in \{1, 2\}$. Lemma 3.1 implies $(B \cap \widehat{M}_{3-i})H = G$. Since $P \cap M_i$ is centralized by $\widehat{M}_{3-i}$, we have PB is a group, a contradiction.

We have proved

(7.0.1) If $\dim W^\perp \geqslant 2$, then

$$P \notin \mathcal{M}(M_1 B_H, B_H) \cup \mathcal{M}(M_2 B_H, B_H).$$

Assume that either $N_{M_1}(S \cap M_1) = B \cap M_1$ or $N_{M_2}(B \cap M_2) = B \cap M_2$. Then Lemma 2.7 combined with (7.0.1) yields $\dim W^\perp = 1$. Since $\dim W^\perp = 1$, B leaves a 1-space invariant and so $V_{-1} \neq 0$ and $n = 2m + 1$ or $2m + 2$. Since P leaves $W^\perp$ invariant, $\widehat{M_2}$ is centralized by P. Plainly we also have $Z(G) \leqslant N_G(P)$. If $n = 2m + 2$, then Lemma 3.5 (iv) implies that $\langle B_H, \widehat{M_2}, Z(G) \rangle = B$ and so PB is a group, a contradiction. Hence $n = 2m+1$ is odd and, as $\widehat{M_2}$ centralizes P, $P\widehat{M_2} \in \mathcal{M}(H\widehat{M_2}, B_H\widehat{M_2})$ and so we may assume that $\widehat{M_2} \leqslant P \leqslant H$. Therefore $\widehat{M_1} \times \widehat{M_2} \cap H = (\widehat{M_1} \cap H) \times \widehat{M_2}$. In particular, $P \cap \widehat{M_1} \in \mathcal{M}(\widehat{M_1} \cap H, B_H \cap \widehat{M_1})$. If $n > 9$, then, by the minimality of n, $P = \widehat{P} \cap H$ for some $\widehat{P} \in \mathcal{W}(\widehat{M_1}, B \cap \widehat{M_1})$ and so PB is a group, a contradiction. Hence $n \in \{7, 9\}$. Suppose that $n = 9$. Since P is not B-invariant and $\widehat{M_1} \cong \mathrm{GO}_8^+(q)$, $P \cap \widehat{M_1}$ is described in part (ii) of the theorem. Thus P is described in (i), which is a contradiction. Assume that $n = 7$. Then M_1 has all its 2-minimal subgroups normalized by B by induction, and so this is not the case. We have now completed the analysis when either $N_{M_1}(B \cap M_1) = B \cap M_1$ or $N_{M_2}(B \cap M_2) = B \cap M_2$.

Assume that $N_{M_1}(B \cap M_1) > B \cap M_1$ and $N_{M_2}(B \cap M_2) > B \cap M_2$. Then $5 \geqslant \dim W \geqslant \dim W^\perp \geqslant 3$ and $q \equiv 3, 5 \pmod 8$ by Lemma 3.13. If $n \in \{9, 10\}$, then $2m \geqslant 8$ and so $m_1 = 2$ and B leaves V_{m_1} invariant. As $\dim V_{m_1} = 2^{m_1+1} = 8$, these cases are impossible.

If $n = 8$, then as P is not irreducible neither is B_H. Hence we get that $n = 2m + 2$, $m = 3$ with $\epsilon = -\theta^{m+1} = -$. Thus

$$V = V_1 + V_0 + V_{-1} = V_1 + V_0 + \langle e_{-1} \rangle + \langle f_{-1} \rangle$$

is a B-invariant decomposition of V. Thus, as $4 \leqslant \dim W \leqslant 5$,

$$W \in \{V_1 + \langle e_{-1} \rangle, V_1 + \langle f_{-1} \rangle, V_1, V_0 + V_{-1}\}$$

and correspondingly

$$W^\perp \in \{V_0 + \langle f_{-1} \rangle, V_0 + \langle e_{-1} \rangle, V_0 + V_{-1}, V_1\}.$$

Since, by Lemma 3.13 we cannot have a factor which is 4-dimensional of $-$-type, we must have one of the first two possibilities. In particular, $\dim W = 5$ and $\dim W^\perp = 3$ and P is normalized by $\langle Z(G), Z(\widehat{M_1}) \rangle$. Using Lemma 3.5, we have $B = \langle B_H, Z(G), Z(\widehat{M_1}) \rangle$ and so P is normalized by B, a contradiction.

Suppose that $n = 7$ and to simplify notation assume that $V = V_1 + V_0 + \langle e_{-1} \rangle$ and $G = \mathrm{GO}_7^\epsilon(q)$ with $\epsilon = \theta$. Then $W = V_1$ or $V_1 + \langle e_{-1} \rangle$ and so P is contained in a subgroup isomorphic to $\mathrm{GO}_4^+(q) \times \mathrm{GO}_3^\theta(q)$ or $\mathrm{GO}_5^+(q) \times \mathrm{GO}_2^\theta(q)$. In the latter case, Lemma 3.6 (i) implies that $\widehat{M_1} \times \widehat{M_2} \cap H$ contains a subgroup isomorphic to $\Omega_5^+(q) \times {}^{-\theta}\mathrm{GO}_2^\theta(q)$ where ${}^{-\theta}\mathrm{GO}_2^\theta(q) \cong \mathrm{Dih}(q - \theta)$ and the second factor has a self-normalizing Sylow 2-subgroup. Hence we can apply Lemma 2.7 to see that

$$P \in \mathcal{M}(M_1 B_H, B_H) \cup \mathcal{M}(M_2 B_H, B_H),$$

contrary to (7.0.1).

Assume then that P is contained in $\mathrm{GO}_4^+(q) \times \mathrm{GO}_3^+(q)$. Then, as $P \notin \mathcal{M}(M_1 B_H, B_H) \cup \mathcal{M}(M_2 B_H, B_H)$ by (7.0.1), Lemma 2.8 implies that $P \leqslant N_H(S_H \cap M_1 M_2)$.

We may suppose that $Z(G) \leqslant H$ and so, as $H < G$, we have $H = {}^+\mathrm{GO}_{2m+1}^\epsilon(q)$ and in addition we know $q \equiv 3, 5 \pmod 8$. We also have

$$N_G(S_H \cap M_1 M_2) = \mathrm{F}_4 \times \mathrm{GO}_1^\theta(q) \wr \mathrm{Sym}(3) = \mathrm{F}_4 \times \mathrm{B}_3.$$

Hence

$$N_G(S_H \cap M_1 M_2)/(S_H \cap M_1 M_2) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3).$$

By Lemma 4.5 (ii) (a), $\mathcal{M}(\mathrm{F}_4, B \cap \mathrm{F}_4)$ has two 2-minimal subgroups while $\mathrm{GO}_1^\theta(q) \wr \mathrm{Sym}(3)$ is itself 2-minimal. Hence $N_H(S_H \cap M_1 M_2)$ has three 2-minimal subgroups which arise as $\widehat{P} \cap H$. Let $D \in \mathrm{Syl}_3(N_G(S_H \cap$

M_1M_2)) and set $\overline{N_G(S_H \cap M_1M_2)} = N_G(S_H \cap M_1M_2)/(S_H \cap M_1M_2)$. Then D is elementary abelian of order 27, $N_H(S \cap M_1M_2) = DS_H$ and $\overline{S_H}$ acts on $\overline{D}$ as an elementary abelian group of order 4. Since $S_H = B_H$, $\overline{S_H}$ does not centralize any non-trivial element of $\overline{D}$ and $\overline{S_H}$ acts on D (after some choice of generators for D) as either $\langle \mathrm{diag}(-1,-1,1), \mathrm{diag}(1,-1,-1)\rangle$ or $\langle \mathrm{diag}(-1,-1,-1), \mathrm{diag}(-1,-1,1)\rangle$. Correspondingly there are either three or five subgroups which are 2-minimal subgroups of $N_H(S_H \cap M_1M_2)$. By Lemma 3.12 (ii),

$$\mathrm{GO}_1^+(q) \wr \mathrm{Sym}(7) \cap H = \mathrm{GO}_1^+(q) \wr \mathrm{Alt}(7)$$

and this group contains $Y = 2^7.(\mathrm{Alt}(4) \times \mathrm{Alt}(3)) : 2$ containing S_H. Visibly Y preserves the decomposition of V into a 4-space and a 3-space and so $Y \leqslant M_1M_2 \cap H$. Since $Y \cap M_1M_2$ has shape $2^7.(\mathrm{Alt}(4) \times \mathrm{Alt}(3))$, Y normalizes $(S_H \cap M_1)(S_H \cap M_2)$. It is easy to calculate that Y has four 2-minimal subgroups only one of which is contained in M_1. Hence $N_H(S_H \cap M_1M_2)$ contains exactly five 2-minimal subgroups only three of which are normalized by B. Thus two are not normalized by B and so there exists a reflection $x \in B$ such that the 2-minimal subgroups in $\mathcal{M}(N_H(S \cap M_1M_2), B_H)$ are

$$\{P, P^x, \widehat{P} \cap H \mid \widehat{P} \in \mathcal{M}(N_G(S \cap M_1M_2), B) \subseteq \mathcal{W}(G, B)\}.$$

In particular, (iii) holds, contrary to P being a counter example.

Suppose that $n = 6$. First assume that $\epsilon = \theta^3$ and $6 = 2m$. The only possibility is that P is contained in $\mathrm{GO}_4^+(q) \times \mathrm{GO}_2^\theta(q)$. By Lemma 3.6 (iv) applied with W_1 of type θ, this group contains a normal subgroup contained in $\Omega_6^\epsilon(q)$ which is isomorphic to $\mathrm{SO}_4^+(q) \times \Omega_2^\theta(q)$. Lemma 3.13 allows us to apply Lemma 2.7 to this subgroup and we obtain P is normalized by B, a contradiction.

If $n = 6$ and $\epsilon = -\theta^3$, then $V = V_1 + V_{-1}$ and Lemma 3.6 (ii) applied with $W_1 = V_{-1}$ of type $-\theta$ yields $\widehat{M_1} \times \widehat{M_2} \cap \Omega(V)$ contains a subgroup isomorphic to $\Omega(V_{-1}) \times \mathrm{SO}(V_1)$, and again Lemma 2.7 applies. This yields a contradiction and completes the discussion of the case when P acts reducibly on V.

Suppose that P acts irreducibly on V and that P preserves a decomposition of V into subspaces of dimension $2^k \geqslant 2$ with k chosen maximally (See Proposition 3.14 (iii)). Then $P \leqslant N_k = \mathrm{GO}_{2^k}^{\epsilon_1}(q) \wr \mathrm{Sym}(n/2^k)$. We shall show that $P = \widehat{P} \cap H$ where $\widehat{P} \in \mathcal{N}(G, B) \cup \mathcal{G}_4^+(G, B)$, which is a contradiction.

By Proposition 5.8 we may also suppose that $P \nleqslant N_1$. Hence $k \geqslant 2$. Lemma 6.5 implies that $P \in \mathcal{M}((L_k \cap H)B_H, B_H)$ and $n = 2^{k+b}$.

The maximal choice of k now gives $2^k = n/2$. Furthermore, as $n > 4$, $V = W_1 \perp W_2$ with $\dim W_i = 2^k$ and W_i have $+$-type (see Lemma 3.9).

Let $\widehat{K_1} = \mathrm{GO}(W_1)$, $K_1 = \Omega(W_1)$, $\widehat{K_2} = \mathrm{GO}(W_2)$, $K_2 = \Omega(W_2)$ and $B_0 = N_B(K_1)$. If $n \geqslant 16$ or $n = 8$ and $q \equiv 1, 7 \pmod 8$, we have $N_{K_1}(S \cap K_1) = S \cap K_1$. By Lemma 2.11, B_0 is contained in a unique maximal subgroup of $(P \cap K_1)B_0$ and of $(P \cap K_2)B_0$. Since $K_1K_2B_0/B_0 \cong \mathrm{GO}^+_{2^k}(q)$, we now have that $P \cap K_1$ is a 2-minimal subgroup of K_1 which is normalized by $B \cap \mathrm{GO}(W_1)$. It follows that $P \cap K_1 = \widehat{P}_1 \cap K_1$ where $\widehat{P}_1 \in \mathcal{N}(\widehat{K_1}, \widehat{K_1} \cap B) \cup \mathcal{G}^+_4(\widehat{K_1}, \widehat{K_1} \cap B)$ and then by definition $\widehat{P} = \widehat{P}_1 \wr T_1 \in \mathcal{W}(G, B)$ and $P = \widehat{P} \cap H$, a contradiction.

Thus we have $n = 8$, $K_1 \cong \Omega^+_4(q)$ and $q \equiv 3, 5 \pmod 8$. Applying Lemma 2.11 we obtain $(P \cap K_1)N_B(K_1) \in \mathcal{M}(K_1N_B(K_1), N_B(K_1))$ or $P \leqslant N_G((K_1 \cap S)(K_2 \cap S))$. In the first case, we argue as above that $P = H \cap \widehat{P}$ with $\widehat{P} \in \mathcal{W}(G, B)$, which is a contradiction. Suppose that $P \leqslant N_G((K_1 \cap S)(K_2 \cap S)) = \mathrm{F}_4 \wr T_1$. This group has order $2^{14}.3^4$ and contains exactly two 2-minimal subgroups $(\mathrm{GO}^{\pm}_1(q) \wr \mathrm{Sym}(4)) \wr T_1) \in \mathcal{G}^+_4(G, B)$. Denote these 2-minimal subgroups by $\widehat{P}_+$ and $\widehat{P}_-$. In particular, if $H = G$, we have a contradiction.

Suppose that $H = \Omega^+_8(q)$. Let $L_+ = \mathrm{GO}^+_1(q) \wr \mathrm{Sym}(8)$ and $L_- = \mathrm{GO}^-_1(q) \wr \mathrm{Sym}(8)$. Then $P_+ \leqslant L_+$ and $P_- \in L_-$. By Lemma 3.12, $L_\pm \cap H \cong 2^7.\mathrm{Alt}(8)$. Since $\mathrm{Alt}(8) \cong \mathrm{SL}_4(2)$ has three parabolic subgroups, we can see that each group $P_+ \cap H$ and $P_- \cap H$ splits into two 2-minimal subgroups (and these are listed in (ii)). Thus $N_H(K_1K_2 \cap S)$ has at least four 2-minimal subgroups. We need to show that there are no more. Set $\overline{N_H(K_1K_2 \cap S)} = N_H(K_1K_2 \cap S)/(K_1K_2 \cap S)$. We may write

$$\overline{N_H(K_1K_2 \cap S)} = \overline{DS_H}$$

with $\overline{D}$ elementary abelian of order 3^4 and $\overline{S_H}$ elementary abelian of order 2^3. We write $\overline{S_H} = \langle h, g_1, g_2\rangle$ with notation chosen so that g_1 and g_2 normalizes $\overline{D} \cap \overline{K_1}$ and $\overline{D} \cap \overline{K_2}$. Then we further define $\delta_1, \delta_2, \delta_3$ and δ_4 such that $\overline{D} = \langle \delta_1, \delta_2, \delta_3, \delta_4\rangle$ and so that, for $i = 1, 2$, g_i centralizes δ_i and δ_{2+i} and inverts δ_{3-i} and δ_{5-i}. We may also assume that h conjugates δ_1 to δ_3 and δ_2 to δ_4. Then we check that the following subgroups are the only 2-minimal subgroups $\langle \overline{S_H}, \delta_1\delta_3\rangle$, $\langle \overline{S_H}, \delta_2\delta_4\rangle$, $\langle \overline{S_H}, \delta_1\delta_3^{-1}\rangle$ and $\langle \overline{S_H}, \delta_2\delta_4^{-1}\rangle$. It follows that each of these groups is already accounted for in the subgroups $P_+ \cap H$ and $P_- \cap H$. We conclude that $H \neq \Omega^+_8(q)$. The contradiction for $H = {}^{\pm}\mathrm{GO}^+_8(q)$ can be calculated from the information presented using Lemma 3.12 to see when $P_\pm \cap H$ remains 2-minimal.

Suppose now that $P \leqslant \mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(n)$ and that P is both irreducible on V and does not preserve any blocks of imprimitivity of dimension greater than 1. Then Lemma 6.2 implies that $n = 2^{m_s+1}+1$ and $P = \widehat{P} \cap H$ where $\widehat{P} \in \mathcal{O}_{3,5}(G,B)$, which is impossible as P is a counter example.

Thus P is both primitive and irreducible. By Proposition 3.14 and Lemma 4.9, P is the normalizer of a subfield subgroup of G. Lemma 6.1 implies that $n = 2^{m_s+1}+1$, $q \equiv 1,7 \pmod 8$ and $PB = \mathrm{GO}_n^{\epsilon}(p^{a_2}) \in \mathcal{O}_{1,7}(G,B) \subseteq \mathcal{W}(G,B)$, which is a contradiction. This completes the proof of the theorem. □

Proof of Corollary 1.1. We have $n = 2k$ and $q^k \equiv \epsilon \pmod 4$. Hence $\theta^k = \epsilon$ and so $k = m$ and $n = 2m$. It follows that $\mathcal{O}_{3,5}(G,B) = \mathcal{O}_{1,7}(G,B) = \mathcal{D}_2(G,B) = \mathcal{G}_3(G,B) = \varnothing$. Now Theorem A implies $\mathcal{M}(G,B) = \mathcal{N}(G,B) \cup \mathcal{G}_4^+(G,B)$. This proves the corollary. □

8. Examples

Example 8.1. *Suppose that* $G = \mathrm{GO}_{10}^-(q)$.

$q \equiv 1 \pmod 4$ *We have* $\theta = +$ *and so* $\theta^{n/2} = +$. *Therefore* $n = 2m+2$. *The* 2*-minimal subgroups are as follows:*

- $\mathcal{N}(G,B)$: $(\mathrm{Dih}(2(q-1)_2) \wr \mathrm{Sym}(4) \times S_{-1}$;
- $\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q-1)_2 f^c) \wr T_2 \times S_{-1}$ *where* $f^c \in \Pi(q-1)$;
- $\mathcal{D}_2(G,B)$: $\mathrm{Dih}(2(q-1)_2) \wr T_2 \times \mathrm{Dih}(2^2 f^c)$ *where* $f^c \in \Pi(q+1)$;
- $\mathcal{O}_{1,7}(G,B)$: $\mathrm{GO}_9^{\pm}(p^{a_2}) \times \mathrm{GO}_1^{\mp}(q)$, *when* $q \equiv 1 \pmod 8$; *and*
- $\mathcal{O}_{3,5}(G,B)$: $\mathrm{GO}_1^{\pm}(q) \wr \mathrm{Sym}(9) \times \mathrm{GO}_1^{\mp}(q) \sim 2^9.\mathrm{Sym}(9) \times 2$ *when* $q \equiv 5 \pmod 8$.
- $\mathcal{G}_4^+(G,B)$: $X \wr T_1 \times S_{-1}$, $X \in \mathcal{X}_4$;

$q \equiv 3 \pmod 4$ *We have* $\theta = -$ *and so* $\theta^{n/2} = -$. *Therefore* $n = 2m$ *and Corollary 1.1 is in play. The* 2*-minimal subgroups are as follows:*

- $\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q+1)_2) \wr \mathrm{Sym}(5)$;
- $\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q+1)_2) \wr \mathrm{Sym}(4) \times \mathrm{Dih}(2(q+1)_2))$;
- $\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q+1)_2 f^c) \wr T_2 \times \mathrm{Dih}(2(q+1)_2)$ *with* $f^c \in \Pi(q+1)$;
- $\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q+1)_2) \wr T_2 \times \mathrm{Dih}(2(q+1)_2 f^c)$ *with* $f^c \in \Pi(q+1)$;
- $\mathcal{G}_4^+(G,B)$: $X \wr T_1 \times \mathrm{Dih}(2(q+1)_2)$, $X \in \mathcal{X}_4$.

In all cases, $\mathcal{M}(H,B_H) = \{\widehat{P} \cap H \mid \widehat{P} \in \mathcal{M}(G,B)\}$.

To depict the similarity between the 2-minimal subgroups in the groups of +-type and the groups of −-type we also have the following example (which illustrates the expected "Ennola"-type duality).

Example 8.2. *Suppose that* $G = \mathrm{GO}_{10}^+(q)$ *and* $q \equiv 1 \pmod 4$. *Then* $\theta = +$ *and* $n = 2m$. *In this case Corollary 1.1 yields that the* 2*-minimal subgroups are as follows:*

$\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q-1)_2) \wr \mathrm{Sym}(5)$;
$\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q-1)_2) \wr \mathrm{Sym}(4) \times \mathrm{Dih}(2(q-1)_2)$;
$\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q-1)_2t^c) \wr T_2 \times \mathrm{Dih}(2(q-1)_2)$ *with* $t^c \in \Pi(q-1)$;
$\mathcal{N}(G,B)$: $\mathrm{Dih}(2(q-1)_2) \wr T_2 \times \mathrm{Dih}(2(q-1)_2t^c)$ *with* $t^c \in \Pi(q-1)$;
$\mathcal{G}_4^+(G,B)$: $X \wr T_1 \times \mathrm{Dih}(2(q-1)_2)$, $X \in \mathcal{X}_4$.

A similar phenomenon emerges when $q \equiv 3 \pmod 4$.

We close with an odd dimensional example which illustrates the unusual behaviour at the prime 5 as well as exhibiting an example with exceptional 2-minimal subgroups.

Example 8.3. *Suppose that* $H = \Omega_7^+(5)$. *Then* $\theta = +$, $m = 3 = 2+1$, $S = S_1 \times S_0 \times S_{-1}$ *and*

$\mathcal{O}_{3,5}(H,B_H)$: $P_1 = (\mathrm{GO}_1^+(5) \wr \mathrm{Sym}(5) \times S_0) \cap H$;
$\mathcal{O}_{3,5}(H,B_H)$: $P_2 = (\mathrm{GO}_1^+(5) \wr \mathrm{Sym}(3) \times S_1) \cap H$;
$\mathcal{N}(H,B_H)$: $P_3 = (\mathrm{Dih}(8) \wr \mathrm{Sym}(3) \times S_{-1}) \cap H$;
$\mathcal{G}_4^+(H,B_H)$: $P_4 = (\mathrm{GO}_1^+(5) \wr \mathrm{Sym}(4) \times S_0 \times S_{-1}) \cap H$;
$\mathcal{G}_4^+(H,B_H)$: $P_5 = (\mathrm{GO}_1^-(5) \wr \mathrm{Sym}(4) \times S_0 \times S_{-1}) \cap H$;
$\mathcal{G}_4^+(H,B_H)$: $P_6 = (\mathrm{GO}_4^+(5) \times S_0 \times S_{-1}) \cap H$;
$\mathcal{G}_3(H,B_H)$: $P_7 = (\mathrm{GO}_3^+(5) \times S_1 \times S_{-1}) \cap H$;
Exceptions: P, P^x *contained in* $(\mathrm{GO}_1^+(5) \wr \mathrm{Sym}(4) \times \mathrm{GO}_1^+(5) \wr \mathrm{Sym}(3)) \cap H \leqslant \mathrm{GO}_1^+(q) \wr \mathrm{Sym}(7) \cap H$ *which has shape* $2^6.\mathrm{Alt}(7)$.

We have $\langle P_2, P_5\rangle$ *has shape* $2^{1+4}.\mathrm{Sym}(6)$ *and* $\langle P_2, P_4\rangle$ *has shape* $2^{1+4+1}.\mathrm{Alt}(6)$. *Together these two groups generate* $\mathrm{GO}_6^+(5) \times \mathrm{GO}_1^+(5) \cap H$. *We have* $\langle P_2, P_5, P\rangle \cong \mathrm{Sp}_6(2) \cong \langle P_2, P_5, P^x\rangle$ *and the corresponding coset geometry is a simplicial complex* $\Delta(P_2, P_5, P, P^x)$ *that Kantor calls a GAB and is described in* [5, Section 5].

An unintuitive aspect of the last example is that $(\mathrm{GO}_1^+(5) \wr \mathrm{Sym}(4) \times \mathrm{GO}_1^+(5) \wr \mathrm{Sym}(3)) \cap H$ contains four 2-minimal subgroups as is transparent from the proof of Theorem A whereas the group which looks very much the same $(\mathrm{GO}_1^-(5) \wr \mathrm{Sym}(4) \times \mathrm{GO}_1^+(5) \wr \mathrm{Sym}(3)) \cap H$ has a quotient $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$ and so contains just two 2-minimal subgroups one of which is in the first group.

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