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Galt, Alexey; Joshi, Vijay; Mamontov, Andrey; Shpectorov, Sergey; Staroletov, Alexey

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Double axes and subalgebras of Monster type in Matsuo algebras

Alexey Galt, Vijay Joshi, Andrey Mamontov,

Sergey Shpectorov, Alexey Staroletov

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Abstract

Axial algebras are a class of commutative non-associative algebras generated by idempotents, called axes, with adjoint action semi-simple and satisfying a prescribed fusion law. Axial algebras were introduced by Hall, Rehren and Shpectorov [17, 18] as a broad generalization of Majorana algebras of Ivanov, whose axioms were derived from the properties of the Griess algebra for the Monster sporadic simple group. The class of axial algebras of Monster type includes Majorana algebras for the Monster and many other sporadic simple groups, Jordan algebras for classical and some exceptional simple groups, and Matsuo algebras corresponding to 3-transposition groups. Thus, axial algebras of Monster type unify several strands in the theory of finite simple groups.

It is shown here that double axes, i.e., sums of two orthogonal axes in a Matsuo algebra, satisfy the fusion law of Monster type $(2\eta, \eta)$. Primitive subalgebras generated by two single or double axes are completely classified and 3-generated primitive subalgebras are classified in one of the three cases. These classifications further lead to the general flip construction outputting a rich variety of axial algebras of Monster type. An application of the flip construction to the case of Matsuo algebras related to the symmetric groups results in three new explicit infinite series of such algebras.

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1 Introduction

Axial algebras are commutative non-associative algebras generated by a set of special idempotents, called axes, satisfying a prescribed fusion law. For example, the $196,884$-dimensional real unital Griess algebra V is generated by a set of idempotents called $2A$-axes that satisfy the fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ shown in Table 1. The meaning of this is that the adjoint action $\mathrm{ad}_a : v \mapsto av$

$*$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1		$\frac{1}{4}$	$\frac{1}{32}$
0		0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 1: Fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$

of any $2A$-axis $a \in V$ is semi-simple (i.e., diagonalizable), with all eigenvalues in the set $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$ and the entries of the table prescribe how eigenvectors of ad_a multiply in V. For example, the entry $\frac{1}{32} * 0 = \{\frac{1}{32}\}$ means that if u is a $\frac{1}{32}$-eigenvector and v is a 0-eigenvector then uv is a $\frac{1}{32}$-eigenvector; the entry $\frac{1}{4} * \frac{1}{4} = \{1, 0\}$ means that the product of a $\frac{1}{4}$-eigenvector and a second $\frac{1}{4}$-eigenvector is a sum of a 1-eigenvector and a 0-eigenvector, and so on.

Historically, the first class of algebras whose axioms included a fusion law were the Majorana algebras of Ivanov [21]. These are real axial algebras satisfying the fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$ plus certain additional axioms. The Griess algebra V is the principal example of a Majorana algebra, and in fact, the axioms of Majorana algebras were derived from the properties of the Griess algebra exploited in the celebrated theorem of Sakuma [35]. Note that Sakuma's theorem was proven in the context of vertex operator algebras (VOAs), where the Griess algebra arises as the weight-two component of the Moonshine VOA $V^\natural$, whose automorphism group is the sporadic simple group M, the Monster.

There are further classes of algebras that exhibit axial behavior. Idempotents in associative algebras satisfy the simplest fusion law involving only $\{1, 0\}$. More interestingly, in Jordan algebras, every idempotent leads to

the so-called Peirce decomposition of the algebra, and this property can be restated in axial terms as the fusion law $\mathcal{J}(\frac{1}{2})$ in Table 2. Motivated by

$*$	1	0	$\frac{1}{2}$
1	1		$\frac{1}{2}$
0		0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

Table 2: Fusion law $\mathcal{J}(\frac{1}{2})$

these and other examples, Hall, Rehren, and Shpectorov [17, 18] introduced axial algebras as a broad generalization of Majorana and Jordan algebras, allowing an arbitrary field, an arbitrary fusion law, and getting rid of all additional "non-axial" axioms. Focussing on a particular fusion law and/or adding additional axioms defines a subclass of axial algebras. In this paper, we work with two subclasses: algebras of Jordan type η satisfying the fusion law $\mathcal{J}(\eta)$ as in Table 2, but with an arbitrary $\eta \neq 1, 0$ in place of $\frac{1}{2}$, and algebras of Monster type (α, β) for the fusion law $\mathcal{M}(\alpha, \beta)$ as in Table 1, but with $\frac{1}{4}$ and $\frac{1}{32}$ substituted with arbitrary α and β distinct from 1, 0 and from each other. The only other axiom that we require is primitivity, which refers to primitivity of each generating axis a of the algebra A, understood as the condition that the 1-eigenspace $A_1(a)$ of ad_a be 1-dimensional spanned by a. (Note that since a is an idempotent, it is a 1-eigenvector for ad_a.)

We note that the class of algebras of Jordan type is a subclass of algebras of Monster type, corresponding to the situation where the α-eigenspace is trivial and $\beta = \eta$.

We have already mentioned that the automorphism group of the Griess algebra is the Monster M. Jordan algebras also admit large automorphism groups, among which we find all classical groups and some exceptional groups of Lie type. The apparent relation between axial algebras of Monster type and (simple) groups is for a good reason: the fusion laws $\mathcal{J}(\eta)$ and $\mathcal{M}(\alpha, \beta)$ are C_2-graded and this results in the axial algebra A admitting, for each axis a, an automorphism τ_a of order two, called the Miyamoto involution of a. The Miyamoto involutions generate the Miyamoto group of A, which is a subgroup of Aut (A). Naturally, the Miyamoto group of the Griess algebra V

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is the Monster group and, similarly, for the Jordan algebras, the Miyamoto groups are the corresponding groups of Lie type. We stress that it is not just the known examples that are highly symmetric: all algebras of Monster type, whether known or not, admit significant automorphism groups. Hence the study of algebras of Monster type is highly interesting and relevant not only for specialists in non-associative algebras, but also for group theorists, because these algebras provide a platform to study sporadic simple groups alongside the groups of Lie type, giving the hope of a possible unified theory of (finite) simple groups.

Ideally, the aim of the theory is a complete classification of algebras of Monster type. For the smaller class of algebras of Jordan type, it is shown in [18] (with a correction in [20]) that, for $\eta \neq \frac{1}{2}$, algebras of Jordan type are exactly the Matsuo algebras, corresponding to 3-transposition groups, or their factor algebras. The class of Matsuo algebras, playing an important rôle in this paper, was first introduced by Matsuo [30] and then reintroduced and generalized by Hall, Rehren, and Shpectorov in [18]. The case $\eta = \frac{1}{2}$ is still open and it is significantly more complex because Jordan algebras appear in this case. However, the axial-style characterization of Jordan algebras by Segev [36] and the recent result of Gorshkov and Staroletov [14], bounding the dimension of a 3-generated algebra of Jordan type, give hope that the classification of algebras of Jordan type can soon be completed.

For the algebras of Monster type, classification results are currently few. Sakuma's Theorem was reproved by Ivanov, Pasechnik, Seress, and Shpectorov in [25] in the context of Majorana algebras. Next, Hall, Rehren, and Shpectorov [17] generalized the theorem by removing several assumptions specific to Majorana algebras leaving only the fusion law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$, primitivity, and the existence of a Frobenius form, a non-zero bilinear form associating with the algebra product. Later Rehren [33, 34] attempted to remove the Frobenius form condition and also generalize to the case of arbitrary 2-generated primitive algebras of Monster type (α, β). He managed to obtain an upper bound of 8 for the dimension of such an algebra, modulo the extra assumptions that $\alpha \neq 2\beta, 4\beta$. In recent preprints [10, 11, 12], Franchi, Mainardis, and Shpectorov make the next step, extending this bound, 8, to one of the exceptional cases, $\alpha = 2\beta$, and finding the real culprit, an infinite-dimensional 2-generated algebras HW[1], in the second exceptional case where

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[1]HW stands for "highwater algebra" because the algebra was discovered in Venice during the disastrous floods in November 2019.

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$\alpha = 4\beta$. They also remove the Frobenius form condition in the case of the law $\mathcal{M}(\frac{1}{4}, \frac{1}{32})$, thus achieving the ultimate classification in this key case. They also classify the generic 2-generated algebras, the ones that exist for arbitrary α and β. In another recent preprint, Yabe [?] (see also the note by Franchi and Mainardis [9]) classifies the symmetric case, where there exists an involution switching the two generators. The general classification of 2-generated algebras of Monster type remains an open and highly attractive problem.

Beyond the 2-generated case, the results have been limited so far to finding subalgebras of the Griess algebra and computing the algebras of type $(\frac{1}{4}, \frac{1}{32})$ for concrete Miyamoto groups by hand or using available GAP and MAGMA programs [25, 27, 26, 22, 35, 3, 23, 24, 7, 39, 32, 31]. (See also the interesting survey [8].) While these computations lead to many new interesting examples of algebras, the current situation is somewhat paradoxical. Intuitively, it is clear that the class of algebras of Monster type should be significantly richer than its subclass of algebras of Jordan type. However, all (finitely many!) known examples outside of this subclass have been of type $(\frac{1}{4}, \frac{1}{32})$. Clearly, the class of algebras of Monster type (α, β) should be suitably populated with examples before one can expect more general classification results.

In this paper we make a step towards remedying this situation. Namely, we introduce the flip construction that results in a rich variety of primitive algebras of Monster type $(2\eta, \eta)$, where η is any scalar outside of $\{0, \frac{1}{2}, 1\}$. It is interesting that $\alpha = 2\eta$ and $\beta = \eta$ is one of the two exceptional situations $(\alpha = 2\beta)$ arising in the work of Rehren [33]. The flip construction starts with a Matsuo algebra $M = M_\eta(\Gamma)$, where Γ is the Fischer space of an arbitrary 3-transposition group. Recall that a 3-transposition group is a pair (G, D) where G is a group generated by a normal subset D of involutions satisfying the condition that $|cd| \leq 3$ for all $c, d \in D$. The Fischer space Γ of (G, D) is the point-line geometry that has the involutions from D as its points and, as lines, all triples $\{c, d, e\} \subseteq D$ satisfying $c^d = e = d^c$. Note that points c and d are collinear in Γ if and only if they do not commute as elements of G.

The Matsuo algebra $M_\eta(\Gamma)$, defined over a field $\mathbb{F}$ of characteristic not equal to 2, has D as its basis, with multiplication defined on this basis as

follows:

$$
c \cdot d = \begin{cases} c, & \text{if } c = d; \\[4pt] 0, & \text{if } c \text{ and } d \text{ are non-collinear;} \\[4pt] \frac{\eta}{2}(c + d - e), & \text{if } c \text{ and } d \text{ lie on a line, } \{c, d, e\}. \end{cases}
$$

Here the parameter η is an arbitrary element of $\mathbb{F}$, not equal to 1 or 0. (For an extended discussion of these concepts, see Section 3.)

The elements of the basis D of $M = M_\eta(\Gamma)$ are the primitive axes of this algebra, satisfying the fusion law $\mathcal{J}(\eta)$ in Table 4. Note that non-collinear axes a and b are orthogonal in M, i.e., $a \cdot b = 0$. In this case, $x = a + b$ is again an idempotent in M. We call the elements x *double axes*, as opposed to single axes, which are the elements of D.

Our first result concerns the fusion law that a double axis satisfies.

Theorem 1.1. *If $\eta \neq \frac{1}{2}$ then the double axis $x = a + b$ satisfies the fusion law $\mathcal{M}(2\eta, \eta)$:*

$\diamond$	1	0	2η	η
1	1		2η	η
0		0	2η	η
2η	2η	2η	$1 + 0$	η
η	η	η	η	$1 + 0 + 2\eta$

(This is proved in Section 4 as Proposition 4.7.)

That is, double axes are of Monster type $(2\eta, \eta)$. Note that single axes also satisfy the above law, with the 2η-eigenspace trivial. Thus, any subalgebra in a Matsuo algebra generated by a collection of single and double axes is an algebra of Monster type $(2\eta, \eta)$. The pinch of salt in this statement is the absence of the word "primitive". Indeed, the double axis $x = a + b$ is not primitive in M, because its 1-eigenspace is 2-dimensional, equal to $\langle a, b \rangle$. However, x can be primitive in a proper subalgebra of M, as long as this subalgebra does not contain a and b. The question is therefore: can we find a large family of subalgebras, where the generating double axes are primitive?

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We first look at the 2-generated case, that is, at subalgebras generated by two axes. If both axes are single, say, $a, b \in D$, then $\langle\!\langle a, b \rangle\!\rangle^2$ is well-known. If a and b are non-collinear then $\langle\!\langle a, b \rangle\!\rangle \cong \mathbb{F}^2$, the algebra known in the axial lingo as $2B$. If a and b are collinear then $\langle\!\langle a, b \rangle\!\rangle$ is a 3-dimensional Matsuo algebra known as $3C(\eta)$.

So we assume that at least one of the two generating axes is a double axis. Accordingly, we consider two cases: either (A) the algebra is generated by a single axis a and a double axis $b+c$; or (B) it is generated by two double axes, $a+b$ and $c+d$. The exact configuration is described by the diagram on the support set $\{a, b, c\}$ or $\{a, b, c, d\}$, the diagram being the graph in which two points from the support set are connected by an edge when they are collinear in the Fischer space Γ. The possible diagrams for cases (A) and (B) are given in Figures 1 and 2. Each diagram defines a Coxeter group, of which the 3-transposition group G (the minimal G, generated just by the support involutions) is a factor group. All these Coxeter groups are well-known, and so it is not difficult to list all possible G and then build the corresponding 2-generated subalgebras. The number of subcases is eliminated by Theorem 5.5 stating that the subalgebra cannot be primitive unless the diagram admits the flip symmetry. By the *flip* we mean the permutation of the support set switching two points within each double axis. This eliminates diagrams A_2, B_2, B_3, and B_5. The remaining symmetric diagrams all lead to primitive 2-generated subalgebras.

We can summarize this discussion as the following result.

Theorem 1.2. *Every 2-generated primitive subalgebra is one of the following:* $2B \cong \mathbb{F}^2$, $3C(\eta)$, $3C(2\eta)$, *the new 4-dimensional algebra* $Q_2(\eta)$, *the new 5-dimensional algebra shown in Table 7, or the new 8-dimensional algebra in Table 8.*

It is interesting to compare this with the list of Sakuma algebras arising for the Monster fusion law. Similarities between the two lists are quite obvious and the main difference is the absence here of an analogue of the Sakuma algebra $5A$ with the Miyamoto group D_{10}. Of course, the algebras above are the ones that live inside Matsuo algebras. Hypothetically, other 2-generated primitive algebras of Monster type $(2\eta, \eta)$ might exist. Hence we are asking the following: Is the list in Theorem 1.2 the complete list of 2-generated primitive algebras of Monster type $(2\eta, \eta)$?

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[2]We use double brackets for algebra generation, to distinguish it from the linear span of the vectors.

We also attempt to determine 3-generated primitive subalgebras of Matsuo algebras. Here again the case of three single axes is well understood. Hence one needs to deal only with the following cases: (C) two single axes and one double axis; (D) one single axis and two double axes; and (E) three double axes. The flip-symmetric diagrams in each of these cases are shown in Figures 3, 4, and 5. In this paper we complete the case (C). Some of the algebras arising here decompose as direct sums of smaller algebras. In the following theorem we only list the indecomposable ones.

Theorem 1.3. *Each 3-generated primitive indecomposable subalgebra in case (C) is one of the following: the new 9-dimensional algebra $Q^3(\eta)$, the new 8-dimensional algebra $2Q_2(\eta)$, the new 12-dimensional algebra $3Q_2(\eta)$, or the new 24-dimensional algebra shown in Table 12.*

The classification of cases (D) and (E) is an ongoing project. For (D), the support set has cardinality 5. So here one can use the classification of 5-generated 3-transposition groups by Hall and Soicher [19]. Unfortunately, there is no classification of 6-generated 3-transposition groups, which is what one would need for (E).

It is remarkable that all algebras found so far for flip-symmetric diagrams turn out to be primitive. This suggests that perhaps the flip symmetry condition is not only necessary for primitivity, but also sufficient. Developing this idea further, we propose in this paper the general flip construction. Consider an arbitrary 3-transposition group (G, D) and an involutive automorphism τ of G such that $D^\tau = D$. Then τ acts on the Fischer space Γ and, consequently, also on the Matsuo algebra $M = M_\eta(\Gamma)$. Let $H = \langle \tau \rangle \leq \text{Aut}(M)$. If τ flips two orthogonal Matsuo axes a and b then the fixed subalgebra M_H of M contains the double axis $a + b$, but not a and b. Hence $x = a + b$ is guaranteed to be primitive in M_H. (This is Proposition 7.4.) Hence we define the *flip algebra* A_τ as the subalgebra of M_H generated by all single and (flipped) double axes contained in M_H. This is always a primitive axial algebra of Monster type $(2\eta, \eta)$.

There exist great many 3-transposition groups and each Fischer space admits many classes of involutive automorphisms. Thus, the flip construction does indeed lead to a rich variety of algebras of Monster type. The work of understanding these algebras is just starting. In this paper, we completely analyze the simplest case of the symmetric group $G = S_n$. The special case of the fixed-point-free involution $\tau \in S_n$ (when n is even, $n = 2k$) gives the first proper infinite series $Q_k(\eta)$ of algebras of Monster type $(2\eta, \eta)$.

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Theorem 1.4. *Suppose that $n = 2k$ and $\tau = (1,2)(3,4)\cdots(2k-1,2k)$. Then the flip algebra $Q_k(\eta) = A_\tau$ is of dimension k^2, spanned by its k single axes and $k^2 - k$ double axes.*

We call the algebra $Q_k(\eta)$ the k^2-*algebra*.

In the general case, the flip algebra decomposes as the direct sum of two smaller algebras.

Theorem 1.5. *If $n = 2k + m$, with $k, m \neq 0$, and $\tau = (1,2)(3,4)\cdots(2k-1,2k)$ then $A_\tau = Q_k(\eta) \oplus M_\eta(\Gamma')$, where the first summand arises on the subset $\{1, 2, \ldots, 2k\}$ and the second, Matsuo summand comes from the fixed subset $\{2k+1, \ldots, n\}$ of τ.*

Note that in Theorem 1.4 the flip algebra $Q_k(\eta)$ coincides with the full fixed subalgebra M_H, but this is not the case in Theorem 1.5.

We also investigate the properties of the algebra $Q_k(\eta)$. In particular, we show the following using the structure theory developed by Khasraw, McInroy, and Shpectorov [29].

Theorem 1.6. *The algebra $Q_k(\eta)$ is simple unless $\eta = -\frac{1}{2(k-1)}$ or $-\frac{1}{k-2}$. For the two exceptional values of η, the algebra has a unique maximal nontrivial ideal (the radical), the factor over which is a simple algebra.*

The dimension of the radical for the special values of η is (almost always) 1 and $k - 1$, respectively.

Towards the end of the paper, we also introduce two further infinite series of algebras, $2Q_k(\eta)$ of dimension $2k^2$ and $3Q_k(\eta)$ of dimension $3k^2$. These arise for 3-transposition groups of the form $2^{n-1} : S_n$ and $3^{n-1} : S_n$ and special choices of τ. For these groups, we do not analyze all possible flips τ. We also do not determine the special values of η, for which $2Q_k(\eta)$ and $3Q_k(\eta)$ are not simple. All these questions are left for another paper.

The idea of double axes came from GAP experiments performed in Novosibirsk. Some of the above theorems appeared in the PhD thesis of Joshi [28] at the University of Birmingham. This includes the fusion law obeyed by double axes and the classification of the 2-generated subalgebras. The infinite series $Q_k(\eta)$ was discovered simultaneously in Birmingham and Novosibirsk (likely within an hour of each other) as part of ongoing collaboration. During nearly two years of work on this paper, several mathematicians having access to our drafts have started looking at different cases of 3-transposition groups.

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Joshi's PhD thesis includes, in addition to the symmetric case, also an almost complete treatment of flips for $G = Sp_{2n}(2)$. Alsaeedi [1] constructed an infinite series $Q^k(\eta)$, dual to our $Q_k(\eta)$, as a subalgebra of the Matsuo algebra for $G = {}^+\Omega^+_{k+1}(3)$. Shi [38] provided a treatment of $G = O^\pm_{2n}(2)$ similar to Joshi's treatment of $Sp_{2n}(2)$. The case of $G = SU_n(2)$ was looked at by Hoffman, Rodrigues, and Shpectorov [15]. Finally, all flip algebras for the sporadic Fischer groups and triality groups were computed by the 3A (the Novosibirsk part of the team) using GAP. All these results, with the exception of [1], are still awaiting publication.

Of course, all this work only concerns the almost simple 3-transposition groups, so it is just the tip of the iceberg. The variety of flip algebras is enormous. In a sense, the flip construction is similar to the construction of twisted groups of Lie type. There, as well, every automorphism defines a fixed subgroup, but some automorphisms are more interesting than others, leading to new simple groups. In the same way, for flip algebras, the question is which of them are the key algebras, the true building blocks, and which are the composites.

Let us now discuss the contents of the present paper. In Section 2, we introduce the basics of axial algebras. In Section 3, we discuss the class of Matsuo algebras corresponding to 3-transposition groups, or in geometric terms, to Fischer spaces. In Section 4, we prove that double axes in Matsuo algebras satisfy the fusion law of Monster type $(2\eta, \eta)$. In Section 5, we classify all 2-generated primitive subalgebras generated by single and double axes. In all but two cases we can find the suitable configuration inside a symmetric group, and so the calculations in these cases are particularly uncomplicated. In the remaining two cases, we operate, instead, in terms of the Fischer spaces for the groups $2^3 : S_4$ and $3^3 : S_4$. In Section 6, we outline the classification of 3-generated primitive subalgebras and complete the case (C). In Section 7, we introduce the flip construction and show that it always leads to a primitive algebra. In Section 8, we analyze the case of the symmetric group group S_n and develop the properties of the new series of algebras $Q_k(\eta)$. Finally, in Section 9, we construct two further infinite series, $2Q_k(\eta)$ and $3Q_k(\eta)$.

2 Axial algebras

In this section we provide the necessary background on axial algebras. Note that algebras are not assumed to be associative. Hence an algebra is just a vector space with a bilinear product.

2.1 Axes and axial algebras

We start with basic definitions.

Definition 2.1. *A* fusion law $\mathcal{F}$ *over a field* $\mathbb{F}$ *consists of a (finite) set* $X \subseteq \mathbb{F}$ *and a symmetric (commutative) product*

$$* : X \times X \to 2^X,$$

where 2^X *denotes the set of all subsets of* X.

Let A be a commutative algebra over $\mathbb{F}$. For $a \in A$, we denote by ad_a the adjoint map $A \to A$ defined by $u \mapsto au$. For $\lambda \in \mathbb{F}$, let $A_\lambda(a)$ denote the λ-eigenspace of ad_a; that is, $A_\lambda(a) = \{u \in A \mid au = \lambda u\}$. Note that this notation makes sense even if λ is not an eigenvalue of ad_a; in this case, we simply have $A_\lambda(a) = 0$. For $L \subset \mathbb{F}$, set $A_L(a) := \oplus_{\lambda \in L} A_\lambda(a)$.

Let $\mathcal{F} = (X, *)$ be a fusion law over $\mathbb{F}$.

Definition 2.2. *A non-zero idempotent* $a \in A$ *is an* ($\mathcal{F}$-)axis *if*

(a) $A = A_X(a)$; *and*

(b) $A_\lambda(a)A_\mu(a) \subseteq A_{\lambda*\mu}(a)$ *for all* $\lambda, \mu \in X$.

Condition (a) means that ad_a is semi-simple and all its eigenvalues are in X. Note that, since a is an idempotent, 1 is an eigenvalue of ad_a and $a \in A_1(a)$. Hence we always assume that $1 \in X$.

Definition 2.3. *An axis* a *is* primitive *if* $A_1(a)$ *is 1-dimensional; that is,* $A_1(a) = \langle a \rangle$.

We will mostly deal with primitive axes. In this case, $A_1(a)A_\lambda(a) = \langle a \rangle A_\lambda(a) = A_\lambda(a)$ if $\lambda \neq 0$ and $A_1(a)A_0(a) = 0$. In view of this, we can assume that $1 * 0 = \emptyset$ (assuming that $0 \in X$) and $1 * \lambda = \{\lambda\}$ for all $0 \neq \lambda \in X$.

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Definition 2.4. *An algebra A over $\mathbb{F}$ is an ($\mathcal{F}$-)*axial *algebra if it is generated as algebra by a set of $\mathcal{F}$-axes. The algebra A is a* primitive ($\mathcal{F}$-)axial *algebra if it is generated by a set of primitive $\mathcal{F}$-axes.*

In principle, an axial algebra should be formally defined as the pair consisting of the algebra and the set of generating axes. However, in practice we will just talk about the algebra and the set of generating axes will be assumed.

Table 3 shows the fusion law $\mathcal{M}$ that is the focus of this paper. Here

$*$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1,0$	β
β	β	β	β	$1,0,\alpha$

Table 3: Fusion law $\mathcal{M}(\alpha, \beta)$

$\alpha, \beta \in \mathbb{F}$ are arbitrary numbers distinct from each other and from 1 and 0. Each cell of the table lists the elements of the corresponding set $\lambda * \mu$. For example, $1 * 0 = \emptyset$ and $\alpha * \alpha = \{1, 0\}$. Primitive axial algebras with this fusion law are called *algebras of Monster type* (α, β).

Often an axial algebra admits a bilinear form that associates with the algebra product.

Definition 2.5. *A* Frobenius form *on an axial algebra A over a field $\mathbb{F}$ is a non-zero bilinear form $(\cdot, \cdot)$ on A such that*

$$(uv, w) = (u, vw)$$

for all $u, v, w \in A$.

The decomposition $A = \oplus_{\lambda \in X} A_\lambda(a)$ of an axial algebra A with respect to an axis $a \in A$ is an orthogonal decomposition, that is,

$$(A_\lambda(a), A_\mu(a)) = 0$$

for all $\lambda, \mu \in X$, $\lambda \neq \mu$.

We say that an axis $a \in A$ is *singular* if $(a, a) = 0$ and a is *non-singular* otherwise. A primitive axis lies in the radical

$$A^\perp = \{u \in A \mid (u, v) = 0 \text{ for all } v \in A\}$$

if and only if it is singular.

2.2 Ideals and simplicity

Methods to find all ideals and, consequently, to check whether A is simple were developed in [29]. Recall that every axial algebra comes with a set of generating axes. According to [29], all ideals in an axial algebra A can be classified into two types: (1) ideals not containing any of the generating axes, and (2) ideals containing generating axes. The first type of ideals are controlled by the radical of the algebra.

Definition 2.6. *The* radical $R(A)$ *of a primitive axial algebra A is the unique maximal ideal not containing any of the generating axes of A.*

If A admits a Frobenius form then it is easy to see that its radical $A^\perp$ is an ideal of A. Recall that a primitive axis $a \in A$ is contained in $A^\perp$ if and only if a is singular.

Theorem 2.7 ([29]). *If A is a primitive axial algebra admitting a Frobenius form with respect to which none of the generating axes of A is singular then the radical $R(A)$ coincides with the radical $A^\perp$ of the Frobenius form.*

In particular, if the Frobenius form is non-degenerate (i.e., $A^\perp = 0$) then A has no non-zero ideals of the first kind. For the ideals of the second type, [29] suggest the following construction. Suppose $a \in A$ is a primitive axis. Since $A = \oplus_{\lambda \in X} A_\lambda(a)$, we can write an arbitrary $u \in A$ as $u = \sum_{\lambda \in X} u_\lambda$, where $u_\lambda \in A_\lambda(a)$ for each λ. Since a is primitive, $u_1 = \alpha a$ for some $\alpha \in \mathbb{F}$. We call u_1 the *projection* of u onto a.

Definition 2.8. *Suppose A is a primitive axial algebra. The* projection graph Δ *of A has as vertices all generating axes of A and Δ has a directed edge from b to a if b has a non-zero projection b_1 onto a.*

It can be shown that if b is contained in an ideal J then every component b_λ is contained in J. In particular, if $b_1 = \alpha a \neq 0$ then $a \in J$. It follows that if Δ is strongly connected (any vertex can be reached from any other vertex via a directed path) then every ideal J of the second type contains all generating axes, and so $J = A$.

When A admits a Frobenius form, the definition of Δ simplifies.

Theorem 2.9 ([29]). *Suppose A is a primitive axial algebra admitting a Frobenius form with all generating axes non-singular. Then the projection graph Δ is a simple (undirected) graph and a and b are adjacent in Δ if and only if $(a, b) \neq 0$.*

In particular, if Δ is connected then A has no proper ideals of the second type. The final result of this section summarizes our discussion.

Theorem 2.10. *Suppose that a primitive axial algebra admits a Frobenius form and all generating axes are non-singular with respect to the form. If the projection graph Δ of A is connected then A is simple if and only if the Frobenius form has zero radical.*

2.3 Grading and the Miyamoto group

When the fusion law $\mathcal{F}$ is graded by an abelian group, every axis in an axial algebra A leads to automorphisms of A.

Definition 2.11. *Suppose $\mathcal{F} = (X, *)$ is a fusion law. A grading of $\mathcal{F}$ by a group T is a partition $\{X_t | t \in T\}$ of the set X (where we allow parts X_t to be empty for some $t \in T$), such that, for all $\lambda, \mu \in X$, if $\lambda \in X_t$ and $\mu \in X_{t'}$ then $\lambda * \mu$ is contained in $X_{tt'}$.*

See [4] for a slightly different, categorical definition.

The fusion law $\mathcal{M}(\alpha, \beta)$ in Table 3 is graded by the group $C_2 = \{1, -1\}$. Here $X_1 = \{1, 0, \alpha\}$ and $X_{-1} = \{\beta\}$.

Suppose $\mathcal{F}$ is a fusion law graded by a group T and A is an $\mathcal{F}$-axial algebra. For an axis $a \in A$ and a linear character ξ of T, define the linear map $\tau_a(\xi) : A \to A$ as follows: on each part $A_{X_t}(a) = \oplus_{\lambda \in X_t} A_\lambda(a)$, the map $\tau_a(\xi)$ acts as scalar $\xi(t)$. (That is, $\tau_a(\xi)(u) = \xi(t)u$ for all $u \in A_{X_t}(a)$.) It is easy to see that $\tau_a(\xi)$ is an automorphism of A and the map $\xi \mapsto \tau_a(\xi)$ is a homomorphism from the group T^* of linear characters of T to $\mathrm{Aut}(A)$. The image of this homomorphism, $T_a = \{\tau_a(\xi) \mid \xi \in T^*\}$ is called the *axial*

subgroup of Aut (A) corresponding to the axis a and the subgroup of Aut (A) generated by the axial subgroups T_a for all generating axes $a \in A$ is called the *Miyamoto group* of A.

When $T = C_2$ (for example, in the case of the fusion law $\mathcal{M}(\alpha, \beta)$), the group T^* is of order two (assuming that the characteristic of $\mathbb{F}$ is not 2). Then the only non-identity character ξ of T produces the element $\tau_a = \tau_a(\xi)$, known as the *Miyamoto involution*. Here the axial subgroup $T_a = \langle \tau_a \rangle$ is of order two (or one if $A_{X_{-1}}(a) = 0$) and the Miyamoto group is simply the subgroup of Aut (A) generated by all Miyamoto involutions τ_a.

3 Matsuo algebras

In this section we introduce the family of axial algebras called the *Matsuo algebras*.

Recall that a 3-*transposition group* is a pair (G, D), where G is a group and D is a normal subset (union of conjugacy classes) of G such that:

(a) D generates G;

(b) every $d \in D$ is an involution (that is, $|d| = 2$); and

(c) for $c, d \in D$, $|cd| \leq 3$.

In many cases, D is unique for a given G, so it is common to talk about the 3-transposition group G, instead of (G, D).

An example of 3-transposition group is given by $G = S_n$, the symmetric group on n symbols, and $D = (1, 2)^G$, the class of transpositions (2-cycles). Finite 3-transposition groups were classified, under some restrictions, by Fischer [6] and, in complete generality, by Cuypers and Hall [2], who used geometric methods.

Suppose (G, D) is a 3-transposition group. The *Fischer space* of (G, D) is a point-line geometry $\Gamma = \Gamma(G, D)$, whose point set is D and where points c and d are collinear if and only if $|cd| = 3$. Furthermore, any two collinear points c and d lie in a unique common line, which consists of c, d, and the third point $e = c^d = d^c$. Note that c, d, and e are the three involutions in the subgroup S_3 that any two of them generate. Thus, lines of Γ are in a bijection with the subgroups S_3 generated by the elements of D.

It is easy to see that the connected components of the Fischer space Γ coincide with the conjugacy classes of G contained in D. In particular, the

Fischer space is connected if and only if D is a single conjugacy class of G. We will also say in this case that (G, D) is *connected*.

The 3-transposition group (G, D) can be recovered from Γ up to the center of G, which, clearly, acts trivially on Γ. The further discussion will be mainly in terms of the Fischer space Γ, but we will keep using D for the point set of Γ.

Let us associate an algebra with the Fischer space Γ. Select a field $\mathbb{F}$ of characteristic not equal to 2 and let $\eta \in \mathbb{F}$, $\eta \neq 0, 1$.

Definition 3.1. *The* Matsuo algebra $M_\eta(\Gamma)$ *over* $\mathbb{F}$, *corresponding to* Γ *and* η, *has the point set* D *as its basis. Multiplication is defined on the basis as follows:*

$$
c \cdot d = \begin{cases} c, & \text{if } c = d; \\ 0, & \text{if } c \text{ and } d \text{ are non-collinear;} \\ \frac{\eta}{2}(c + d - e), & \text{if } c \text{ and } d \text{ lie on a line, } \{c, d, e\}. \end{cases}
$$

Here we use the dot for the algebra product to distinguish it from the multiplication in the 3-transposition group G. In what follows we skip the dot as long as this causes no confusion.

The following small cases will appear prominently in the remainder of the paper, so we need special names for them, coming from [17] and generalizing the names of the Norton-Sakuma algebras [25]. First of all, if the Fischer space Γ consists of a single point, say c, then $M \cong \mathbb{F}$ is referred to as the algebra $1A$. If Γ consists of two non-collinear points then $M \cong \mathbb{F}^2$ and it is referred to as the algebra $2B$. Finally, if Γ consists of three points forming a line then M is 3-dimensional called $3C(\eta)$. (The value of η is irrelevant for $1A$ and $2B$.)

Let us record now some properties of Matsuo algebras.

Proposition 3.2. *If* Γ *is disconnected with components* Γ_i, $i \in I$, *then* $M_\eta(\Gamma) = \oplus_{i \in I} M_\eta(\Gamma_i)$.

Included in this statement is the property that $M_\eta(\Gamma_i) M_\eta(\Gamma_j) = 0$ for all $i, j \in I$, $i \neq j$.

Clearly, each $c \in D$ is an idempotent in $M_\eta(\Gamma)$. Furthermore, it can be shown that c satisfies the fusion law $\mathcal{J}(\eta)$ in Table 4. This means that Matsuo algebras are *algebras of Jordan type* η. This class of axial algebras

$*$	1	0	η
1	1		η
0		0	η
η	η	η	$1,0$

Table 4: Fusion law $\mathcal{J}(\eta)$

was introduced in [17], where a partial classification, for $\eta \neq \frac{1}{2}$, was achieved. (See also a correction and further details in [20].)

Proposition 3.3 ([17, 20]). *Any algebra of Jordan type $\eta \neq \frac{1}{2}$ is either a Matsuo algebra or a factor algebra of a Matsuo algebra.*

The Matsuo algebra admits a Frobenius form $(\cdot, \cdot)$ such that $(c, c) = 1$ for each axis $c \in D$. This form is given on the basis D by the following:

$$(c, d) = \begin{cases} 1, & \text{if } c = d; \\ 0, & \text{if } |cd| = 2; \\ \frac{\eta}{2}, & \text{if } |cd| = 3. \end{cases}$$

The fusion law $\mathcal{J}(\eta)$ is C_2-graded, with $X_1 = \{1, 0\}$ and $X_{-1} = \{\eta\}$. The action of the Miyamoto involution τ_c on $M_\eta(\Gamma)$ agrees with the action of the 3-transposition c on D by conjugation. Correspondingly, the Miyamoto group of $M_\eta(\Gamma)$ is isomorphic to the factor group $G/Z(G)$ of the 3-transposition group G (from which the Fischer space Γ was obtained) over its center.

Another way to describe the action of the Miyamoto involution τ_c is in terms of the Fischer space Γ. Namely, τ_c fixes c and all points non-collinear with c and it switches the two points other than c on any line through c. This provides a very efficient way of computing the image under τ_c.

4 Double axes

Suppose $M = M_\eta(\Gamma)$ is the Matsuo algebra for the Fischer space Γ corresponding to a 3-transposition group (G, D).

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Definition 4.1. *Axes* $a, b \in D$ *are called* orthogonal *if* $ab = 0$*. In this case we call the idempotent* $x := a + b$ *a* double axis.

Indeed, $(a + b)^2 = a^2 + ab + ba + b^2 = a + 0 + 0 + b = a + b$, and so $a + b$ is an idempotent.

To distinguish double axes from the Matsuo axes in D, we will sometimes call the latter *single axes*. Thus, single axes satisfy the fusion law $\mathcal{J}(\eta)$ and each double axis is made out of two orthogonal single axes.

The goal of this section is to show that double axes satisfy a specific nice fusion law and so they are indeed nice axes in the sense of axial algebras. Throughout this section, $a, b \in D$ are orthogonal single axes and $x = a + b$.

Note that a linear transformation ϕ is semisimple if and only if its minimal polynomial, say f, has no multiple roots. If U is a subspace invariant under ϕ then, clearly, $f(\phi) = 0$ on U. In other words, the minimal polynomial of ϕ in its action on U is a divisor of f, and in particular, it cannot have multiple roots. Hence ϕ is also semisimple on each invariant subspace U. We apply this to $\phi = \mathrm{ad}_b$ acting on the eigenspaces of ad_a. First of all, note that the eigenspaces $M_\lambda(a)$ are indeed invariant under ad_b. This is because $b \in M_0(a)$ and, according to the fusion law $\mathcal{J}(\eta)$, we have $0 * \lambda \subseteq \{\lambda\}$ for all $\lambda \in X = \{1, 0, \eta\}$. Hence $\mathrm{ad}_b(M_\lambda(a)) = bM_\lambda(a) \subseteq M_\lambda(a)$.

We use the following notation: for $\lambda, \mu \in \{1, 0, \eta\}$, $M_{\lambda,\mu} := M_\lambda(a) \cap M_\mu(b)$.

Lemma 4.2. *For* $\lambda \in \{1, 0, \eta\}$*,* $M_\lambda(a) = \oplus_{\mu \in \{1,0,\eta\}} M_{\lambda,\mu}$*.*

Proof. We have already seen that $M_\lambda(a)$ is invariant under ad_b. Since ad_b is semisimple on A, it is also semisimple on $M_\lambda(a)$, and so $M_\lambda(a)$ decomposes as the direct sum of eigenspaces with respect to ad_b. Manifestly, $M_{\lambda,\mu} = M_\lambda(a) \cap M_\mu(b)$ are the only possible nontrivial eigenspaces of ad_b in $M_\lambda(a)$. This yields the desired decomposition. $\square$

Since $M = \oplus_{\lambda \in \{1,0,\eta\}} M_\lambda(a)$, we also obtain the following.

Corollary 4.3. $M = \oplus_{\lambda,\mu \in \{1,0,\eta\}} M_{\lambda,\mu}$*.*

We next observe that ad_x acts on each $M_{\lambda,\mu}$ as a scalar.

Lemma 4.4. *For* $u \in M_{\lambda,\mu}$*, we have* $xu = (\lambda + \mu)u$*.*

Proof. Indeed, $xu = (a + b)u = au + bu = \lambda u + \mu u = (\lambda + \mu)u$, since u lies in $M_\lambda(a)$ and in $M_\mu(b)$. $\square$

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At this point, we already see that ad_x is semisimple on M. Next, we investigate which eigenvalues arise and identify the eigenspaces.

In total there are $3^2 = 9$ pieces $M_{\lambda,\mu}$. We first note that three of them are trivial and further two are just the 1-dimensional spaces $\langle a \rangle$ and $\langle b \rangle$.

Lemma 4.5. (a) $M_{1,1} = M_{1,\eta} = M_{\eta,1} = 0$; and

(b) $M_{1,0} = M_1(a) = \langle a \rangle$ and $M_{0,1} = M_1(b) = \langle b \rangle$.

Proof. Since $M_1(a) = \langle a \rangle$ is 1-dimensional and since $a \in M_0(b)$, we conclude that $M_{1,1} = 0$, $M_{1,0} = \langle a \rangle$, and $M_{1,\eta} = 0$. Similarly, since $M_1(b) = \langle b \rangle$ and since $b \in M_0(a)$, we also have that $M_{0,1} = \langle b \rangle$, and $M_{\eta,1} = 0$. $\square$

So in total we have at most six nontrivial pieces $M_{\lambda,\mu}$. By Lemma 4.4, $M_{1,0} \oplus M_{0,1} \subseteq M_1(x)$, $M_{0,0} \subseteq M_0(x)$, $M_{\eta,\eta} \subseteq M_{2\eta}(x)$, and $M_{0,\eta} \oplus M_{\eta,0} \subseteq M_\eta(x)$. Combining this with Corollary 4.3, yields that all these inclusions are equalities. Hence we have the following.

Proposition 4.6. $M = M_1(x) \oplus M_0(x) \oplus M_{2\eta}(x) \oplus M_\eta(x)$, where

(a) $M_1(x) = M_{1,0} \oplus M_{0,1} = \langle a, b \rangle$;

(b) $M_0(x) = M_{0,0}$;

(c) $M_{2\eta}(x) = M_{\eta,\eta}$; and

(d) $M_\eta(x) = M_{\eta,0} \oplus M_{0,\eta}$.

From this point and until the end of the paper we assume that $\eta \neq \frac{1}{2}$. This is in addition to our earlier assumption that $\eta \neq 1, 0$. This guarantees that $2\eta \notin \{1, 0, \eta\}$.

We have established that the eigenvalues of ad_x are contained in $\{1, 0, 2\eta, \eta\}$. It remains to see which fusion law is satisfied.

Proposition 4.7. *The double axis x satisfies the fusion law $\mathcal{M}(\alpha, \beta)$ with $\alpha = 2\eta$ and $\beta = \eta$ (see Table 5).*

Note that we consider the fusion law for x in parallel with the Jordan type fusion law for a and b. Hence we use a different symbol, $\diamond$, in the table for x.

We start with two lemmas.

$\diamond$	1	0	2η	η
1	1		2η	η
0		0	2η	η
2η	2η	2η	$1+0$	η
η	η	η	η	$1+0+2\eta$

Table 5: Fusion law for the double axis x

Lemma 4.8. $M_{\lambda,\mu}M_{\gamma,\delta} \subseteq M_{\lambda*\gamma}(a) \cap M_{\mu*\delta}(b)$.

Proof. Indeed,

$$
\begin{aligned}
M_{\lambda,\mu}M_{\gamma,\delta} &= (M_\lambda(a) \cap M_\mu(b))(M_\gamma(a) \cap M_\delta(b)) \\
&\subseteq (M_\lambda(a)M_\gamma(a)) \cap (M_\mu(b)M_\delta(b)) \\
&\subseteq M_{\lambda*\gamma}(a) \cap M_{\mu*\delta}(b).
\end{aligned}
$$

$\square$

The second lemma is an extension of Corollary 4.3.

Lemma 4.9. *Suppose* $S,T \subseteq \{1,0,\eta\}$. *Then* $M_S(a) \cap M_T(b) = \oplus_{\lambda \in S, \mu \in T} M_{\lambda,\mu}$.

Proof. Clearly, $M_S(a) \cap M_T(b) \supseteq \oplus_{\lambda \in S, \mu \in T} M_{\lambda,\mu}$. Hence we just need to show the reverse inclusion.

Define $g \in \mathbb{F}[z]$ to be the product $\prod_{\lambda \in S}(z - \lambda)$. Then $u \in M$ lies in $M_S(a)$ if and only if $g(\mathrm{ad}_a)u = 0$. Let $u \in M_S(a) \cap M_T(b)$ and write it as $u = \sum_{\mu \in T} u_\mu$, where each u_μ lies in $M_\mu(b)$.

Note that $0 = g(\mathrm{ad}_a)u = \sum_{\mu \in T} g(\mathrm{ad}_a)u_\mu$ and that each component $g(\mathrm{ad}_a)u_\mu$ is contained in the corresponding $M_\mu(b)$, since the latter is invariant under ad_a. It follows that $g(\mathrm{ad}_a)u_\mu = 0$ for each $\mu \in T$, that is, $u_\mu \in M_S(a) \cap M_\mu(b)$.

Now we repeat this argument switching the rôles of a and b. Namely, we decompose u_μ as $u_\mu = \sum_{\lambda \in S} u_{\lambda,\mu}$, where $u_{\lambda,\mu} \in M_\lambda(a)$. Taking $h \in \mathbb{F}[z]$ equal to $z - \mu$, we see that $0 = h(\mathrm{ad}_b)u_\mu = \sum_{\lambda \in S} h(\mathrm{ad}_b)u_{\lambda,\mu}$ and, as above, we conclude that $h(\mathrm{ad}_b)u_{\lambda,\mu} = 0$ for all $\lambda \in S$, that is, $u_{\lambda,\mu} \in M_\lambda(a) \cap M_\mu(b) = M_{\lambda,\mu}$. Thus, $u = \sum_{\mu \in T} u_\mu = \sum_{\lambda \in S, \mu \in T} u_{\lambda,\mu} \in \oplus_{\lambda \in S, \mu \in T} M_{\lambda,\mu}$. We have shown that the reverse inclusion does indeed hold. $\square$

We can now proceed with the proof of Proposition 4.7.

Proof. We have already established that ad_x has eigenvalues in $\{1, 0, 2\eta, \eta\}$. Since M is commutative, we only need to check the upper triangular part of the fusion law.

We start with the first row of the table:

- $M_1(x)M_1(x) = \langle a, b \rangle \langle a, b \rangle \subseteq \langle aa, ab, bb \rangle = \langle a, b \rangle = M_1(x)$, since $aa = a$, $ab = 0$, and $bb = b$. Thus, $1 \diamond 1 = \{1\}$.

- $M_1(x)M_0(x) = \langle a, b \rangle M_{0,0} \subseteq aM_{0,0} + bM_{0,0} \subseteq aM_0(a) + bM_0(b) = 0 + 0 = 0$. Thus, $1 \diamond 0 = \emptyset$.

- $M_1(x)M_{2\eta}(x) = \langle a, b \rangle M_{\eta,\eta} \subseteq aM_{\eta,\eta} + bM_{\eta,\eta} = M_{\eta,\eta} + M_{\eta,\eta} = M_{\eta,\eta} = M_{2\eta}(x)$, since $M_{\eta,\eta} = M_\eta(a) \cap M_\eta(b)$ and so $aM_{\eta,\eta} = M_{\eta,\eta} = bM_{\eta,\eta}$. Thus, $1 \diamond 2\eta = \{2\eta\}$.

- $M_1(x)M_\eta(x) = \langle a, b \rangle (M_{\eta,0} + M_{0,\eta}) \subseteq aM_{\eta,0} + aM_{0,\eta} + bM_{\eta,0} + bM_{0,\eta} = M_{\eta,0} + 0 + 0 + M_{0,\eta} = M_{\eta,0} + M_{0,\eta} = M_\eta(x)$. Thus, $1 \diamond \eta = \{\eta\}$.

We now turn to the second row and here we will start using Lemma 4.8:

- $M_0(x)M_0(x) = M_{0,0}M_{0,0} \subseteq M_{0*0}(a) \cap M_{0*0}(b) = M_0(a) \cap M_0(b) = M_{0,0} = M_0(x)$. Thus, $0 \diamond 0 = \{0\}$.

- $M_0(x)M_{2\eta}(x) = M_{0,0}M_{\eta,\eta} \subseteq M_{0*\eta}(a) \cap M_{0*\eta}(b) = M_\eta(a) \cap M_\eta(b) = M_{\eta,\eta} = M_{2\eta}(x)$. Hence, $0 \diamond 2\eta = \{2\eta\}$.

- $M_0(x)M_\eta(x) = M_{0,0}(M_{\eta,0} + M_{0,\eta}) \subseteq M_{0,0}M_{\eta,0} + M_{0,0}M_{0,\eta} \subseteq (M_{0*\eta}(a) \cap M_{0*0}(b)) + (M_{0*0}(a) \cap M_{0*\eta}(b)) = (M_\eta(a) \cap M_0(b)) + (M_0(a) \cap M_\eta(b)) = M_{\eta,0} + M_{0,\eta} = M_\eta(x)$. We have shown that $0 \diamond \eta = \{\eta\}$.

Next, the third row:

- $M_{2\eta}(x)M_{2\eta}(x) = M_{\eta,\eta}M_{\eta,\eta} \subseteq M_{\eta*\eta}(a) \cap M_{\eta*\eta}(b) = (M_1(a) + M_0(a)) \cap (M_1(b) + M_0(b)) = M_{1,1} + M_{1,0} + M_{0,1} + M_{0,0} = M_1(x) + M_0(x)$. (Here we used Lemma 4.9 for the first time.) Therefore, $2\eta \diamond 2\eta = \{1, 0\}$.

- $M_{2\eta}(x)M_\eta(x) = M_{\eta,\eta}(M_{\eta,0} + M_{0,\eta}) \subseteq M_{\eta,\eta}M_{\eta,0} + M_{\eta,\eta}M_{0,\eta} \subseteq (M_{\eta*\eta}(a) \cap M_{\eta*0}(b)) + (M_{\eta*0}(a) \cap M_{\eta*\eta}(b)) = ((M_1(a) + M_0(a)) \cap M_\eta(b)) + (M_\eta(a) \cap (M_1(b) + M_0(b))) = M_{1,\eta} + M_{0,\eta} + M_{\eta,1} + M_{\eta,0} = M_{0,\eta} + M_{\eta,0} = M_\eta(x)$. Thus, $2\eta \diamond \eta = \{\eta\}$.

Finally, there is only one entry in the fourth row for us to check:

- $M_\eta(x)M_\eta(x) = (M_{\eta,0} + M_{0,\eta})(M_{\eta,0} + M_{0,\eta}) \subseteq M_{\eta,0}M_{\eta,0} + M_{\eta,0}M_{0,\eta} + M_{0,\eta}M_{\eta,0} + M_{0,\eta}M_{0,\eta} \subseteq (M_{\eta*\eta}(a) \cap M_{0*0}(b)) + (M_{\eta*0}(a) \cap M_{0*\eta}(b)) + (M_{0*\eta}(a) \cap M_{\eta*0}(b)) + (M_{0*0}(a) \cap M_{\eta*\eta}(b)) = ((M_1(a) + M_0(a)) \cap M_0(b)) + (M_\eta(a) \cap M_\eta(b)) + (M_\eta(a) \cap M_\eta(b)) + (M_0(a) \cap (M_1(b) + M_0(b))) = M_{1,0} + M_{0,0} + M_{\eta,\eta} + M_{\eta,\eta} + M_{0,1} + M_{0,0} = M_1(x) + M_0(x) + M_{2\eta}(x)$.
  Therefore, $\eta \diamond \eta = \{1, 0, 2\eta\}$.

This completes the proof of the proposition. $\qquad\square$

The fusion law $\mathcal{M}(2\eta, \eta)$, like any law $\mathcal{M}(\alpha, \beta)$, is graded by the group C_2. Hence for each double axis $x \in M$ we have the corresponding Miyamoto involution τ_x.

Proposition 4.10. *For a double axis $x = a + b$, we have $\tau_x = \tau_a\tau_b$.*

Proof. The involution τ_a acts as identity on $M_1(a) + M_0(a)$ and as minus identity on $M_\eta(a)$. Similarly, τ_b acts as identity on $M_1(b) + M_0(b)$ and as minus identity on $M_\eta(b)$. Also, τ_x acts as identity on $M_1(x) + M_0(x) + M_{2\eta}(x)$ and as minus identity on $M_\eta(x)$.

Let us compare the action of τ_x and $\tau_a\tau_b$. Clearly, τ_x, τ_a and τ_b act as identity on $M_1(x) + M_0(x) = M_{1,0} + M_{0,1} + M_{0,0}$ and so the actions agree here. On $M_{2\eta}(x) = M_{\eta,\eta}$, τ_x acts as identity, while both τ_a and τ_b act as minus identity. However, this means that $\tau_a\tau_b$ acts as identity, and so τ_x and $\tau_a\tau_b$ agree on $M_{2\eta}(x)$. Finally, τ_x acts as minus identity on $M_\eta(x) = M_{\eta,0} + M_{0,\eta}$. On the first summand, τ_a acts as minus identity and τ_b as identity. Hence, $\tau_a\tau_b$ acts as minus identity, agreeing with τ_x. Similarly, on the second summand, τ_a acts as identity and τ_b as minus identity. Hence again $\tau_a\tau_b$ agrees with τ_x. $\qquad\square$

Note that double axes are not primitive, as $M_1(x) = \langle a, b \rangle$ is 2-dimensional. However, x may be primitive within a proper subalgebra of the algebra M. We will explore this idea in the remaining sections.

5   2-Generated subalgebras

Manifestly, the Jordan type fusion law $\mathcal{J}(\eta)$ is a minor (sublaw) of the Monster type fusion law $\mathcal{M}(2\eta, \eta)$. This means that single axes satisfy the same

fusion law $\mathcal{M}(2\eta, \eta)$ as double axes, except that the 2η-eigenspace for them is trivial. Therefore, any subalgebra A of M that is generated by single and double axes is an algebra of Monster type $(2\eta, \eta)$ and it is primitive as long as the double axes we use are primitive in A. The principal goal of this section is to find all primitive subalgebras in Matsuo algebras M, generated by two axes, at least one of which is a double axis.

As in Section 4, $M = M_\eta(\Gamma)$ is a Matsuo algebra coming from a Fischer space Γ, and (G, D) is the 3-transposition group leading to Γ. Recall from Section 3 that M admits a Frobenius form $(\cdot, \cdot)$ and, clearly, this form is inherited by any subalgebra $A \subseteq M$.

5.1   Flip

We start by showing that primitivity of the subalgebra A forces a certain symmetry in the underlying set of single axes.

Definition 5.1. *The support* $\mathrm{supp}(x)$ *of a single axis* $x = a$ *is the set* $\{a\}$ *and the support* $\mathrm{supp}(y)$ *of a double axis* $y = b + c$ *is the set* $\{b, c\}$. *More generally, the* support $\mathrm{supp}(Y)$ *of a set* Y *of single and double axes is the union of supports of all axes from* Y.

Manifestly, $\mathrm{supp}(Y)$ is a subset of the basis D of M. By the *diagram* on the set $Z \subseteq D$ we mean the graph (denoted by the same symbol Z) whose vertex set is Z and where two vertices are adjacent if and only if they are collinear as points of the Fischer space Γ.

In what follows, we use double angular brackets $\langle\!\langle \ \rangle\!\rangle$ to indicate subalgebra generation, leaving single brackets for the linear span.

Proposition 5.2. *Suppose that* $x = a + b$ *is a double axis and* y *is any axis. If* x *is primitive in* $A = \langle\!\langle x, y \rangle\!\rangle$ *then* $(a, y) = (b, y)$.

Proof. Decompose y with respect to ad_x as $y = y_1 + y_0 + y_{2\eta} + y_\eta$, where each y_λ is contained in $M_\lambda(x)$. In particular, $y_1 \in M_1(x) = \langle a, b \rangle$. Hence $y_1 = \alpha a + \beta b$. Since $a \in M_1(x)$ and the eigenspaces $M_\lambda(x)$ are pairwise orthogonal, we deduce that $(a, y) = (a, y_1) = (a, \alpha a + \beta b) = \alpha(a, a) + \beta(a, b) = \alpha$, since $(a, a) = 1$ and $(a, b) = 0$ (because $b \in M_0(a)$). Similarly, $(b, y) = \beta$.

Now recall that every component y_λ is contained in every subalgebra containing x and y. In particular, $y_1 \in A_1(x)$. If x is primitive then $A_1(x) = \langle x \rangle = \langle a + b \rangle$. This means that $y_1 = \alpha a + \beta b$ is a multiple of $a + b$, and so $\alpha = \beta$, implying that $(a, y) = (b, y)$. $\qquad \square$

Corollary 5.3. *Under the assumptions of Proposition 5.2, if $x \neq y$ and x is primitive in $A = \langle\!\langle x, y \rangle\!\rangle$ then $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$ and, furthermore, a and b are adjacent in the diagram on $Z = \operatorname{supp}(\{x, y\})$ to the same number of vertices from $\operatorname{supp}(y)$.*

Proof. Indeed, if $u \in \{a, b\}$ is contained in $\operatorname{supp}(y)$ then $(u, y) = 1$; if u is adjacent to one single axis in $\operatorname{supp}(y)$ then $(u, y) = \frac{\eta}{2}$; and if u is adjacent to two single axes in $\operatorname{supp}(y)$ (when y is a double axis) then $(u, y) = \eta$. This follows from the description of the Frobenius form on M given in Section 3.

Finally, we remark that if both a and b are contained in $\operatorname{supp}(y)$ then $x = y$, a contradiction. $\qquad\square$

We can now show the symmetry forced by primitivity.

Definition 5.4. *Let Y be a set of single and double axes such that $\langle\!\langle Y \rangle\!\rangle$ is primitive. The* flip *is the permutation of $Z = \operatorname{supp}(Y)$ that fixes every single axis from Y and switches the two single axes in the support of each double axis from Y.*

Note that by Corollary 5.3 different axes from Y have disjoint supports. So the flip is well-defined.

Theorem 5.5. *If the subalgebra $A = \langle\!\langle Y \rangle\!\rangle$ is primitive then the flip is an automorphism of the diagram on $Z = \operatorname{supp}(Y)$.*

Proof. Consider a double axis $x = a + b \in Y$ and let y be any other axis from Y. According to Corollary 5.3, a and b are adjacent in the diagram to the same number $s \in \{0, 1, 2\}$ of single axes from $\operatorname{supp}(y)$. If $s = 0$ or 2 then clearly the flip preserves edges between $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$. If $s = 1$ then the only non-symmetric situation is where y is a double axis and a and b are adjacent to the same single axis in $\operatorname{supp}(y)$. However, switching the rôles of x and y, we see that this situation is impossible. $\qquad\square$

Clearly, this theorem restricts the diagrams that can lead to primitive subalgebras. We now turn to 2-generated primitive subalgebras.

5.2 Setup

Let $A \subseteq M$ be generated by two axes x and y. When both x and y are single, we are in the Matsuo algebra situation and so we know what $A = \langle\!\langle x, y \rangle\!\rangle$ is; namely, $A \cong 2B$ or $3C(\eta)$.

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Thus, we focus on the situation, where at least one axis is a double axis. There are two cases to consider: (A) $x = a$ is a single axis and $y = b+c$ is a double axis; and (B) both $x = a+b$ and $y = c+d$ are double axes.

Let (H, C) be the 3-transposition group, where $H = \langle \operatorname{supp}(\{x, y\}) \rangle$ and so $H = \langle a, b, c \rangle$ in case (A) and $H = \langle a, b, c, d \rangle$ in case (B). The normal set of involutions C is given by $C = \{a^H\} \cup \{b^H\} \cup \{c^H\}$ and $C = \{a^H\} \cup \{b^H\} \cup \{c^H\} \cup \{d^H\}$ in the respective cases. Let Σ be the Fischer space of (H, C). It is clear that the subalgebra $A = \langle\!\langle x, y \rangle\!\rangle$ is fully contained in the Matsuo subalgebra $M_\eta(\Sigma)$ of M. In view of this, we can assume that $\Gamma = \Sigma$ and so $(G, D) = (H, C)$.

Then the group G, being generated by at most four 3-transpositions, belongs to a short list of small groups. They have been classified by Fischer [6] for three generators and by Zara [41] for four generators. (Also available is the list of 5-generated 3-transposition groups due to Hall and Soicher [19].) We will look at all possible configurations, organizing them according to the diagram on the set $Z = \operatorname{supp}(\{x, y\})$. In view of Theorem 5.5, we only need to consider diagrams that admit the flip symmetry.

The diagram represents a Coxeter group $\hat{G}$ given by the presentation encoded in the diagram. Clearly, G is a factor group of $\hat{G}$ and, in many cases, $G = \hat{G}$. These will be the easier cases, and in the remaining harder cases we will have to find additional relations identifying G as a factor group of the Coxeter group $\hat{G}$.

Lastly, where possible, we will embed the group G into the symmetric group S_n, so that the involutions $a, b, \ldots$ fall into the class of transpositions. This embeds the Fischer space Γ into the Fischer space of S_n and it will significantly simplify our notation.

5.3 Type A

Here we deal with case (A). Recall that $x = a$, $y = b + c$, $Z = \{a, b, c\}$, $H = \langle Z \rangle = \langle a, b, c \rangle$. Since $y = b + c$ is a double axis, the axes b and c satisfy $bc = 0$, that is, they are not collinear in the Fischer space Γ. Thus, we only have diagrams on Z shown in Figure 1.

The second diagram, A_2, is not symmetric with respect to the flip, hence we discard it. Let us look at each of the remaining two diagrams in turn.

Diagram A_1: Here the Coxeter group $\hat{G}$ is an elementary abelian 2-group of rank 3 and $G = \hat{G}$ or G is the unique factor of $\hat{G}$ of order 2^2, in which
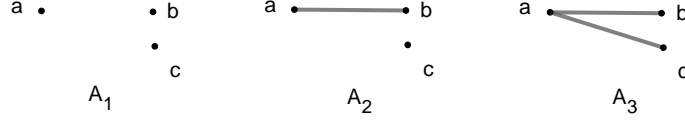
Figure 1: A single axis and a double axis

the three generators of $\hat{G}$ remain distinct. In either case, $xy = a(b+c) = ab + ac = 0 + 0 = 0$, so $A = 2B \cong \mathbb{F}^2$ and y is primitive in A.

Diagram A_3: Here $\hat{G} \cong S_4$ and $G = \hat{G}$, as G cannot be isomorphic to the factor groups S_3 and C_2 of S_4. We identify G with S_4 by equating a with $(1,2)$, b with $(1,3)$ and c with $(2,4)$.

The algebra A is invariant under $\tau_x = \tau_a$, acting on D as conjugation by $(1,2)$ (c.f. the discussion in the penultimate paragraph of Section 3), and under $\tau_y = \tau_b \tau_c$, acting as conjugation by $(1,3)(2,4)$ (see Proposition 4.10). Hence $z := x^{\tau_y} = (1,2)^{(1,3)(2,4)} = (3,4)$ is in A and also $t := y^{\tau_x} = ((1,3) + (2,4))^{(1,2)} = (2,3) + (1,4) = (1,4) + (2,3) \in A$. We claim that $\{x, z, y, t\}$ is a basis of A. Manifestly, these four vectors are linearly independent in M as their supports are disjoint. Hence we just need to check that their span is closed for the algebra product. Throughout the paper and especially in the multiplication tables, it will be convenient to use the notation s_i for single axes and d_i for double axes. Here we set $s_1 := x = (1,2)$, $s_2 := z = (3,4)$, $d_1 := y = (1,3) + (2,4)$ and $d_2 := t = (1,4) + (2,3)$.

Clearly, all four axes are idempotents and $s_1 s_2 = 0$. Also, $s_1 d_1 = (1,2)((1,3) + (2,4)) = \frac{\eta}{2}((1,2) + (1,3) - (2,3)) + \frac{\eta}{2}((1,2) + (2,4) - (1,4)) = \eta s_1 + \frac{\eta}{2} d_1 - \frac{\eta}{2} d_2 = \frac{\eta}{2}(2s_1 + d_1 - d_2)$. Applying the automorphisms τ_x (switching d_1 and d_2) and τ_y (switching s_1 and s_2), we immediately obtain the values of $s_1 d_2$, $s_2 d_1$, and $s_2 d_2$. Finally, $d_1 d_2 = ((1,3) + (2,4))((1,4) + (2,3)) = \frac{\eta}{2}((1,3) + (1,4) - (3,4)) + \frac{\eta}{2}((1,3) + (2,3) - (1,2)) + \frac{\eta}{2}((2,4) + (1,4) - (1,2)) + \frac{\eta}{2}((2,4) + (2,3) - (3,4)) = \eta(-s_1 - s_2 + d_1 + d_2)$. Thus, indeed A is 4-dimensional with basis $\{s_1, s_2, d_1, d_2\}$ and the multiplication table is as in Table 6. Clearly, $b, c \notin A$, so $d_1 = y = b + c$ is primitive in A. We will denote this algebra $Q_2(\eta)$. It is part of an infinite series developed in Section 8.

Let us briefly discuss the properties of this new algebra. We will see in Section 8 that the algebras $Q_k(\eta)$ are simple except when η belongs to a short list of exceptional values. In this case, the only such value is $-\frac{1}{2}$.

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	s_1	s_2	d_1	d_2
s_1	s_1	0	$\frac{\eta}{2}(2s_1 + d_1 - d_2)$	$\frac{\eta}{2}(2s_1 + d_2 - d_1)$
s_2	0	s_2	$\frac{\eta}{2}(2s_2 + d_1 - d_2)$	$\frac{\eta}{2}(2s_2 + d_2 - d_1)$
d_1	$\frac{\eta}{2}(2s_1 + d_1 - d_2)$	$\frac{\eta}{2}(2s_2 + d_1 - d_2)$	d_1	$\eta(-s_1 - s_2 + d_1 + d_2)$
d_2	$\frac{\eta}{2}(2s_1 + d_2 - d_1)$	$\frac{\eta}{2}(2s_2 + d_2 - d_1)$	$\eta(-s_1 - s_2 + d_1 + d_2)$	d_2

Table 6: The 4-dimensional algebra $Q_2(\eta)$

When $\eta = -\frac{1}{2}$, the algebra $Q_2(\eta)$ has a 1-dimensional radical and the 3-dimensional factor algebra over the radical is simple. Note also that, when F is of characteristic 3, we have that $-\frac{1}{2} = 1$, and so this situation cannot occur.

To summarize, in case (A) we found two primitive algebras: diagram A_1 leads to the familiar 2-dimensional algebra $2B$ and diagram A_3 leads to the new 4-dimensional algebra $Q_2(\eta)$.

5.4   Type B

Now we deal with case (B), where $x = a + b$, $y = c + d$, and $Z = \{a, b, c, d\}$. Here a and b are non-collinear in Γ and also c and d are non-collinear. Hence the diagram on Z belongs to the list shown in Figure 2.
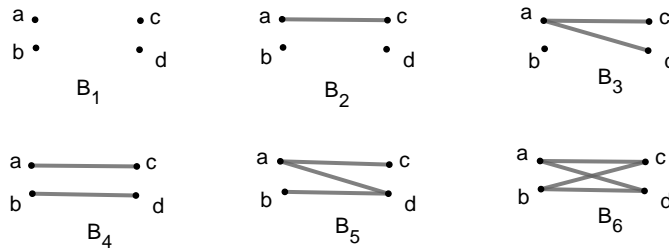

Figure 2: Two double axes

Diagrams B_2, B_3, and B_5 do not admit the flip. Hence we discard them. Again, we do the remaining diagrams one at a time.

Diagram B$_1$: In this first case, $\hat{G}$ is an elementary abelian 2-group of rank 4 and $G = \hat{G}$ or a factor group of $\hat{G}$ of rank 3, in which the four generators of $\hat{G}$ remain distinct. Furthermore, in all these cases, $xy = (a+b)(c+d) = ac + ad + bc + bd = 0 + 0 + 0 + 0 = 0$, and so A is the familiar algebra $2B$.

Diagram B$_4$: Here Γ has two components, lines $\{a, c, e\}$ and $\{b, d, f\}$. Note that $z := x^{\tau_y} = (a+b)^{\tau_c \tau_d} = e + f = (c+d)^{\tau_a \tau_b} = y^{\tau_x}$ must be in A. Furthermore, $xy = (a+b)(c+d) = ac + ad + bc + bd = ac + 0 + 0 + bd = \frac{\eta}{2}(a + c - e) + \frac{\eta}{2}(b + d - f) = \frac{\eta}{2}(x + y - z)$. Applying τ_x and τ_y, we also obtain equalities $xz = \frac{\eta}{2}(x + z - y)$ and $yz = \frac{\eta}{2}(y + z - x)$, showing that A is isomorphic to the 3-dimensional Matsuo algebra $3C(\eta)$. Clearly, x and y are both primitive in A.

Diagram B$_6$: Here the Coxeter group $\hat{G}$ defined by the diagram is infinite and so we need an extra relation to identify G. This extra relation comes from the fact that the order p of $a^d b^c$ must also be in $\{1, 2, 3\}$. Let $\hat{G}(p)$ be the factor group of $\hat{G}$ over the normal subgroup generated by $(a^d b^c)^p$. All three groups $\hat{G}(p)$, $p \in \{1, 2, 3\}$, are finite and their structure is well-known. In fact, they are all 3-transposition groups. We note that $\hat{G}(1) \cong S_4$ is the factor group of both $\hat{G}(2)$ and $\hat{G}(3)$. So we just need to describe the latter two groups. Let $p \in \{2, 3\}$. Consider the 4-dimensional permutational module V of S_4 over $\mathbb{F}_p$. Let the vectors e_i, $i \in \{1, 2, 3, 4\}$, form the basis of V permuted by S_4 and $U \subseteq V$ be the 3-dimensional "sum-zero" submodule of V. Then $\hat{G}(p)$ is isomorphic to the semi-direct product $U : S_4$. Note that, for $p = 2$, U contains a 1-dimensional "all-one" submodule D, which is the center of $\hat{G}(2)$. When $p = 3$, U is irreducible. In both cases, U is the unique minimal non-central normal subgroup of $\hat{G}(p)$ and $\hat{G}(p)/U \cong \hat{G}(1) \cong S_4$. Since S_4 does not have proper factor groups containing commuting involutions, we conclude that, up to the center (which does not influence the Fischer space of G), we have that $G = \hat{G}(p)$ for $p = 1$, 2, or 3.

We consider these possibilities in turn.

For $p = 1$, there is a unique, up to conjugation, identification of the generators a, b, c, and d of G with involutions in S_4. We set $a = (1, 2)$, $b = (3, 4)$, $c = (1, 3)$, and $d = (2, 4)$. Then $x = (1, 2) + (3, 4)$, $y = (1, 3) + (2, 4)$. We also set $z := (1, 4) + (2, 3)$. Then x, y, and z span A, which is isomorphic to the 3-dimensional Matsuo algebra $3C(2\eta)$. Indeed, $x \cdot y = ((1, 2) + (3, 4)) \cdot ((1, 3) + (2, 4)) = (1, 2) \cdot (1, 3) + (1, 2) \cdot (2, 4) + (3, 4) \cdot (1, 3) + (3, 4) \cdot (2, 4) =$

$\frac{\eta}{2}((1,2)+(1,3)-(2,3))+\frac{\eta}{2}((1,2)+(2,4)-(1,4))+\frac{\eta}{2}((3,4)+(1,3)-(1,4))+$
$\frac{\eta}{2}((3,4)+(2,4)-(2,3))=\eta(x+y-z)$, and similarly for the other products.
Clearly, x and y are primitive in $A=3C(2\eta)$.

For $p=2$, the Fischer space $\Gamma(2)$ of $\hat{G}(2)=U:S_4\leq E:S_4$ consists of $2\cdot 6=12$ points: $b_{i,j}=(i,j)$ and $c_{i,j}=(e_i+e_j)(i,j)$, for $1\leq i<j\leq 4$; and $4\cdot 4=16$ lines, where each "b" line $\{b_{i,j},b_{i,k},b_{j,k}\}$, $1\leq i<j<k\leq 4$, is complemented by three "bc" lines $\{b_{i,j},c_{i,k},c_{j,k}\}$, $\{b_{i,k},c_{i,j},c_{j,k}\}$, and $\{b_{j,k},c_{i,j},c_{i,k}\}$. We can identify our generators a, b, c, and d with involutions in $\hat{G}(2)$ as follows: $a=c_{1,2}=(e_1+e_2)(1,2)$, $b=b_{3,4}=(3,4)$, $c=b_{1,3}=(1,3)$ and $d=b_{2,4}=(2,4)$. Then $a^d b^c=e_1+e_4$ is of order 2. Therefore, this indeed provides the required isomorphism from G onto $\hat{G}(2)$.

Recall from the last paragraph of Section 3 how the Miyamoto involutions act on the Fischer space. Since A is invariant under τ_x and τ_y, the following elements lie in A:

$d_1 := x = c_{1,2}+b_{3,4}$,
$d_2 := x^{\tau_y} = ((c_{1,2}+b_{3,4})^{\tau_c})^{\tau_d} = (c_{2,3}+b_{1,4})^{\tau_d} = c_{3,4}+b_{1,2} = b_{1,2}+c_{3,4}$,
$d_3 := y = b_{1,3}+b_{2,4}$,
$d_4 := y^{\tau_x} = ((b_{1,3}+b_{2,4})^{\tau_a})^{\tau_b} = (c_{2,3}+c_{1,4})^{\tau_b} = c_{2,4}+c_{1,3} = c_{1,3}+c_{2,4}$.

Additionally, A is closed for the algebra product and so it also contains:

$$
\begin{aligned}
w := &-\frac{2}{\eta}xy+2(x+y)\\
= &-\frac{2}{\eta}(c_{1,2}+b_{3,4})(b_{1,3}+b_{2,4})+2(c_{1,2}+b_{3,4}+b_{1,3}+b_{2,4})\\
= &-((c_{1,2}+b_{1,3}-c_{2,3})+(c_{1,2}+b_{2,4}-c_{1,4})+(b_{3,4}+b_{1,3}-b_{1,4})\\
&+(b_{3,4}+b_{2,4}-b_{2,3}))+2(c_{1,2}+b_{3,4}+b_{1,3}+b_{2,4})\\
= &c_{2,3}+c_{1,4}+b_{1,4}+b_{2,3}=b_{1,4}+c_{1,4}+b_{2,3}+c_{2,3}.
\end{aligned}
$$

The five elements above have disjoint support and so they are linearly independent. We skip the straightforward calculation of the products and simply present them in Table 7. In particular, A is 5-dimensional with basis $\{d_1,d_2,d_3,d_4,w\}$. It is a new algebra and both x and y are primitive in it, since A contains none of the single axes a, b, c, and d.

We check this algebra for simplicity using GAP [13] and the structure theory from Subsection 2.2. We will provide details for this algebra and skip

		d_1	d_2	d_3	d_4	w
d_1		d_1	0	$\frac{\eta}{2}(2d_1+2d_3-w)$	$\frac{\eta}{2}(2d_1+2d_4-w)$	$\eta(2d_1-d_3-d_4+w)$
d_2		0	d_2	$\frac{\eta}{2}(2d_2+2d_3-w)$	$\frac{\eta}{2}(2d_2+2d_4-w)$	$\eta(2d_2-d_3-d_4+w)$
d_3		$\frac{\eta}{2}(2d_1+2d_3-w)$	$\frac{\eta}{2}(2d_2+2d_3-w)$	d_3	0	$\eta(2d_3-d_1-d_2+w)$
d_4		$\frac{\eta}{2}(2d_1+2d_4-w)$	$\frac{\eta}{2}(2d_2+2d_4-w)$	0	d_4	$\eta(2d_4-d_1-d_2+w)$
w		$\eta(2d_1-d_3-d_4+w)$	$\eta(2d_2-d_3-d_4+w)$	$\eta(2d_3-d_1-d_2+w)$	$\eta(2d_4-d_1-d_2+w)$	w

Table 7: The 5-dimensional algebra

them for the later examples. The Gram matrix for the Frobenius form with respect to the basis $\{d_1, d_2, d_3, d_4, w\}$ can easily be found and it is as follows:

$$
\begin{pmatrix}
2 & 0 & 2\eta & 2\eta & 4\eta \\
0 & 2 & 2\eta & 2\eta & 4\eta \\
2\eta & 2\eta & 2 & 0 & 4\eta \\
2\eta & 2\eta & 0 & 2 & 4\eta \\
4\eta & 4\eta & 4\eta & 4\eta & 4
\end{pmatrix}.
$$

Manifestly, the projection graph on $\{x, y\} = \{d_1, d_3\}$ is connected. This means that the algebra has no proper ideals containing axes. The determinant of the above Gram matrix is $1024\eta^3 - 768\eta^2 + 64$ and it has roots $\frac{1}{2}$ (twice) and $-\frac{1}{4}$. As $\eta \neq \frac{1}{2}$ by assumption, the only special value is $\eta = -\frac{1}{4}$, for which the radical of the Frobenius form is 1-dimensional. The factor over the radical is a simple 4-dimensional algebra.

Note that if the characteristic of $\mathbb{F}$ is 3 then $-\frac{1}{4} = \frac{1}{2}$ and if the characteristic is 5 then $-\frac{1}{4} = 1$. So for these characteristics the 5-dimensional algebra is always simple.

Let, finally, $p = 3$. We start again by describing the Fischer space $\Gamma(3)$ of $\hat{G}(3) = U : S_4 \leq E : S_4$, where E is now the permutational module over $\mathbb{F}_3$. The Fischer space has $3 \cdot 6 = 18$ points. Namely, for each point $b_{i,j} = (i, j) = b_{j,i}$, contained in the complement S_4, there are two further points $c_{i,j} = (e_i - e_j)(i, j)$ and $c_{j,i} = (e_j - e_i)(i, j)$. The lines in $\Gamma(3)$ are of several types. First, for each $1 \leq i < j \leq 4$, the triple (1) $\{b_{i,j}, c_{i,j}, c_{j,i}\}$ is a line. Secondly, for distinct i, j, and k in $\{1, 2, 3, 4\}$, the triples (2) $\{b_{i,j}, b_{i,k}, b_{j,k}\}$, (3) $\{b_{i,j}, c_{i,k}, c_{j,k}\}$, (4) $\{b_{j,k}, c_{i,j}, c_{i,k}\}$, and (5) $\{c_{i,j}, c_{j,k}, c_{k,i}\}$

are lines. This gives the total of 42 lines of $\Gamma(3)$, including six lines of type (1), four lines of type (2), twelve lines of type (3), twelve lines of type (4), and eight lines of type (5). This information is used to multiply in M and to act by the Miyamoto involutions, as described in Section 3.

We identify G with $\hat{G}(3)$ by taking $a = c_{1,2}$, $b = b_{3,4}$, $c = b_{1,3}$, and $d = c_{2,4}$. Then $a^d b^c = c_{1,2}^{\tau_d} b_{3,4}^{\tau_c} = c_{4,1} b_{1,4} = (e_4 - e_1)(1,4)(1,4) = e_4 - e_1$ is of order 3, so the required relation hold. Since A is invariant under τ_x and τ_y, the following elements lie in A:

$$d_1 := x = c_{1,2} + b_{3,4},$$
$$d_2 := x^{\tau_y} = ((c_{1,2} + b_{3,4})^{\tau_c})^{\tau_d} = (c_{3,2} + b_{1,4})^{\tau_d} = c_{4,3} + c_{2,1} = c_{2,1} + c_{4,3},$$
$$d_3 := d_2^{\tau_x} = ((c_{2,1} + c_{4,3})^{\tau_a})^{\tau_b} = (b_{1,2} + c_{4,3})^{\tau_b} = b_{1,2} + c_{3,4},$$
$$d_4 := y = b_{1,3} + c_{2,4},$$
$$d_5 := y^{\tau_x} = ((b_{1,3} + c_{2,4})^{\tau_a})^{\tau_b} = (c_{3,2} + c_{4,1})^{\tau_b} = c_{4,2} + c_{3,1} = c_{3,1} + c_{4,2},$$
$$d_6 := d_5^{\tau_y} = ((c_{3,1} + c_{4,2})^{\tau_c})^{\tau_d} = (c_{1,3} + c_{4,2})^{\tau_d} = c_{1,3} + b_{2,4},$$

$$u := -\frac{1}{\eta} x d_6 + x + d_6$$
$$= -\frac{1}{\eta}(c_{1,2} + b_{3,4})(c_{1,3} + b_{2,4}) + c_{1,2} + b_{3,4} + c_{1,3} + b_{2,4}$$
$$= -\frac{1}{2}((c_{1,2} + c_{1,3} - b_{2,3}) + (c_{1,2} + b_{2,4} - c_{1,4}) + (b_{3,4} + c_{1,3} - c_{1,4})$$
$$\quad + (b_{3,4} + b_{2,4} - b_{2,3})) + c_{1,2} + b_{3,4} + c_{1,3} + b_{2,4}$$
$$= c_{1,4} + b_{2,3},$$

$$w := -\frac{2}{\eta} xy + 2(x + y)$$
$$= -\frac{2}{\eta}(c_{1,2} + b_{3,4})(b_{1,3} + c_{2,4}) + 2(c_{1,2} + b_{3,4} + b_{1,3} + c_{2,4})$$
$$= -((c_{1,2} + b_{1,3} - c_{3,2}) + (c_{1,2} + c_{2,4} - c_{4,1}) + (b_{3,4} + b_{1,3} - b_{1,4})$$
$$\quad + (b_{3,4} + c_{2,4} - c_{2,3})) + 2(c_{1,2} + b_{3,4} + b_{1,3} + c_{2,4})$$
$$= c_{3,2} + c_{4,1} + b_{1,4} + c_{2,3} = b_{1,4} + c_{4,1} + c_{2,3} + c_{3,2}.$$

These eight elements have disjoint support hence they are linearly independent. Calculations show that the pairwise products of these elements are as shown in Table 8. So this is a basis of the (new) subalgebra A. Since a, b, c, and d are not in A, both x and y are primitive in A.

We build in GAP the Gram matrix of the Frobenius form and find its determinant, which turns out to be $-448\eta^8 + 4864\eta^7 - 21808\eta^6 + 51664\eta^5 - $

	d_1	d_2	d_3	d_4	d_5	d_6	u	w
d_1	d_1	$\frac{\eta}{2}(d_1+d_2-d_3)$	$\frac{\eta}{2}(d_1+d_3-d_2)$	$\frac{\eta}{2}(2d_1+2d_4-w)$	$\frac{\eta}{2}(2d_1+2d_5-w)$	$\eta(d_1+d_6-u)$	$\eta(d_1-d_6+u)$	$\eta(2d_1+w)$ $-\eta(d_4+d_5)$
d_2	$\frac{\eta}{2}(d_1+d_2-d_3)$	d_2	$\frac{\eta}{2}(d_2+d_3-d_1)$	$\frac{\eta}{2}(2d_2+2d_4-w)$	$\eta(d_2+d_5-u)$	$\frac{\eta}{2}(2d_2+2d_6-w)$	$\eta(d_2-d_5+u)$	$\eta(2d_2+w)$ $-\eta(d_4+d_6)$
d_3	$\frac{\eta}{2}(d_1+d_3-d_2)$	$\frac{\eta}{2}(d_2+d_3-d_1)$	d_3	$\eta(d_3+d_4-u)$	$\frac{\eta}{2}(2d_3+2d_5-w)$	$\frac{\eta}{2}(2d_3+2d_6-w)$	$\eta(d_3-d_4+u)$	$\eta(2d_3+w)$ $-\eta(d_6+d_5)$
d_4	$\frac{\eta}{2}(2d_1+2d_4-w)$	$\frac{\eta}{2}(2d_2+2d_4-w)$	$\eta(d_4+d_3-u)$	d_4	$\frac{\eta}{2}(d_4+d_5-d_6)$	$\frac{\eta}{2}(d_4+d_6-d_5)$	$\eta(d_4-d_3+u)$	$\eta(2d_4+w)$ $-\eta(d_1+d_2)$
d_5	$\frac{\eta}{2}(2d_1+2d_5-w)$	$\eta(d_2+d_5-u)$	$\frac{\eta}{2}(2d_3+2d_5-w)$	$\frac{\eta}{2}(d_4+d_5-d_6)$	d_5	$\frac{\eta}{2}(d_5+d_6-d_4)$	$\eta(d_5-d_2+u)$	$\eta(2d_5+w)$ $-\eta(d_1+d_3)$
d_6	$\eta(d_1+d_6-u)$	$\frac{\eta}{2}(2d_2+2d_6-w)$	$\frac{\eta}{2}(2d_6+2d_3-w)$	$\frac{\eta}{2}(d_4+d_6-d_5)$	$\frac{\eta}{2}(d_5+d_6-d_4)$	d_6	$\eta(d_6-d_1+u)$	$\eta(2d_6+w)$ $-\eta(d_2+d_3)$
u	$\eta(d_1-d_6+u)$	$\eta(d_2-d_5+u)$	$\eta(d_3-d_4+u)$	$\eta(d_4-d_3+u)$	$\eta(d_5-d_2+u)$	$\eta(d_6-d_1+u)$	u	ηu
w	$\eta(2d_1+w)$ $-\eta(d_4+d_5)$	$\eta(2d_2+w)$ $-\eta(d_4+d_6)$	$\eta(2d_3+w)$ $-\eta(d_6+d_5)$	$\eta(2d_4+w)$ $-\eta(d_1+d_2)$	$\eta(2d_5+w)$ $-\eta(d_1+d_3)$	$\eta(2d_6+w)$ $-\eta(d_2+d_3)$	ηu	$(\eta+1)w-\eta u$

Table 8: The 8-dimensional algebra

$68320\eta^4+48256\eta^3-14848\eta^2+256\eta+512$. This has roots $-\frac{1}{7}$, $\frac{1}{2}$ of multiplicity 2, and 2 of multiplicity 5. Consequently, the algebra is simple unless $\eta = -\frac{1}{7}$ or $\eta = 2$. In the former case, the radical is 1-dimensional and the latter case it is 5-dimensional.

If the characteristic of $\mathbb{F}$ is 3 then both $-\frac{1}{7}$ and 2 are equal to $\frac{1}{2}$, so the algebra is always simple, as $\eta \neq \frac{1}{2}$. Another special characteristic for this algebra is 5, because then $-\frac{1}{7} = 2$.

5.5 Discussion

The list of 2-generated algebras of Monster type $(2\eta, \eta)$ found in this section consists of three versions of the algebra $2B = \mathbb{F}^2$ (generated by two single axes, one single and one double axes, and two double axes), two versions of $3C(\eta)$ (generated by two single axes and by two double axes), algebra $3C(2\eta)$ (two double axes only), the new 4-dimensional algebra $Q_2(\eta)$ (single and double axes), the new 5-dimensional algebra (two double axes), and the new 8-dimensional algebra (two double axes). We put forward the following question.

Question 5.6. *Is it true that the above is the complete list of 2-generated primitive algebras of Monster type $(2\eta, \eta)$?*

It is interesting to compare this with the list of eight Norton-Sakuma algebras, which is the complete list of 2-generated primitive algebras of Monster type $(\frac{1}{4}, \frac{1}{32})$. The most glaring difference is the absence of an equivalent of the 6-dimensional algebra $5A$, where $|\tau_x \tau_y| = 5$. Hence we also put forward the following partial case of the above question.

Question 5.7. *Is it true that there is no 2-generated primitive algebra $\langle\!\langle x, y \rangle\!\rangle$ of Monster type $(2\eta, \eta)$ satisfying $|\tau_x \tau_y| = 5$?*

In our list, we operate in terms of single and double axis, but these are only fully defined in the context of an enveloping Matsuo algebra M.

Question 5.8. *In an arbitrary primitive algebra of Monster type $(2\eta, \eta)$, can the axes be always split into single and double?*

In the presence of a Frobenius form, a possible approach to this question is via the length of the axes. Namely, it might be possible to scale the form so that all "single" axes x satisfy $(x, x) = 1$ and all "double" axes x satisfy $(x, x) = 2$.

Another idea is to base the distinction between "single" and "double" on the absence or presence of the 2η-eigenspace.

6   Some 3-generated subalgebras

The natural next step is to try and classify 3-generated primitive subalgebras. Again, we can assume that at least one of the three generators is a double axis. This gives us three further cases: (C) two single axes and one double axis; (D) one single axis and two double axes; and (E) three double axes.

It is easy to list the relevant diagrams on the support Z of the set $Y = \{x, y, z\}$ of three generating axes. Note that here we only need to list the diagrams admitting the flip as only they can lead to primitive algebras.

All possible symmetric diagrams for case (C) are shown in Figure 3. In this section we will completely enumerate the primitive algebras of Monster type $(2\eta, \eta)$ arising in this case. We also discuss the properties of these algebras.

The full list of symmetric diagrams arising in cases (D) and (E) are shown in Figures 4 and 5, respectively. These cases constitute an ongoing project and we are hoping to report on the results in a separate paper.
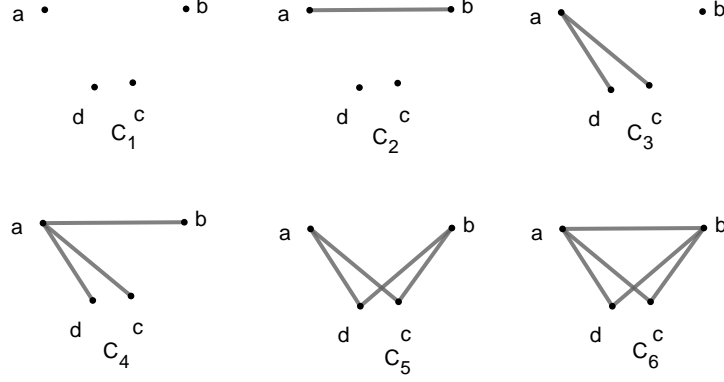
Figure 3: Two single axes and one double axis

6.1 Type C

In this case, $x = a$ and $y = b$ are single axes while $z = c + d$ is a double axis. Therefore, $Z = \mathrm{supp}(\{x, y, z\}) = \{a, b, c, d\}$.

Diagram C_1: In this first case, $\hat{G}$ is an elementary abelian group of order 2^4 and $G = \hat{G}$ or a factor group of $\hat{G}$ of order 2^3, in which the images of the four generators of $\hat{G}$ remain distinct. Futhermore, $xy = 0$, $xz = 0$, $yz = 0$, which means that A is isomorphic to $\mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} = \mathbb{F}^3$.

Diagram C_2: Here $\hat{G} \cong \langle a, b \rangle \times \langle c \rangle \times \langle d \rangle \cong S_3 \times C_2 \times C_2$ and it is easy to see that $G = \hat{G}$, as it cannot be any of the proper factor groups. Since $xz = 0$ and $yz = 0$, the algebra $A = \langle\!\langle x, y, z \rangle\!\rangle$ decomposes as the direct sum $\langle\!\langle x, y \rangle\!\rangle \oplus \langle\!\langle z \rangle\!\rangle \cong 3C(\eta) \oplus \mathbb{F}$.

Diagram C_3: In this case, $\hat{G} = \langle a, c, d \rangle \times \langle b \rangle \cong S_4 \times C_2$. Again, G can only be the full $\hat{G}$. The algebra A decomposes, in turn, as $\langle\!\langle x, z \rangle\!\rangle \oplus \langle\!\langle y \rangle\!\rangle \cong Q_2(\eta) \oplus \mathbb{F}$, where $Q_2(\eta)$ is the algebra from case A_3.

Diagram C_4: Here the diagram on Z coincides with the Coxeter diagram D_4 and hence $\hat{G}$ is isomorphic to the Weyl group $W(D_4) \cong 2^3 : S_4$ of order 192. We claim that $G = \hat{G}$ or $\hat{G}/Z(\hat{G})$. Indeed, $G_b := \langle a, c, d \rangle \leq G$ is a factor group of $W(A_3) \cong S_4$ containing distinct commuting transpositions. Hence $G_b \cong S_4$ and, similarly, $G_c := \langle a, b, d \rangle \cong S_4$ and $G_d := \langle a, b, c \rangle \cong S_4$.
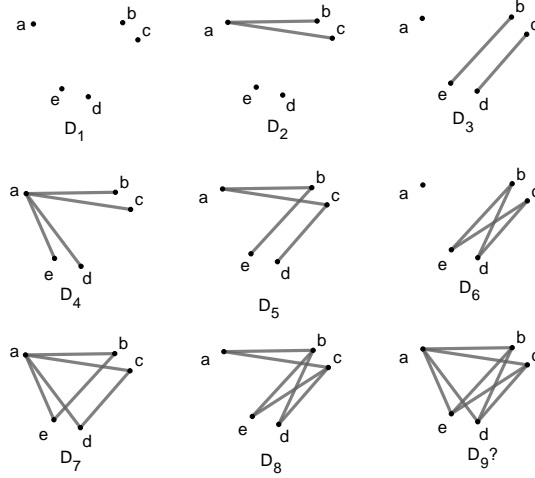
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Figure 4: One single axis and two double axes

Now we claim that $G_b \cap G_c = \langle a, d \rangle$. If not, then $G_b = G_c$, as $\langle a, d \rangle \cong S_3$ is maximal in both groups. Hence $b \in G_b$. We can identify a, c, and d with transpositions $(1,2)$, $(1,3)$, and $(2,4)$, respectively. Then b must be a transposition commuting with c and d, so b must be c or d; a contradiction.

Thus, $|G| \geq 24 + 24 - 6 = 42$ and hence G is a factor group of $\hat{G}$ over a normal subgroup of order at most $\frac{192}{42} < 5$. However, the only normal subgroup of $\hat{G}$ of order at most 4 is the center $Z(\hat{G})$, and so, indeed, $G = \hat{G}$ or $\hat{G}/Z(\hat{G})$. Note that these two groups have the same Fischer space, so we can select whichever one we prefer. Thus, we may assume that $G = \hat{G}$.

Note that the present $G = \hat{G}$ appeared in Section 5, in the case of diagram B_6, as the group $\hat{G}(2)$. In particular, $\hat{G}$ is isomorphic to the index 2 subgroup $U : S_4$ in the group $E : S_4$. Here E is the permutational module for S_4, with the basis $\{e_1, e_2, e_3, e_4\}$ permuted by the complement S_4, and U is the "sum-zero" submodule of E.

Recall the Fischer space of $\hat{G}(2)$ consisting of $2 \cdot 6 = 12$ points: $b_{i,j} = (i,j)$ and $c_{i,j} = (e_i + e_j)(i,j)$, for $1 \leq i < j \leq 4$; and $4 \cdot 4 = 16$ lines $\{b_{i,j}, b_{i,k}, b_{j,k}\}$, $\{b_{i,j}, c_{i,k}, c_{j,k}\}$, $\{b_{i,k}, c_{i,j}, c_{j,k}\}$, and $\{b_{j,k}, c_{i,j}, c_{i,k}\}$, for $1 \leq i < j < k \leq 4$.

We have already identified a, c, and d with $b_{1,2} = (1,2)$, $b_{1,3} = (1,3)$, and $b_{2,4} = (2,4)$, respectively. We can, for example, identify b with $c_{1,3} = (e_1 + e_3)(1,3)$, and then all relations are satisfied.
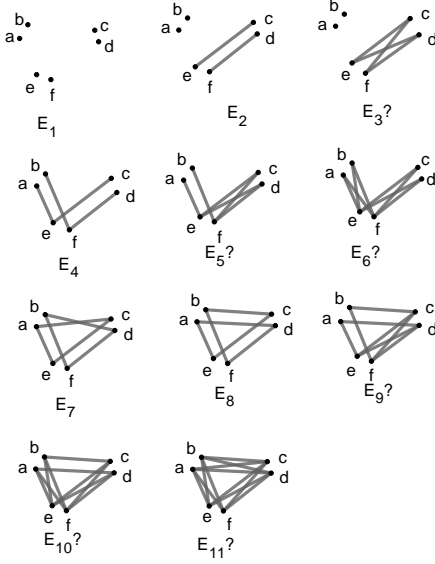
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Figure 5: Three double axes

Recall that $A = \langle\!\langle x, y, z \rangle\!\rangle$ is invariant under $\tau_x = \tau_a$, $\tau_y = \tau_b$, and $\tau_z = \tau_c\tau_d$. Recall also that the Miyamoto involution τ_f, for a point f, fixes f and every point non-collinear with f and it switches the two points other than f on each line through f.

Applying this, the algebra $A = \langle\!\langle x, y, z \rangle\!\rangle$ contains the following axes:

$$s_1 := x = a = b_{1,2},$$
$$s_2 := y = b = c_{1,3},$$
$$s_3 := x^{\tau_y} = y^{\tau_x} = c_{2,3},$$
$$s_4 := s_3^{\tau_z} = (c_{2,3}^{\tau_c})^{\tau_d} = c_{1,2}^{\tau_d} = c_{1,4},$$
$$s_5 := s_4^{\tau_x} = c_{1,4}^{\tau_x} = c_{2,4},$$
$$s_6 := x^{\tau_z} = (x^{\tau_c})^{\tau_d} = b_{2,3}^{\tau_d} = b_{3,4},$$
$$d_2 := z = b_{1,3} + b_{2,4},$$
$$d_3 := z^{\tau_x} = b_{2,3} + b_{1,4} = b_{1,4} + b_{2,3},$$
$$d_1 := d_3^{\tau_y} = c_{3,4} + c_{1,2} = c_{1,2} + c_{3,4}.$$

Here, as in Section 5, $s_1, \ldots, s_6$ are single axes and d_1, d_2, d_3 are double axes. (The order of the double axes is altered so that the multiplication

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table looks more symmetric.) We claim that the nine axes form a basis of A. The supports of the axes are disjoint and so they are certainly linearly independent. It remains to see that the subspace they span is closed for the algebra product. It is a straightforward calculation using the description of lines above and utilizing the substantial symmetry we have. Note that the six single axes span a subalgebra isomorphic to the Matsuo algebra of S_4, and so these products are immediate. Here is a sample calculation involving double axes: $d_1 d_3 = (c_{1,2} + c_{3,4})(b_{1,4} + b_{2,3}) = c_{1,2}b_{1,4} + c_{1,2}b_{2,3} + c_{3,4}b_{1,4} + c_{3,4}b_{2,3} = \frac{\eta}{2}(c_{1,2}+b_{1,4}-c_{2,4})+\frac{\eta}{2}(c_{1,2}+b_{2,3}-c_{1,3})+\frac{\eta}{2}(c_{3,4}+b_{1,4}-c_{1,3})+\frac{\eta}{2}(c_{3,4}+b_{2,3}-c_{2,4}) = \eta(d_1+d_3-s_2-s_5)$. By the way, this calculation indicates, using the symmetry, that the three double axes span a subalgebra $3C(2\eta)$.

The complete table of products is in Table 9. As a consequence, we

	s_1	s_2	s_3	s_4	s_5	s_6	d_1	d_2	d_3
s_1	s_1	$\frac{\eta}{2}(s_1 + s_2 - s_3)$	$\frac{\eta}{2}(s_1 + s_3 - s_2)$	$\frac{\eta}{2}(s_1 + s_4 - s_5)$	$\frac{\eta}{2}(s_1 + s_5 - s_4)$	0	0	$\frac{\eta}{2}(2s_1 + d_2 - d_3)$	$\frac{\eta}{2}(2s_1 + d_3 - d_2)$
s_2	$\frac{\eta}{2}(s_1 + s_2 - s_3)$	s_2	$\frac{\eta}{2}(s_2 + s_3 - s_1)$	$\frac{\eta}{2}(s_2 + s_4 - s_6)$	0	$\frac{\eta}{2}(s_2 + s_6 - s_4)$	$\frac{\eta}{2}(2s_2 + d_1 - d_3)$	0	$\frac{\eta}{2}(2s_2 + d_3 - d_1)$
s_3	$\frac{\eta}{2}(s_1 + s_3 - s_2)$	$\frac{\eta}{2}(s_2 + s_3 - s_1)$	s_3	0	$\frac{\eta}{2}(s_3 + s_5 - s_6)$	$\frac{\eta}{2}(s_3 + s_6 - s_5)$	$\frac{\eta}{2}(2s_3 + d_1 - d_2)$	$\frac{\eta}{2}(2s_3 + d_2 - d_1)$	0
s_4	$\frac{\eta}{2}(s_1 + s_4 - s_5)$	$\frac{\eta}{2}(s_2 + s_4 - s_6)$	0	s_4	$\frac{\eta}{2}(s_4 + s_5 - s_1)$	$\frac{\eta}{2}(s_4 + s_6 - s_2)$	$\frac{\eta}{2}(2s_4 + d_1 - d_2)$	$\frac{\eta}{2}(2s_4 + d_2 - d_1)$	0
s_5	$\frac{\eta}{2}(s_1 + s_5 - s_4)$	0	$\frac{\eta}{2}(s_3 + s_5 - s_6)$	$\frac{\eta}{2}(s_4 + s_5 - s_1)$	s_5	$\frac{\eta}{2}(s_5 + s_6 - s_3)$	$\frac{\eta}{2}(2s_5 + d_1 - d_3)$	0	$\frac{\eta}{2}(2s_5 + d_3 - d_1)$
s_6	0	$\frac{\eta}{2}(s_2 + s_6 - s_4)$	$\frac{\eta}{2}(s_3 + s_6 - s_5)$	$\frac{\eta}{2}(s_4 + s_6 - s_2)$	$\frac{\eta}{2}(s_5 + s_6 - s_3)$	s_6	0	$\frac{\eta}{2}(2s_6 + d_2 - d_3)$	$\frac{\eta}{2}(2s_6 + d_3 - d_2)$
d_1	0	$\frac{\eta}{2}(2s_2 + d_1 - d_3)$	$\frac{\eta}{2}(2s_3 + d_1 - d_2)$	$\frac{\eta}{2}(2s_4 + d_1 - d_2)$	$\frac{\eta}{2}(2s_5 + d_1 - d_3)$	0	d_1	$\eta(d_1 + d_2 - s_3 - s_4)$	$\eta(d_1 + d_3 - s_2 - s_5)$
d_2	$\frac{\eta}{2}(2s_1 + d_2 - d_3)$	0	$\frac{\eta}{2}(2s_3 + d_2 - d_1)$	$\frac{\eta}{2}(2s_4 + d_2 - d_1)$	0	$\frac{\eta}{2}(2s_6 + d_2 - d_3)$	$\eta(d_1 + d_2 - s_3 - s_4)$	d_2	$\eta(d_2 + d_3 - s_1 - s_6)$
d_3	$\frac{\eta}{2}(2s_1 + d_3 - d_2)$	$\frac{\eta}{2}(2s_2 + d_3 - d_1)$	0	0	$\frac{\eta}{2}(2s_5 + d_3 - d_1)$	$\frac{\eta}{2}(2s_6 + d_3 - d_2)$	$\eta(d_1 + d_3 - s_2 - s_5)$	$\eta(d_2 + d_3 - s_1 - s_6)$	d_3

Table 9: The 9-dimensional algebra $Q^3(\eta)$

infer that this 9-dimensional subalgebra[3] is primitive, since otherwise c and d would belong to the subalgebra.

The Gram matrix of the Frobenius form was computed in GAP and its determinant is $128\eta^3 - 96\eta^2 + 8$, having roots $\frac{1}{2}$ (with multiplicity two) and $-\frac{1}{4}$. Hence the algebra is not simple only when $\eta = -\frac{1}{4}$. In characteristics 3 and 5, $-\frac{1}{4} = \frac{1}{2}$ and 1, respectively, so for such fields $\mathbb{F}$, the algebra is always simple.

Diagram C_5: Here the diagram on Z is the same as in case B_6 in Section 5. Correspondingly, we again need to consider the 3-transposition factor groups $\hat{G}(p)$, $p = 1, 2,$ or 3.

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[3]This algebra has been since included by Alsaeedi [1] in an infinite series $Q^k(\eta)$ that is dual, in a sense, to our series $Q_k(\eta)$ from Section 8. Also in this new series is the algebra $Q_2(\eta) = Q^2(\eta)$ from the case of diagram A_3.

First let $p = 1$. As $G = \hat{G}(1) \cong S_4$, we can identify a, c, and d with $(1, 2)$, $(1, 3)$, and $(2, 4)$, respectively. Since b is a transposition commuting with a, we must have that $b = (3, 4)$. Now, the subalgebra $\langle\!\langle x, z \rangle\!\rangle = \langle\!\langle a, c + d \rangle\!\rangle$ falls into the case A_3 of Section 5. Hence this subalgebra is none other than the algebra $Q_2(\eta)$ and it also contains $y = b = x^{\tau_z}$. Thus, in this case, $\langle\!\langle x, y, z \rangle\!\rangle \cong Q_2(\eta)$.

Suppose now that $p = 2$. The group $G = \hat{G} = U : S_4 \leq E : S_4$ is the same group that we met for the diagrams B_6 and C_4. In particular, we can reuse the description of the Fischer space of G. We can identify a, b, c, and d with $b_{1,2}$, $c_{3,4}$, $b_{1,3}$, and $b_{2,4}$, respectively. Applying Miyamoto involutions, we obtain the following axes in $A = \langle\!\langle x, y, z \rangle\!\rangle$:

$$s_1 := x = a = b_{1,2},$$
$$s_2 := y^{\tau_z} = (c_{3,4}^{\tau_c})^{\tau_d} = c_{1,4}^{\tau_d} = c_{1,2},$$
$$s_3 := x^{\tau_z} = (b_{1,2}^{\tau_c})^{\tau_d} = b_{2,3}^{\tau_d} = b_{3,4},$$
$$s_4 := y = b = c_{3,4},$$
$$d_1 := z = c + d = b_{1,3} + b_{2,4},$$
$$d_3 := z^{\tau_x} = (b_{1,3} + b_{2,4})^{\tau_a} = b_{2,3} + b_{1,4} = b_{1,4} + b_{2,3},$$
$$d_4 := z^{\tau_y} = (b_{1,3} + b_{2,4})^{\tau_b} = c_{1,4} + c_{2,3},$$
$$d_2 := d_4^{\tau_x} = (c_{1,4} + c_{2,3})^{\tau_a} = c_{2,4} + c_{1,3} = c_{1,3} + c_{2,4}.$$

(We selected the order of the double axes that exhibits the symmetry of the multiplication table.)

Again, it is immediate to see that the supports of these axes partition the Fischer space. In particular, the axes are linearly independent. To see that the 8-dimensional subspace they span is the subalgebra $A = \langle\!\langle x, y, z \rangle\!\rangle$, we compute the products. The results of the computation in Table 10 show that A is indeed 8-dimensional spanned by the above eight axes. In particular, we note that c and d are not in A, and so z is primitive in A.

The determinant of the Gram matrix of the Frobenius form on A was computed in GAP, and it is $256\eta^3 - 192\eta^2 + 16$. The roots of this polynomial $\frac{1}{2}$ (of multiplicity 2) and $-\frac{1}{4}$. So the only value of η for which the algebra is not simple is, as in the case of diagram C_4, the value $\eta = -\frac{1}{4}$. When $\mathbb{F}$ is of characteristic 3 or 5, this cannot happen and so the algebra is always simple.

Finally, we need to consider the case $p = 3$. Again, we can identify a, b, c, and d, respectively, with $b_{1,2}$, $c_{3,4}$, $b_{1,3}$, and $b_{2,4}$, using the notation

	s_1	s_2	s_3	s_4	d_1	d_2	d_3	d_4
s_1	s_1	0	0	0	$\frac{\eta}{2}(2s_1+d_1-d_3)$	$\frac{\eta}{2}(2s_1+d_2-d_4)$	$\frac{\eta}{2}(2s_1+d_3-d_1)$	$\frac{\eta}{2}(2s_1+d_4-d_2)$
s_2	0	s_2	0	0	$\frac{\eta}{2}(2s_2+d_1-d_4)$	$\frac{\eta}{2}(2s_2+d_2-d_3)$	$\frac{\eta}{2}(2s_2+d_3-d_2)$	$\frac{\eta}{2}(2s_2+d_4-d_1)$
s_3	0	0	s_3	0	$\frac{\eta}{2}(2s_3+d_1-d_3)$	$\frac{\eta}{2}(2s_3+d_2-d_4)$	$\frac{\eta}{2}(2s_3+d_3-d_1)$	$\frac{\eta}{2}(2s_3+d_4-d_2)$
s_4	0	0	0	s_4	$\frac{\eta}{2}(2s_4+d_1-d_4)$	$\frac{\eta}{2}(2s_4+d_2-d_3)$	$\frac{\eta}{2}(2s_4+d_3-d_2)$	$\frac{\eta}{2}(2s_4+d_4-d_1)$
d_1	$\frac{\eta}{2}(2s_1+d_1-d_3)$	$\frac{\eta}{2}(2s_2+d_1-d_4)$	$\frac{\eta}{2}(2s_3+d_1-d_3)$	$\frac{\eta}{2}(2s_4+d_1-d_4)$	d_1	0	$\eta(-s_1-s_3+d_1+d_3)$	$\eta(-s_2-s_4+d_1+d_4)$
d_2	$\frac{\eta}{2}(2s_1+d_2-d_4)$	$\frac{\eta}{2}(2s_2+d_2-d_3)$	$\frac{\eta}{2}(2s_3+d_2-d_4)$	$\frac{\eta}{2}(2s_4+d_2-d_3)$	0	d_2	$\eta(-s_2-s_4+d_2+d_3)$	$\eta(-s_1-s_3+d_2+d_4)$
d_3	$\frac{\eta}{2}(2s_1+d_3-d_1)$	$\frac{\eta}{2}(2s_2+d_3-d_2)$	$\frac{\eta}{2}(2s_3+d_3-d_1)$	$\frac{\eta}{2}(2s_4+d_3-d_2)$	$\eta(-s_1-s_3+d_1+d_3)$	$\eta(-s_2-s_4+d_2+d_3)$	d_3	0
d_4	$\frac{\eta}{2}(2s_1+d_4-d_2)$	$\frac{\eta}{2}(2s_2+d_4-d_1)$	$\frac{\eta}{2}(2s_3+d_4-d_2)$	$\frac{\eta}{2}(2s_4+d_4-d_1)$	$\eta(-s_2-s_4+d_1+d_4)$	$\eta(-s_1-s_3+d_2+d_4)$	0	d_4

Table 10: The 8-dimensional algebra $2Q_2(\eta)$

$b_{i,j} = (i,j) = b_{j,i}$, $c_{i,j} = (e_i - e_j)(i,j) \neq c_{j,i} = (e_j - e_i)(i,j)$ established in Section 5, where we discussed the group $\hat{G}(3)$ and its Fischer space.

Again we act by $\tau_x = \tau_a$, $\tau_y = \tau_b$, and $\tau_z = \tau_c \tau_d$ to find all axes contained in A. They are:

$$s_1 := a = b_{1,2},$$
$$s_2 := b^{\tau_z} = (c_{3,4}^{\tau_c})^{\tau_d} = c_{1,4}^{\tau_d} = c_{1,2},$$
$$s_3 := s_2^{\tau_a} = c_{1,2}^{\tau_a} = c_{2,1},$$
$$s_4 := a^{\tau_z} = (b_{1,2}^{\tau_c})^{\tau_d} = b_{2,3}^{\tau_d} = b_{3,4},$$
$$s_5 := b = c_{3,4},$$
$$s_6 := s_4^{\tau_b} = b_{3,4}^{\tau_b} = c_{4,3},$$
$$d_1 := c + d = b_{1,3} + b_{2,4},$$
$$d_2 := ((c+d)^{\tau_a})^{\tau_b} = ((b_{1,3} + b_{2,4})^{\tau_a})^{\tau_b} = (b_{2,3} + b_{1,4})^{\tau_b}$$
$$= c_{2,4} + c_{3,1} = c_{3,1} + c_{2,4},$$
$$d_3 := d_2^{\tau_z} = ((c_{3,1} + c_{2,4})^{\tau_c})^{\tau_d} = (c_{1,3} + c_{2,4})^{\tau_d} = c_{1,3} + c_{4,2},$$
$$d_4 := d_1^{\tau_a} = (b_{1,3} + b_{2,4})^{\tau_a} = b_{2,3} + b_{1,4} = b_{1,4} + b_{2,3},$$
$$d_5 := d_2^{\tau_a} = (c_{3,1} + c_{2,4})^{\tau_a} = c_{3,2} + c_{1,4} = c_{1,4} + c_{3,2},$$
$$d_6 := d_3^{\tau_a} = (c_{1,3} + c_{4,2})^{\tau_a} = c_{2,3} + c_{4,1} = c_{4,1} + c_{2,3}.$$

As in all previous cases, these axes have disjoint support and so they are linearly independent. The products are shown in Table 11. It demonstrates that $A = \langle\!\langle x, y, z \rangle\!\rangle$ is 12-dimensional spanned by the above twelve axes. The axis z is primitive in A since A does not contain c and d. The determinant of the Gram matrix of the Frobenius form on this algebra, computed in GAP, is $7\eta^{12} - 111\eta^{11} + \frac{3051}{4}\eta^{10} - 2939\eta^9 + \frac{27153}{4}\eta^8 - 8964\eta^7 + 4452\eta^6 + 5040\eta^5 - 10152\eta^4 + 6976\eta^3 - 1920\eta^2 + 64$. Its roots are 2 (with multiplicity 8), $\frac{1}{2}$ (with

Table 11: The 12-dimensional algebra $3Q_2(\eta)$

	s_1	s_2	s_3	s_4	s_5	s_6	d_1	d_2	d_3	d_4	d_5	d_6
s_1	s_1	$\frac{\eta}{2}(s_1+s_2-s_3)$	$\frac{\eta}{2}(s_1+s_3-s_2)$	0	0	0	$\frac{\eta}{2}(2s_1+d_1-d_4)$	$\frac{\eta}{2}(2s_1+d_2-d_5)$	$\frac{\eta}{2}(2s_1+d_3-d_6)$	$\frac{\eta}{2}(2s_1+d_4-d_1)$	$\frac{\eta}{2}(2s_1+d_5-d_2)$	$\frac{\eta}{2}(2s_1+d_6-d_3)$
s_2	$\frac{\eta}{2}(s_1+s_2-s_3)$	s_2	$\frac{\eta}{2}(s_2+s_3-s_1)$	0	0	0	$\frac{\eta}{2}(2s_2+d_1-d_5)$	$\frac{\eta}{2}(2s_2+d_2-d_6)$	$\frac{\eta}{2}(2s_2+d_3-d_4)$	$\frac{\eta}{2}(2s_2+d_4-d_3)$	$\frac{\eta}{2}(2s_2+d_5-d_1)$	$\frac{\eta}{2}(2s_2+d_6-d_2)$
s_3	$\frac{\eta}{2}(s_1+s_3-s_2)$	$\frac{\eta}{2}(s_2+s_3-s_1)$	s_3	0	0	0	$\frac{\eta}{2}(2s_3+d_1-d_6)$	$\frac{\eta}{2}(2s_3+d_2-d_4)$	$\frac{\eta}{2}(2s_3+d_3-d_5)$	$\frac{\eta}{2}(2s_3+d_4-d_2)$	$\frac{\eta}{2}(2s_3+d_5-d_3)$	$\frac{\eta}{2}(2s_3+d_6-d_1)$
s_4	0	0	0	s_4	$\frac{\eta}{2}(s_4+s_5-s_6)$	$\frac{\eta}{2}(s_4+s_6-s_5)$	$\frac{\eta}{2}(2s_4+d_1-d_4)$	$\frac{\eta}{2}(2s_4+d_2-d_6)$	$\frac{\eta}{2}(2s_4+d_3-d_5)$	$\frac{\eta}{2}(2s_4+d_4-d_1)$	$\frac{\eta}{2}(2s_4+d_5-d_3)$	$\frac{\eta}{2}(2s_4+d_6-d_2)$
s_5	0	0	0	$\frac{\eta}{2}(s_4+s_5-s_6)$	s_5	$\frac{\eta}{2}(s_5+s_6-s_4)$	$\frac{\eta}{2}(2s_5+d_1-d_5)$	$\frac{\eta}{2}(2s_5+d_2-d_4)$	$\frac{\eta}{2}(2s_5+d_3-d_6)$	$\frac{\eta}{2}(2s_5+d_4-d_2)$	$\frac{\eta}{2}(2s_5+d_5-d_1)$	$\frac{\eta}{2}(2s_5+d_6-d_3)$
s_6	0	0	0	$\frac{\eta}{2}(s_4+s_6-s_5)$	$\frac{\eta}{2}(s_5+s_6-s_4)$	s_6	$\frac{\eta}{2}(2s_6+d_1-d_6)$	$\frac{\eta}{2}(2s_6+d_2-d_5)$	$\frac{\eta}{2}(2s_6+d_3-d_4)$	$\frac{\eta}{2}(2s_6+d_4-d_3)$	$\frac{\eta}{2}(2s_6+d_5-d_2)$	$\frac{\eta}{2}(2s_6+d_6-d_1)$
d_1	$\frac{\eta}{2}(2s_1+d_1-d_4)$	$\frac{\eta}{2}(2s_2+d_1-d_5)$	$\frac{\eta}{2}(2s_3+d_1-d_6)$	$\frac{\eta}{2}(2s_4+d_1-d_4)$	$\frac{\eta}{2}(2s_5+d_1-d_5)$	$\frac{\eta}{2}(2s_6+d_1-d_6)$	d_1	$\frac{\eta}{2}(d_1+d_2-d_3)$	$\frac{\eta}{2}(d_1+d_3-d_2)$	$\eta(d_1+d_4-s_1-s_4)$	$\eta(d_1+d_5-s_2-s_5)$	$\eta(d_1+d_6-s_3-s_6)$
d_2	$\frac{\eta}{2}(2s_1+d_2-d_5)$	$\frac{\eta}{2}(2s_2+d_2-d_6)$	$\frac{\eta}{2}(2s_3+d_2-d_4)$	$\frac{\eta}{2}(2s_4+d_2-d_6)$	$\frac{\eta}{2}(2s_5+d_2-d_4)$	$\frac{\eta}{2}(2s_6+d_2-d_5)$	$\frac{\eta}{2}(d_1+d_2-d_3)$	d_2	$\frac{\eta}{2}(d_2+d_3-d_1)$	$\eta(d_2+d_4-s_3-s_5)$	$\eta(d_2+d_5-s_1-s_6)$	$\eta(d_2+d_6-s_2-s_4)$
d_3	$\frac{\eta}{2}(2s_1+d_3-d_6)$	$\frac{\eta}{2}(2s_2+d_3-d_4)$	$\frac{\eta}{2}(2s_3+d_3-d_5)$	$\frac{\eta}{2}(2s_4+d_3-d_5)$	$\frac{\eta}{2}(2s_5+d_3-d_6)$	$\frac{\eta}{2}(2s_6+d_3-d_4)$	$\frac{\eta}{2}(d_1+d_3-d_2)$	$\frac{\eta}{2}(d_2+d_3-d_1)$	d_3	$\eta(d_3+d_4-s_2-s_6)$	$\eta(d_3+d_5-s_3-s_4)$	$\eta(d_3+d_6-s_1-s_5)$
d_4	$\frac{\eta}{2}(2s_1+d_4-d_1)$	$\frac{\eta}{2}(2s_2+d_4-d_3)$	$\frac{\eta}{2}(2s_3+d_4-d_2)$	$\frac{\eta}{2}(2s_4+d_4-d_1)$	$\frac{\eta}{2}(2s_5+d_4-d_2)$	$\frac{\eta}{2}(2s_6+d_4-d_3)$	$\eta(d_1+d_4-s_1-s_4)$	$\eta(d_2+d_4-s_3-s_5)$	$\eta(d_3+d_4-s_2-s_6)$	d_4	$\frac{\eta}{2}(2d_4+d_5-d_6)$	$\frac{\eta}{2}(2d_4+d_6-d_5)$
d_5	$\frac{\eta}{2}(2s_1+d_5-d_2)$	$\frac{\eta}{2}(2s_2+d_5-d_1)$	$\frac{\eta}{2}(2s_3+d_5-d_3)$	$\frac{\eta}{2}(2s_4+d_5-d_3)$	$\frac{\eta}{2}(2s_5+d_5-d_1)$	$\frac{\eta}{2}(2s_6+d_5-d_2)$	$\eta(d_1+d_5-s_2-s_5)$	$\eta(d_2+d_5-s_1-s_6)$	$\eta(d_3+d_5-s_3-s_4)$	$\frac{\eta}{2}(2d_5+d_4-d_6)$	d_5	$\frac{\eta}{2}(2d_5+d_6-d_4)$
d_6	$\frac{\eta}{2}(2s_1+d_6-d_3)$	$\frac{\eta}{2}(2s_2+d_6-d_2)$	$\frac{\eta}{2}(2s_3+d_6-d_1)$	$\frac{\eta}{2}(2s_4+d_6-d_2)$	$\frac{\eta}{2}(2s_5+d_6-d_3)$	$\frac{\eta}{2}(2s_6+d_6-d_1)$	$\eta(d_1+d_6-s_3-s_6)$	$\eta(d_2+d_6-s_2-s_4)$	$\eta(d_3+d_6-s_1-s_5)$	$\frac{\eta}{2}(2d_6+d_4-d_5)$	$\frac{\eta}{2}(2d_6+d_5-d_4)$	d_6

multiplicity 2), $-\frac{1}{7}$, and -1. So the algebra is not simple only if $\eta = 2$, $-\frac{1}{7}$, or -1. If $\mathbb{F}$ is of characteristic 3 then $2 = -\frac{1}{7} = -1 = \frac{1}{2}$, so the algebra is always simple. If the characteristic of $\mathbb{F}$ is 5 then $2 = -\frac{1}{7}$, so for such fields, there are only two values of η, for which the the algebra is not simple.

Diagram C_6: The Coxeter group defined by the diagram C_6 is infinite. So we need to involve additional relations coming from the 3-transposition property of $G = \langle a, b, c, d \rangle$.

Consider $e := a^{\tau_z}$, which is a single axis in A. Furthermore, e is contained in $\langle\!\langle a, c + d \rangle\!\rangle$. According to our analysis of the diagram A_3, the latter algebra is the 4-dimensional algebra $Q_2(\eta)$, and we see from Table 6 that the two single axes in this algebra, a and e, are not collinear. In particular, $e \neq b$, since b is collinear with a. Clearly, $A = \langle\!\langle a, b, c + d \rangle\!\rangle = \langle\!\langle e, b, c + d \rangle\!\rangle$. Indeed, we saw that $e = a^{\tau_z} \in A$ and, similarly, $a = e^{\tau_z} \in \langle\!\langle e, b, c + d \rangle\!\rangle$. If e is non-collinear with b then the diagram on the support set $\{e, b, c, d\}$ is C_5 and A must be one of the algebras we obtained in that case. Thus, we may assume without loss of generality that e is collinear with b. As an element of G, the axis e is equal to $a^{cd} \in D$. So we get the additional relation $(a^{cd}b)^3 = 1$.

Next consider the single axis $f := a^{\tau_b}$, the third point on the line through a and b. Clearly, $f \in A = \langle\!\langle x, y, z \rangle\!\rangle = \langle\!\langle a, b, c + d \rangle\!\rangle$. Note that $f \notin \{c, d\}$. Indeed, if, say, $f = c$ then A contains c and $z = c + d$, and hence all of $M_1(z) = \langle c, d \rangle$ lies in A, which contradicts primitivity of z. Thus, $f \notin \{c, d\}$. Since A is primitive, so is also the subalgebra $\langle\!\langle f, c + d \rangle\!\rangle$. By Theorem 5.5, the diagram on the support set $\{f, c, d\}$ has to be flip-symmetric. That is, either f is collinear to both c and d, or it is non-collinear to both c and d. Note that $A = \langle\!\langle a, b, c + d \rangle\!\rangle = \langle\!\langle f, b, c, d \rangle\!\rangle$, since $a = f^{\tau_b} \in \langle\!\langle f, b, c + d \rangle\!\rangle$. If f is non-collinear to c and d then the diagram on the support set $\{f, b, c, d\}$ is our C_4, and so it follows that $A = Q^3(\eta)$. Hence, we may assume without loss of generality that f is collinear to c and d. In terms of G, the axis f is the involution $a^b \in D$, and so we get two additional relations: $(a^b c)^3 = 1 = (a^b d)^3$.

Combining the Coxeter presentation coming with the diagram C_6 and the three additional relations $(a^{cd}b)^3 = (a^b c)^3 = (a^b d)^3 = 1$, we obtain via a coset enumeration in GAP that G is a factor group of a group $\hat{G} \cong 2^{1+6} : SU_3(2)'$ in the notation of [19], the (unique) irreducible 3-transposition group generated by four involutions and having the class of 3-transpositions of cardinality 36. (This also follows from the presentation (A.8) in the appendix of [19].) The subgroup 2^{1+6} is the unique minimal normal non-central subgroup. It is easy

to check that G cannot be the factor group $SU_3(2)'$, we can assume that $G = \hat{G}$.

We do not currently have a good description of the Fischer space of this group and it is a bit too big to do computations by hand in any case. Using GAP, we found that $A = \langle\!\langle x, y, z \rangle\!\rangle = \langle\!\langle a, b, c + d \rangle\!\rangle$ is a primitive 24-dimensional subalgebra of the 36-dimensional Matsuo algebra of G. It is spanned by its 12 single axes and 12 double axes, which are as follows:

$$s_1 := a, s_2 := a^{\tau_y},$$
$$s_3 := b, s_4 := a^{\tau_z},$$
$$s_5 := s_2^{\tau_z}, s_6 := s_4^{\tau_y},$$
$$s_7 := s_3^{\tau_z}, s_8 := s_5^{\tau_x},$$
$$s_9 := s_5^{\tau_y}, s_{10} := s_8^{\tau_z},$$
$$s_{11} := s_7^{\tau_x}, s_{12} := s_{11}^{\tau_y},$$
$$d_1 := z^{\tau_x}, d_3 := z = c + d,$$
$$d_4 := d_1^{\tau_y}, d_2 := d_4^{\tau_x},$$
$$d_5 := d_3^{\tau_y}, d_6 := d_5^{\tau_x},$$
$$d_7 := d_6^{\tau_z}, d_8 := d_4^{\tau_z},$$
$$d_9 := d_7^{\tau_x}, d_{10} := d_8^{\tau_x},$$
$$d_{11} := d_8^{\tau_y}, d_{12} := d_{11}^{\tau_x},$$

They are, clearly, forming a basis of A. The multiplication table of A, also found in GAP, is in Table 12.

We also computed in GAP the determinant of the Gram matrix, which is $16777216\eta^9 - 66060288\eta^8 + 113246208\eta^7 - 110100480\eta^6 + 66060288\eta^5 - 24772608\eta^4 + 5505024\eta^3 - 589824\eta^2 + 4096$. The roots of this polynomial are $\frac{1}{2}$ (of multiplicity 8) and $-\frac{1}{16}$. Hence the algebra is simple unless $\eta = -\frac{1}{16}$. In characteristic 3, $-\frac{1}{16} = \frac{1}{2}$ and, in characteristic 17, $-\frac{1}{16} = 1$. Hence, for fields $\mathbb{F}$ in these two characteristics, the algebra is always simple.

7 Flip subalgebras

Let us return to Theorem 5.5, stating that a subalgebra in a Matsuo algebra M, generated by a set of single and double axes Y, can only be primitive

Table 12: The 24-dimensional algebra of type C_6 (all non-diagonal entries have an additional factor $\frac{\eta}{2}$ omitted here)

when the diagram on the support of Y admits a specific automorphism of order two, the flip. In this section, we shift focus from the set of generators (and in particular, from the number of generators) onto the flip itself. In doing so, we uncover an "industrial" method of building primitive algebras of Monster type $(2\eta, \eta)$.

As before, let $M = M_\eta(\Gamma)$ be a Matsuo algebra over a field $\mathbb{F}$, corresponding to the Fischer space Γ coming from a 3-transposition group (G, D). The group $\operatorname{Aut}\Gamma$ acts on M and it is the full group of automorphisms of M preserving the basis D consisting of all single axes.

7.1 Fixed subalgebra

Suppose $H \leq \operatorname{Aut}\Gamma$.

Definition 7.1. *The* fixed subalgebra *of H in M is defined as*

$$M_H := \{u \in M \mid u^h = u \text{ for all } h \in H\}.$$

Clearly, M_H is a subspace and it is closed for the algebra product: if $u, v \in M_H$ then $(uv)^h = u^h v^h = uv$ for each $h \in H$. Thus, M_H is indeed a subalgebra of M.

Definition 7.2. *For an H-orbit $O \subseteq D$, define the* orbit vector *as*

$$u_O := \sum_{c \in O} c.$$

Manifestly, u_O is fixed by H and so $u_O \in M_H$. In fact, the orbit vectors provide us with a nice basis of M_H.

Proposition 7.3. *Let $O_1, \ldots, O_k$ be all the H-orbits on D. Then the vectors $u_i := u_{O_i}$ form a basis of the fixed subalgebra M_H.*

Proof. First of all, the vectors $u_1, \ldots, u_k$ are linearly independent as their supports in the basis D of M (*i.e.*, the orbits O_i) partition D and hence are disjoint.

Consider an arbitrary $u \in M_H$, say, $u = \sum_{c \in D} \alpha_c c$ for some scalars $\alpha_c \in \mathbb{F}$. If $d, e \in D$ are in the same H-orbit, then $e = d^h$ for some $h \in H$, and so $u = u^h = \sum_{c \in D} \alpha_c c^h$ and this shows that $\alpha_d = \alpha_e$. That is, the coefficients α_c stay constant on the H-orbits O_i. Clearly, this means that, for each orbit O_i, there is a single value α_i such that $\alpha_c = \alpha_i$ for all $c \in O_i$. Consequently, u can be written as $u = \sum_{i=1}^{k} \alpha_i u_i$, and so $u_1, \ldots, u_k$ also spans M_H. $\square$

7.2 Flip subalgebra

Select an element $\tau \in \operatorname{Aut}\Gamma$ with $|\tau| = 2$. Let $H = \langle\tau\rangle$. Then the H-orbits $O_1, \ldots, O_k$ have each length 1 or 2. Let us classify the orbit vectors u_i into three groups:

- $u_i = a \in D$, corresponding to orbits $O_i = \{a\}$ of length 1;

- $u_i = a + b$, corresponding to orbits $O_i = \{a, b\}$ with $ab = 0$ (orthogonal orbits);

- $u_i = a + b$, with $O_i = \{a, b\}$ satisfying $ab \neq 0$ (non-orthogonal orbits).

We call the u_i of the first kind *singles*, of the second kind *doubles*, and of the third kind *extras*.

It is easy to see that singles are simply all single axes from M that are contained in M_H.

Proposition 7.4. *Each double $u_i = a + b$ is a double axis and it is primitive in M_H. In fact, the set of doubles consists of all double axes that are contained in M_H and are primitive in it.*

Proof. Manifestly, u_i is a double axis, so we just need to see that u_i is primitive in $B := M_H$. Note that $B_1(u_i) = M_1(u_i) \cap B = \langle a, b\rangle \cap M_H$. We know that $u_i = a + b \in M_H$. On the other hand, $a \notin M_H$, because $a^\tau = b \neq a$. Thus, $B_1(u_i) = \langle u_i\rangle$ is 1-dimensional and so u_i is indeed primitive in $B = M_H$.

For the second claim, if a double axis $x = c + d$ is contained in M_H then H acts on the support $\{c, d\}$ of x. If the action is trivial then both c and d are in M_H, and so x is not primitive in M_H. On the other hand, if $c^\tau = d$ then $\{c, d\}$ is one of the orbits O_i and $x = u_i$. Hence the claim follows. $\quad\square$

We are ready for the key definition.

Definition 7.5. *The* flip subalgebra $A = A(\tau)$ *corresponding to an involution $\tau \in \operatorname{Aut}\Gamma$ is the subalgebra of M_H (where $H = \langle\tau\rangle$) generated by all singles and all doubles.*

Singles are Matsuo axes and so they are primitive in all of M. According to Proposition 7.4, doubles are primitive in M_H and so they are also primitive in $A(\tau)$. Thus, each $A = A(\tau)$ is a primitive algebra of Monster type $(2\eta, \eta)$.

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Let Y be the set of generators of A, that is, Y consists of all singles and all doubles. We note that τ is the flip of the diagram on $Z = \mathrm{supp}(Y)$, and so we will call τ the flip, even though τ is an arbitrary involution in $\mathrm{Aut}\,\Gamma$.

If τ and τ' are conjugate in $\mathrm{Aut}\,\Gamma$, say $\tau' = \tau^\sigma$ for some $\sigma \in \mathrm{Aut}\,\Gamma$, then $A(\tau') = A(\tau)^\sigma$ and so these flip subalgebras are isomorphic. Hence, we just need to deal with possible flips up to conjugation in $\mathrm{Aut}\,\Gamma$.

The extras $x = a + b$, where $ab \neq 0$, are not idempotents, and in particular, they are not axes. This is why we discard them here. However, it is not impossible that the extras may lead to interesting idempotents/axes in a different, more intricate way. Hence we record the following.

Question 7.6. *Is there a way to produce interesting idempotents/axes from the extras? What kind of fusion law do such new axes satisfy?*

8 Symmetric group

Until now, there have been only a handful of known Monster type axial algebras, all of type $(\frac{1}{4}, \frac{1}{32})$, that were outside of the smaller class of algebras of Jordan type. The results of Section 7 allow us to populate the class of algebras of Monster type with many new interesting axial algebras and to show that this class is truly much larger than the class of algebras of Jordan type.

There are many families of 3-transposition groups and for each there are many conjugacy classes of flips τ. In this section we study the simplest family of 3-transposition groups, the symmetric groups. For them, we identify the flip subalgebra $A(\tau)$ for each flip τ. Hence, in this section, $G = S_n$, the symmetric group on n symbols, and $D = (1,2)^G$ is the conjugacy class of transpositions (2-cycles). We note that D, the point set of Γ and the basis of $M = M_\eta(\Gamma)$, has cardinality $\binom{n}{2} = \frac{n(n-1)}{2}$. We also note that, in this case, $\mathrm{Aut}\,\Gamma$ coincides with $G = S_n$. The classes of involutions τ in S_n are identified by a single parameter, the number k, $1 \leq k \leq \frac{n}{2}$, of 2-cycles in the decomposition of τ as a product of independent cycles.

8.1 Infinite series

We first deal with a special situation, where $n = 2k$. Without loss of generality we may assume that

$$\tau = (1,2)(3,4)\dots(2k-1,2k).$$

As above, let $H = \langle \tau \rangle$ be the cyclic subgroup generated by τ. Associated with τ and H is a partition $\mathcal{P}$ of $\{1, 2, \dots, n\}$ into k parts of size two, each part $P_i = \{2i - 1, 2i\}$ being an orbit of H. Each $s \in \{1, 2, \dots, n\}$ lies in some part P_i. Let $\bar{s}$ denote the other number in this P_i. For example, $1 \in P_1 = \{1, 2\}$ and so $\bar{1} = 2$. Note that τ sends every s to $\bar{s}$.

We need to classify singles, doubles and extras, and we do it in terms of the above partition $\mathcal{P}$.

Proposition 8.1. *When* $n = 2k$*, the fixed subalgebra* M_H *has dimension* k^2*. Among the orbit vectors, there are (a)* k *singles* $(s, \bar{s})$*; (b)* $k^2 - k$ *doubles* $(s, t) + (\bar{s}, \bar{t})$*, with* s *and* t *contained in different parts of* $\mathcal{P}$*; and (c) no extras.*

Proof. Let O be the orbit a^H, where $a = (s, t) \in D$. If s and t are in the same part of the partition $\mathcal{P}$ then $t = \bar{s}$ and so $(s, t)^\tau = (t, s) = (s, t)$. Hence in this case we obtain k orbits of length 1. These are the k singles appearing in (a) above.

Let us now consider the complementary case, where s and t belong to different parts of $\mathcal{P}$. In this case, $(s, t)^\tau = (\bar{s}, \bar{t})$ contains no s and no t. So $O = \{(s, t), (\bar{s}, \bar{t})\}$ is of length two and of orthogonal type, since the two transpositions in O are independent and hence commute, which corresponds to product zero in M. Thus, all such orbit vectors are doubles. Let us count them. Suppose that $s \in P_i$ and $t \in P_j$. This pair of parts gives us two doubles: $(s, t) + (\bar{s}, \bar{t})$ and $(s, \bar{t}) + (\bar{s}, t)$. There are $\binom{k}{2} = \frac{k(k-1)}{2}$ pairs $\{i, j\}$, each leading to two doubles. Hence the total number of doubles is $2\frac{k(k-1)}{2} = k^2 - k$, as claimed in (b).

We have already covered all possibilities for $O = (s, t)^H$, hence there is no room for extras.

For the claim on the dimension of M_H, recall from Proposition 7.3 that the orbit vectors form a basis of M_H. Adding k to $k^2 - k$ gives the dimension equal to k^2. $\square$

Corollary 8.2. *When* $n = 2k$*, the flip algebra* $A(\tau)$ *coincides with the fixed subalgebra* M_H*. In particular, its dimension is* k^2 *and it has a basis consisting of singles and doubles.*

Proof. Indeed, as there are no extras, $A(\tau)$ contains the whole basis of M_H and thus coincides with it. $\qquad\square$

We call this algebra $A(\tau)$ the k^2-algebra and denote it $Q_k(\eta)$. The 4-dimensional algebra $Q_2(\eta)$, we found in Section 5 for the diagram A_3, belongs in this series. (The algebra $Q_1(\eta)$ is 1-dimensional generated by the single axis $(1, 2)$.)

As we have here a whole infinite series of algebras, one for each value of k, we cannot show the multiplication table as we did earlier. Instead, we will now describe the 2-generated subalgebras $\langle\!\langle x, y \rangle\!\rangle$ involved in $A(\tau)$. Clearly, all of them must be on the list we obtained in Section 5. There are several cases for the pair $Y = \{x, y\}$. We do not need to provide a formal proof in each case, because all one needs to do is to identify the diagram arising on $Z = \mathrm{supp}(Y)$.

- If both x and y are singles then $xy = 0$ and $\langle\!\langle x, y \rangle\!\rangle \cong 2B$.

- Suppose that $x = (s, \bar{s})$ is a single, with $\{s, \bar{s}\} = P_m$, and y is a double corresponding to a pair $\{i, j\}$.

  - If $m \notin \{i, j\}$ then the diagram on Z has no edges (diagram A_1) and so $\langle\!\langle x, y \rangle\!\rangle \cong 2B$.

  - If $m \in \{i, j\}$ then the diagram on Z is A_3, and so $\langle\!\langle x, y \rangle\!\rangle$ is the 2^2-algebra $Q_2(\eta)$ arising for that diagram.

- Finally suppose that both x and y are doubles, corresponding to pairs $\{i, j\}$ and $\{i', j'\}$ respectively.

  - If $\{i, j\} \cap \{i', j'\} = \emptyset$ then $xy = 0$ and again $\langle\!\langle x, y \rangle\!\rangle \cong 2B$. (This is diagram B_1.)

  - If $i = i'$, but $j \neq j'$, then the diagram on Z is B_4 and hence $\langle\!\langle x, y \rangle\!\rangle \cong 3C(\eta)$.

  - Lastly, if $i = i'$ and $j = j'$ then $x = (s, t) + (\bar{s}, \bar{t})$ and $y = (s, \bar{t}) + (\bar{s}, t)$ for some $s \in P_i$ and $t \in P_j$. Here the diagram is B_6 and, according to the discussion of this case in Section 5, we need to determine the order p of $(s, t)^{(\bar{s}, t)}(\bar{s}, \bar{t})^{(s, \bar{t})}$. Since $(s, t)^{(\bar{s}, t)} = (s, \bar{s})$ and $(\bar{s}, \bar{t})^{(s, \bar{t})} = (\bar{s}, s) = (s, \bar{s})$, the product is equal to the identity, that is, $p = 1$. Therefore, in this last case, $\langle\!\langle x, y \rangle\!\rangle \cong 3C(2\eta)$.

Now the formula for the product xy, when it is non-zero, can be looked up in the multiplication table of the corresponding algebra $\langle\!\langle x, y \rangle\!\rangle$.

8.2 General case

In this subsection we deal with the general case: arbitrary n and k. Let $m := n - 2k$.

Theorem 8.3. *For arbitrary n and k, the flip algebra $A(\tau)$ is a direct sum of the k^2-algebra $Q_k(\eta)$ and the Matsuo algebra $M_\eta(\Gamma')$ where Γ' is the Fischer space of $G' \cong S_m$ acting on the m-element set $\{2k+1, \ldots, n\}$.*

Proof. We split the set $\{1, 2, \ldots, n\}$ as a disjoint union $R \cup S$, where $R = \{1, 2, \ldots, 2k\}$ and $S = \{2k+1, \ldots, n\}$. Clearly, $H = \langle \tau \rangle$ acts on R with r orbits of length 2 and it acts on S trivially. Consider an arbitrary single axis $a = (s, t) \in D$. If $s \in R$ and $t \in S$ then $a^\tau = (\bar{s}, t)$ (we adapt the bar notation from the preceding subsection). Clearly, all such orbits $a^H = \{a, a^\tau\}$ are non-orthogonal and so the corresponding orbit vectors are extras. Hence each axis of $A(\tau)$, whether a single or a double, comes fully from R or fully from S. The former kind clearly span the k^2-algebra $Q_k(\eta)$ and the latter kind are all singles, because H acts trivially on S, and they span the Matsuo algebra $M_\eta(\Gamma')$ of the symmetric group of S.

Finally, since R and S are disjoint, any transposition involved in an axis x of the first kind is independent from any axis y of the second kind. Therefore, $xy = 0$, proving that $A(\tau)$ is a direct sum, as claimed in the theorem. $\quad\square$

8.3 Simplicity

Here we tackle the following question: for which values of η is the k^2-algebra $A = Q_k(\eta)$ not simple and what is then the dimension of the ideal?

Recall from Subsection 2.2 that ideals in an axial algebra are classified into two types: the ones containing a generating axis and the ones contained in the radical. For the first kind, we need to know the projection graph Δ (see Definition 2.8). The vertices of Δ are the generating axes and the edges, in the presence of a Frobenius form, are pairs of generating axes, $\{a, b\}$, $a \neq b$, with $(a, b) \neq 0$. So let us describe the Gram matrix of the Frobenius form on the set of axes.

Let us give all axes standard names in terms of the partition $\mathcal{P}$. First of all, we have the single $a_i = (2i - 1, 2i)$ corresponding to the part P_i,

$i = 1, \ldots, k$. Also, for $1 \leq i < j \leq k$, the two doubles corresponding to $\{i, j\}$ are $b_{i,j} := (2i - 1, 2j - 1) + (2i, 2j)$ and $c_{i,j} := (2i - 1, 2j) + (2i, 2j - 1)$. Now, let us compute the entries of the Gram matrix. First of all, $(a_i, a_i) = 1$ and $(a_i, a_j) = 0$ for all $i \neq j$. Next, $(b_{i,j}, b_{i,j}) = 2 = (c_{i,j}, c_{i,j})$ and $(b_{i,j}, c_{i,j}) = 4\frac{\eta}{2} = 2\eta$. Taking a single $x = a_i$ and a double $y = b_{s,t}$ or $c_{s,t}$, we obtain that $(x, y) = 0$ if $i \notin \{s, t\}$ and $(x, y) = 2\frac{\eta}{2} = \eta$ if $i \in \{s, t\}$. Finally, taking two doubles $x = b_{i,j}$ or $c_{i,j}$ and, similarly, $y = b_{s,t}$ or $c_{s,t}$, we calculate that $(x, y) = 0$ if $\{i, j\}$ and $\{s, t\}$ are disjoint and $(x, y) = 2\frac{\eta}{2} = \eta$ if $\{i, j\}$ and $\{s, t\}$ meet in one element.

Proposition 8.4. *The projection graph Δ of $A = Q_k(\eta)$ is connected. In particular, A has no proper ideals containing any of the generating axes.*

Proof. From the above description of the Gram matrix, it is clear that a_1 is adjacent to all $b_{1,j}$ and $c_{1,j}$, $2 \leq j \leq k$, and these are adjacent to all the remaining vertices of Δ. According to the discussion around Theorem 2.9, connectedness of Δ means that A has no proper ideals containing axes. $\square$

Therefore, every proper ideal of A is contained in the radical $R(A)$. Furthermore, by Theorem 2.7, since none of the axes is singular, $R(A)$ coincides with $A^\perp$, the radical of the Frobenius form. Therefore, A is not simple exactly when the determinant of the Gram matrix is zero.

We next compute this determinant. Our plan is to find a basis with respect to which the Gram matrix splits into several blocks. Let us define $d_{i,j} := \frac{b_{i,j} - c_{i,j}}{2}$ and $e_{i,j} := b_{i,j} + c_{i,j}$. Then the transition matrix from the standard basis to the basis, consisting of all a_i and all $d_{i,j}$ and $e_{i,j}$, is block-diagonal with $\binom{k}{2}$ blocks of size 2×2 and determinant one. Hence the new Gram matrix has the same determinant as the original one.

From the values of the Frobenius form listed above, we note that $(b_{i,j}, x) = (c_{i,j}, x)$ for all axes x excluding $b_{i,j}$ and $c_{i,j}$. This means that each $d_{i,j}$ is orthogonal to all vectors in the new basis apart from itself and, possibly, $e_{i,j}$. We compute that $(d_{i,j}, d_{i,j}) = \frac{1}{4}(b_{i,j} - c_{i,j}, b_{i,j} - c_{i,j}) = \frac{1}{4}(2 - 2(2\eta) + 2) = 1 - \eta$, non-zero since $\eta \neq 1$. However, $(d_{i,j}, e_{i,j}) = \frac{1}{2}(b_{i,j} - c_{i,j}, b_{i,j} + c_{i,j}) = \frac{1}{2}(2 + 2\eta - 2\eta - 2) = 0$. Therefore, the subspace D spanned by all vectors $d_{i,j}$ is non-degenerate (the Gram matrix on D is $(1 - \eta)I$, where I is the identity matrix) and it is orthogonal to the subspace E spanned by all a_i and all $e_{i,j}$.

Let us now focus on the subspace E. Each $e_{i,j}$ is already orthogonal to all a_s with $s \notin \{i, j\}$. Let us amend $e_{i,j}$ so that the corrected vectors are also orthogonal to a_i and a_j. Consider $f_{i,j} := e_{i,j} - \alpha(a_i + a_j)$. Then

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$(f_{i,j}, a_i) = (b_{i,j}+c_{i,j}-\alpha a_i-\alpha a_j, a_i) = \eta+\eta-\alpha-0$. Let us select $\alpha = 2\eta$. Then $f_{i,j}$ is orthogonal to a_i and, symmetrically, to a_j. The transition matrix is again of determinant one, and the subspace W spanned by all a_i is orthogonal to the subspace F spanned by all $f_{i,j}$. The Gram matrix on W is the identity matrix, so W is non-degenerate. Hence the radical of A coincides with the radical of the subspace F and so we can focus on the latter.

Let us compute the Gram matrix on F with respect to the basis consisting of all $f_{i,j}$. The basis vectors are in a bijection with subsets $\{i, j\}$ and so we have three cases to consider. First of all, if $\{i,j\}$ and $\{s,t\}$ are disjoint then $(f_{i,j}, f_{s,t}) = 0$. If $\{i,j\}$ and $\{s,t\}$ meet in one element, say, $i = s$ then $(f_{i,j}, f_{i,t}) = (b_{i,j}+c_{i,j}-2\eta a_i-2\eta a_j, b_{i,t}+c_{i,t}-2\eta a_i-2\eta a_t) = \eta+\eta-2\eta\eta+0+\eta+\eta-2\eta\eta+0-2\eta\eta-2\eta\eta+4\eta^2+0+0+0+0+0 = 4\eta-4\eta^2 = 4(-\eta^2+\eta) = 4\eta(1-\eta)$. Finally, $(f_{i,j}, f_{i,j}) = (b_{i,j} + c_{i,j} - 2\eta a_i - 2\eta a_j, b_{i,j} + c_{i,j} - 2\eta a_i - 2\eta a_j) = 2 + 2 + 4\eta^2 + 4\eta^2 + 2(2\eta - 2\eta\frac{\eta}{2} - 2\eta\frac{\eta}{2} - 2\eta\frac{\eta}{2} - 2\eta\frac{\eta}{2} + 0) = 4 + 4\eta - 8\eta^2 = 4(-2\eta^2 + \eta + 1) = 4(1 - \eta)(2\eta + 1)$.

Let $m := \binom{k}{2} = \frac{k(k-1)}{2}$ be the dimension of F. Let I be the identity matrix of this size and J be the adjacency matrix of the Johnson graph $J(k, 2)$, the graph on 2-element subsets of $\{1, 2, \ldots, k\}$, where two 2-subsets are adjacent when they meet in one element. Then the Gram matrix of F with respect to the basis formed by the vectors $f_{i,j}$ is equal to $4((-2\eta^2+\eta+1)I+(-\eta^2+\eta)J)$. Over a field of characteristic zero, the eigenvalues of J and their multiplicities are well-known, since $J(k, 2)$ is strongly regular. They are the degree, $\theta_0 = 2(k - 2)$, with multiplicity 1, $\theta_1 = k - 4$ with multiplicity $k - 1$, and $\theta_2 = -2$ with multiplicity $\frac{k(k-3)}{2}$.

Proposition 8.5. *The determinant of the Gram matrix of the Frobenius form on $A = Q_k(\eta)$ is equal to $(2(1-\eta))^{k^2-k}(2(k-1)\eta+1)((k-2)\eta+1)^{k-1}$.*

Proof. We first assume that $\mathbb{F}$ is of characteristic zero. Since the Gram matrix on F coincides with $4((-2\eta^2 + \eta + 1)I + (-\eta^2 + \eta)J)$, it has eigenvalues $\kappa_h = 4\eta(1-\eta)\theta_h + 4(1-\eta)(2\eta+1) = 4(1-\eta)(\eta\theta_h + 2\eta + 1)$, $h = 0, 1, 2$, with the corresponding multiplicities. This gives $\kappa_0 = 4(1-\eta)(2(k-1)\eta+1)$, $\kappa_1 = 4(1-\eta)((k-2)\eta+1)$, and $\kappa_2 = 4(1-\eta)$. Since the determinant of a square matrix is the product of its eigenvalues taken with their multiplicities, we conclude that the determinant of the Gram matrix on F is $(4(1-\eta))^m(2(k-1)\eta+1)((k-2)\eta+1)^{k-1}$. Combining this with the determinant of the Gram matrix on D, equal to $(1-\eta)^m$, and with the determinant of the Gram matrix on W, equal to 1, we obtain the polynomial claimed in the proposition.

While our computation was over a field of characteristic zero, the result is a polynomial with integral coefficients. Therefore, it readily transfers into any odd characteristic and so our claim holds for any field $\mathbb{F}$. $\qquad\square$

In this proof and in the discussion before the proposition, we glazed over the small cases $k = 1$ and 2, where some of the eigenvalues disappear and the multiplicity formula does not apply. However, by inspection, the formula for the determinant remains true also in these small cases.

As a consequence, we have the following statement.

Theorem 8.6. *The k^2-algebra $Q_k(\eta)$ is simple unless $2(k-1)\eta + 1 = 0$ (equivalently, $\eta = -\frac{1}{2(k-1)}$) or $(k-2)\eta + 1 = 0$ (equivalently, $\eta = -\frac{1}{k-2}$).*

We now turn to the dimension of the radical for the special values of η. In zero characteristic, the multiplicity of κ_0, equal to 1, (respectively, of κ_1, equal to $k-1$) gives the dimension of the radical when $\eta = -\frac{1}{2(k-1)}$ (respectively, $\eta = -\frac{1}{k-2}$). In positive characteristic, the same applies as long as $\kappa_0 \neq \kappa_1$. This can only happen when $k = 0$ (i.e., the characteristic of $\mathbb{F}$ divides k) and then the special value of η is $\eta = \frac{1}{2}$.

We should also comment on the small cases. The above analysis fully applies when $k \geq 3$. When $k = 1$, $A = Q_1(\eta) \cong \mathbb{F}$ and so is simple. When $k = 2$, only the eigenvalue κ_0 can be zero, and so $A = Q_2(\eta)$ has a non-zero radical only for $\eta = -\frac{1}{2(2-1)} = -\frac{1}{2}$. Then the radical is 1-dimensional (cf. Section 5).

9 Two further series

In this final section of the paper we construct two further infinite series of algebras of Monster type generalizing two examples we found in Section 6.

The 3-transposition groups that we consider here are generalizations of the groups $\hat{G}(p)$ that we first encountered in the case of diagram B_6 in Section 5. Both of these groups can be obtained via a more general construction due to Zara [41, 42] in wreath products $K \wr S_n$, where K has all elements of order at most 3. Our two groups arise for $K = C_2$ or C_3.

9.1 Series $2Q_k(\eta)$

First, consider the semi-direct product $\hat{G}_n = 2^n : S_n$ of the symmetric group $S = S_n$ and its permutational module $E = 2^n$ over $\mathbb{F}_2$. The class C of

transpositions from S extends to a larger conjugacy class D in $\hat{G}_n$. We claim that D is again a class of 3-transpositions. Let $e_1, \ldots, e_n$ be the basis of E permuted by S. Note that we retain the additive notation for the operation in E. The transposition $\sigma := (i,j) \in C$ acts as a transvection on E with $[E, \sigma] = \langle e_i + e_j \rangle$ and $C_E(\sigma) = \langle e_i + e_j \rangle + \langle e_s \mid s \neq i, j \rangle$. Therefore, $|D| = 2|C| = n(n-1)$; namely, the coset $E\sigma$, in addition to σ, contains the second element from D, $\tilde{\sigma} = (e_i + e_j)\sigma$. Note that this description of D implies that $G_n = \langle D \rangle$ is of index 2 in $\hat{G}_n$, namely, $G_n \cap E \cong 2^{n-1}$ is the "sum-zero" submodule of E.

Manifestly, all elements of D are involutions. Any two distinct elements from D involve no more than four indices from $\{1, 2, \ldots, n\}$. Therefore, they are contained in a subgroup isomorphic to the group $\hat{G}(2) \cong 2^3 : S_4$ from Section 5, where they are contained in the class of 3-transpositions. Hence D is also a class of 3-transpositions and (G_n, D) is a 3-transposition group. Let Γ be the Fischer space of (G_n, D) and $M = M_\eta(\Gamma)$.

Let $n \geq 2k$ and let $\tau = (1,2)(3,4) \cdots (2k-1, 2k)$ be the flip, as in Section 8. Let $H = \langle \tau \rangle$.

We start with a special case.

Proposition 9.1. *If $n = 2k$ then the fixed subalgebra M_H is of dimension $2k^2$. Among the orbit vectors, there are $2k$ singles, $2(k^2 - k)$ doubles, and no extras.*

Proof. The singles are the points $(2i-1, 2i)$ and $(e_{2i-1} + e_{2i})(2i-1, 2i)$. All other orbit sums are of the form $\{(s,t) + (\bar{s}, \bar{t})\}$ or $\{(e_s + e_t)(s,t) + (e_{\bar{s}} + e_{\bar{t}})(\bar{s}, \bar{t})\}$, and they are all doubles. Here s and t are in different parts of the partition $P = \{\{1,2\}, \{3,4\}, \ldots, \{2k-1, 2k\}\}$. Note that we adopt the bar notation from Section 8 for the complementary elements in the parts.

Comparing with Proposition 8.1, we have twice as many singles here and twice as many doubles, so the claim follows. $\square$

Since M_H is spanned by singles and doubles, it coincides with the flip algebra $A(\tau)$. We use $2Q_k(\eta)$ to denote this algebra M. The 8-dimensional example from the case C_5 from Section 6 is isomorphic to $2Q_2(\eta)$.

Let us also include a statement analogous to Theorem 8.3. We note, however, that, unlike there, we cannot claim now that this result covers all possible flips.

Proposition 9.2. *If $n = 2k + m$ then $A(\tau)$ is the direct sum of $2Q_k(\eta)$ arising on the subset $R := \{1, 2, \ldots, 2k\}$ and the Matsuo algebra $M_\eta(\Gamma')$,*

where Γ' is the Fischer space of the group $G_m \cong 2^{m-1} : S_m$ arising on the subset $S := \{2k+1, \ldots, n\}$.

We skip the proof as it is quite analogous to that of Theorem 8.3.

Concerning the question of simplicity of $2Q_k(\eta)$, it is easy to see that the projection graph on the set of single and double axes in $2Q_k(\eta)$, $k \geq 2$, is connected. Hence $2Q_k(\eta)$ contains no proper ideals containing axes and so it is simple for all but a finite number of values of η. We leave determination of these special values of η until another time.

9.2 Series $3Q_k(\eta)$

This is similar to the first case, except now we take $\hat{G}_n$ to be the semi-direct product of $S = S_n$ and the permutational module $E = 3^n$ over $\mathbb{F}_3$. Again, let $e_1, \ldots, e_n$ be the basis permuted by S. Let D be the conjugacy class of $\hat{G}_n$ containing the class C of transpositions from S.

Taking $\sigma = (i, j)$, it is easy to see that $[E, \sigma] = \langle e_i - e_j \rangle$ and $C_E(\sigma) = \langle e_i + e_j \rangle + \langle e_s \mid s \neq i, j \rangle$, that is, σ acts on E as a reflection. From here we deduce that the coset $E\sigma$ contains three elements from D, namely, σ and two further elements, $(e_i - e_j)\sigma$ and $(e_j - e_i)\sigma$. Thus, $|D| = 3|C| = \frac{3}{2}n(n-1)$. Also, it follows that any two elements from D involve no more than four indices from $\{1, 2, \ldots, n\}$ and so they are contained in a subgroup $\hat{G}(3) = 3^3 : S_4$ as in Section 5. Since $D \cap \hat{G}(3)$ is a class of 3-transpositions in $\hat{G}(3)$, we deduce that D is a class of 3-transpositions. Also similarly to the first case, $G_n = \langle D \rangle$ is a proper, index 3 subgroup of $\hat{G}_n$, with $G_n \cap E \cong 3^{n-1}$ being the "sum-zero" submodule in E.

Let Γ be the Fischer space of the 3-transposition group (G_n, D) and let $M = M_\eta(\Gamma)$. Take $k \leq \frac{n}{2}$. Our flip τ is again based on $\tau_0 = (1, 2)(3, 4) \cdots (2k-1, 2k)$. However, we need a correcting factor to achieve the subalgebra we want. Let z be the automorphism of $\hat{G}_n$, centralizing the complement S_n and inverting every element of E. We set $\tau = z\tau_0$. Since z and τ_0 commute, τ is an involution.

Let $H = \langle \tau \rangle$.

Proposition 9.3. *Suppose that $n = 2k$. Then the fixed subalgebra M_H is of dimension $3k^2$ and spanned by $3k$ singles, $3(k^2 - k)$ doubles, and no extras.*

Proof. We claim that the points $(2i - 1, 2i)$, $(e_{2i-1} - e_{2i})(2i - 1, 2i)$, and $(e_{2i} - e_{2i-1})(2i - 1, 2i)$, where $i = 1, \ldots, k$, are singles. Indeed, $(2i - 1, 2i)^\tau =$

$((2i-1, 2i)^z)^{\tau_0} = (2i-1, 2i)^{\tau_0} = (2i-1, 2i)$, so $(2i-1, 2i)$ is a single. Also, $((e_{2i-1}-e_{2i})(2i-1, 2i))^\tau = (((e_{2i-1}-e_{2i})(2i-1, 2i))^z)^{\tau_0} = ((-e_{2i-1}+e_{2i})(2i-1, 2i))^{\tau_0} = (-e_{2i}+e_{2i-1})(2i-1, 2i) = (e_{2i-1}-e_{2i})(2i-1, 2i)$, so this is also a single. The calculation for $(e_{2i}-e_{2i-1})(2i-1, 2i)$ is quite similar.

All other orbit vectors are of the form $(s, t) + (\bar{s}, \bar{t})$ or $(e_s - e_t)(s, t) + (e_{\bar{t}} - e_{\bar{s}})(\bar{s}, \bar{t})$, where s and t are in different parts of the partition $P = \{\{1, 2\}, \{3, 4\}, \dots, \{2k-1, 2k\}\}$ associated with τ. Note that the two summands in these orbit vectors have disjoint support, so they are all doubles, leaving no room for extras.

Note also that $(e_t - e_s)(s, t) = (e_t - e_s)(t, s)$ and so our discussion above indeed covers all orbit vectors. $\qquad\square$

From this we see that the flip algebra $A(\tau)$ coincides with M_H. Our notation for this flip algebra is $3Q_k(\eta)$. The 12-dimensional examples from the case C_5 from Section 6 is $3Q_2(\eta)$.

For a general $n = 2k + m$ and $\tau = z\tau_0$, we again get a direct sum decomposition.

Proposition 9.4. *Suppose that $n = 2k + m$. Then $A(\tau)$ is the direct sum of $3Q_k(\eta)$ arising on the subset $\{1, 2, \dots, 2k\}$ and the Matsuo algebra $M_\eta(\Gamma')$, where Γ' is the Fischer space of the group S_m acting on the subset $\{2k+1, \dots, n\}$.*

We omit the proof. Note, however, that the Matsuo summand $M_\eta(\Gamma')$ here is not for the group G_m, as in the first case, but rather for the group S_m, as in Theorem 8.3. This is because of the correcting factor z.

The projection graph of $3Q_k(\eta)$ is again easily seen to be connected, and so $3Q_k(\eta)$ is simple for all but a finite number of special values of η.

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Alexey Galt, Sobolev Institute of Mathematics, Novosibirsk, Russia;
Novosibirsk State University, Novosibirsk, Russia;
E-mail address: `galt84@gmail.com`

Vijay Joshi, School of Mathematics, University of Birmingham, Watson Building, Edgbaston, Birmingham B15 2TT, United Kingdom;
E-mail address: `vijay6d@gmail.com`

Andrey Mamontov, Sobolev Institute of Mathematics, Novosibirsk, Russia;
Novosibirsk State University, Novosibirsk, Russia;
E-mail address: `andreysmamontov@gmail.com`

Sergey Shpectorov, School of Mathematics, University of Birmingham, Watson Building, Edgbaston, Birmingham B15 2TT, United Kingdom;
E-mail address: `s.shpectorov@bham.ac.uk`

Alexey Staroletov, Sobolev Institute of Mathematics, Novosibirsk, Russia;
Novosibirsk State University, Novosibirsk, Russia;
E-mail address: `staroletov@math.nsc.ru`