# Finite groups with large Chebotarev invariant

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# FINITE GROUPS WITH LARGE CHEBOTAREV INVARIANT*

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ABSTRACT

A subset $\{g_1, \ldots, g_d\}$ of a finite group $G$ is said to invariably generate $G$ if the set $\{g_1^{x_1}, \ldots, g_d^{x_d}\}$ generates $G$ for every choice of $x_i \in G$. The Chebotarev invariant $C(G)$ of $G$ is the expected value of the random variable $n$ that is minimal subject to the requirement that $n$ randomly chosen elements of $G$ invariably generate $G$. The authors recently showed that for each $\epsilon > 0$, there exists a constant $c_\epsilon$ such that $C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon$. This bound is asymptotically best possible. In this paper we prove a partial converse: namely, for each $\alpha > 0$ there exists an absolute constant $\delta_\alpha$ such that if $G$ is a finite group and $C(G) > \alpha\sqrt{|G|}$, then $G$ has a section $X/Y$ such that $|X/Y| \geq \delta_\alpha\sqrt{|G|}$, and $X/Y \cong \mathbb{F}_q \rtimes H$ for some prime power $q$, with $H \leq \mathbb{F}_q^\times$.

## 1. Introduction

Following [10] and [5], we say that a subset $\{g_1, g_2, \ldots, g_d\}$ of a group $G$ *invariably generates* $G$ if $\{g_1^{x_1}, g_2^{x_2}, \ldots, g_d^{x_d}\}$ generates $G$ for each $d$-tuple $(x_1, x_2 \ldots, x_d) \in G^d$. The *Chebotarev invariant* $C(G)$ of $G$ is the expected value of the random

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variable $n$ which is minimal subject to the requirement that $n$ randomly chosen elements of $G$ invariably generate $G$.

Motivated by the problem of finding field extensions $K/F$ such that a fixed finite group $G$ occurs as the Galois group of $K/F$, E. Kowalski and D. Zywina carried out a detailed investigation of the invariant $C(G)$ in [12]. Amongst many interesting results, they show that $C(G)$ can be quite large in comparison to $|G|$. More precisely, it is shown that if $G \cong G_q := \mathbb{F}_q \rtimes \mathbb{F}_q^\times$, then

$$C(G) = q - \sum_{1 \neq d | q-1} \frac{\mu(d)}{q(1 - d^{-1})(1 - d^{-1} + q^{-1})}.$$

In particular, $C(G_q) \sim \sqrt{|G_q|}$ as $q \to \infty$. It was also conjectured in [12] that these are the "worst" cases: that is, that $C(G) = O(\sqrt{|G|})$ as $|G| \to \infty$. The conjecture was proved by the first author in [15], and was later improved in [17] where it is shown that for each $\epsilon > 0$, there exists a constant $c_\epsilon$ such that $C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon$. Furthermore, one has $C(G) \leq \frac{5}{3}\sqrt{|G|}$ when $G$ is soluble.

In this paper, we prove a partial converse. Informally, we prove that the the only examples where $C(G)$ is a constant times $\sqrt{|G|}$ are those groups with a "large" section isomorphic to a subgroup of $G_q$, for some prime power $q$. Our main result reads as follows.

THEOREM 1: *Fix a constant $\alpha > 0$. There exists absolute constants $\beta_\alpha$, $\gamma_\alpha$, $\delta_\alpha$ and $k_\alpha$, depending only on $\alpha$, such that whenever $G$ is a finite group with the property that $C(G) > \alpha\sqrt{|G|}$, then $G$ has a factor group $\overline{G}$ such that*

  (i) *$\overline{G} \cong V \rtimes H$, with $V \cong \mathbb{F}_q^k$, and $H \leq \Gamma L_1(q) \wr \mathrm{Sym}(k)$, with $q$ a prime power and $k \leq k_\alpha$;*
  (ii) *$|\overline{G}| \geq \delta_\alpha \sqrt{|G|}$; and*
  (iii) *$\beta_\alpha |V| \leq |H| \leq \gamma_\alpha |V|$.*

Our approach utilises the theory of crowns in finite groups, which we describe in Section 2. We also require a characterisation of those irreducible linear groups $H \leq GL(V)$ such that the set $H^*(V) := \{h \in H : v^h = v \text{ for some } v \in V \setminus \{0\}\}$ is bounded above by an absolute constant, and this is the content of Section 3. Finally, Section 4 is reserved for the proof of Theorem 1.
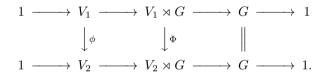
## 2. Crowns in finite groups

Before defining the notion of a crown in a finite group, we require some termi-
nology. First, let $L$ be a monolithic primitive group. That is, $L$ is a finite group
with a unique minimal normal subgroup $V \not\leq \mathrm{Frat}(L)$. For each positive integer
$k$, write $L^k$ for the $k$-fold direct product of $L$. The *crown-based power of $L$ of
size $k$* is the subgroup $L_k$ of $L^k$ defined by

$$L_k = \{(l_1, \ldots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \bmod V\}.$$

Equivalently, $L_k = V^k \, \mathrm{Diag} \, L^k$.

Next, let $G$ be a finite group. We say that a group $V$ is a *$G$-group* if $G$ acts
on $V$ via automorphisms. Following [9], we say that two irreducible $G$-groups
$V_1$ and $V_2$ are *$G$-equivalent* and we put $V_1 \sim_G V_2$, if there are isomorphisms
$\phi : V_1 \to V_2$ and $\Phi : V_1 \rtimes G \to V_2 \rtimes G$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & V_1 & \longrightarrow & V_1 \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Phi} & & \| & & \\
1 & \longrightarrow & V_2 & \longrightarrow & V_2 \rtimes G & \longrightarrow & G & \longrightarrow & 1.
\end{array}
$$

Note that two $G$-isomorphic $G$-groups are $G$-equivalent. In the abelian case,
the converse is true: if $V_1$ and $V_2$ are abelian and $G$-equivalent, then $V_1$ and
$V_2$ are also $G$-isomorphic. It is proved (see for example [9, Proposition 1.4])
that two chief factors $V_1$ and $V_2$ of $G$ are $G$-equivalent if and only if either
they are $G$-isomorphic, or there exists a maximal subgroup $M$ of $G$ such that
$G/\mathrm{Core}_G(M)$ has two minimal normal subgroups $N_1$ and $N_2$ $G$-isomorphic to
$V_1$ and $V_2$ respectively. For example, the minimal normal subgroups of a crown-
based power $L_k$ are all $L_k$-equivalent.

Let $V = X/Y$ be a chief factor of $G$. A complement $U$ to $V$ in $G$ is a
subgroup $U$ of $G$ such that $UV = G$ and $U \cap X = Y$. We say that $V = X/Y$
is a Frattini chief factor if $X/Y$ is contained in the Frattini subgroup of $G/Y$;
this is equivalent to saying that $V$ is abelian and there is no complement to $V$
in $G$. The number of non-Frattini chief factors $G$-equivalent to $V$ in any chief
series of $G$ does not depend on the series, and so this number is well-defined:
we will write it as $\delta_V(G)$. We now define $L_V$, the monolithic primitive group

associated to $V$, by

$$L_V := \begin{cases} V \rtimes (G/C_G(V)) & \text{if } V \text{ is abelian,} \\ G/C_G(V) & \text{otherwise.} \end{cases}$$

If $V$ is a non-Frattini chief factor of $G$, then $L_V$ is a homomorphic image of $G$. More precisely, there exists a normal subgroup $N$ of $G$ such that $G/N \cong L_V$ and $\mathrm{soc}(G/N) \sim_G V$. Consider now all the normal subgroups $N$ of $G$ with the property that $G/N \cong L_V$ and $\mathrm{soc}(G/N) \sim_G V$: the intersection $R_G(V)$ of all these subgroups has the property that $G/R_G(V)$ is isomorphic to the crown-based power $(L_V)_{\delta_V(G)}$. The socle $I_G(V)/R_G(V)$ of $G/R_G(V)$ is called the $V$-crown of $G$ and it is a direct product of $\delta_V(G)$ minimal normal subgroups $G$-equivalent to $V$.

We now record a lemma and two propositions which will be crucial in our proof of Theorem 1. The lemma reads as follows.

LEMMA 2: [1, Lemma 1.3.6] *Let $G$ be a finite group with trivial Frattini subgroup. There exists a chief factor $V$ of $G$ and a non trivial normal subgroup $U$ of $G$ such that $I_G(V) = R_G(V) \times U$.*

To state the propositions, we need some additional notation. For a finite group $G$, and an abelian chief factor $V$ of $G$, set $H_V = H_V(G) := G/C_G(V)$, $m = m_V = m_V(G) := \dim_{\mathrm{End}_G(V)} H^1(H_V, V)$, and write $H^* = H^*(V) = H_G^*(V)$ for the set of elements $h$ of $H_V$ which fix a non-zero vector in $V$. Also, let $\delta_V = \delta_V(G)$, and set $\theta_V = \theta_V(G) = 0$ if $\delta_V = 1$, and $\theta_V = 1$ otherwise. Finally, let $q_V = q_V(G) := |\mathrm{End}_G(V)|$ and $n_V = n_V(G) := \dim_{\mathrm{End}_G(V)} V$. Note that $\mathrm{End}_G(V)$ is a finite field, since $V$ is finite and irreducible.

PROPOSITION 3: [17, Proposition 8 and the Proof of Theorem 1] *Let $G$ be a finite group with trivial Frattini subgroup, and let $U$, $V$ and $R = R_G(V)$ be as in Lemma 2. If $U$ is non-abelian, then there exists absolute constants $b_1$, $b_2$ and $b_3$ such that*

$$C(G) \le C(G/U) + \lceil b_3 (\log|G|)^2 \rceil + \frac{b_1}{b_2} \sqrt{|G|^3} \log|G| (1 - b_2/\log|G|)^{\lceil b_3(\log|G|)^2 \rceil}.$$

PROPOSITION 4: [17, Proposition 8 and the Proof of Theorem 1] *Let $G$ be a finite group with trivial Frattini subgroup, and let $U$, $V$ and $R = R_G(V)$ be as in Lemma 2. Suppose that $V$ is abelian, and write $q = q_V$, $n = n_V$ and $H = H_V$, $H^* = H^*(V)$ and $m = m_V$. Also, set $\delta = \delta_V$ and $\theta = \theta_V$. Set*

$$\alpha_U := \begin{cases} \sum_{0 \leq i \leq \delta-1} \frac{q^{\delta}}{q^{\delta}-q^{i}} \leq \delta + \frac{q}{(q-1)^2} & \text{if } H = 1, \\ \min\left\{\left(\delta \cdot \theta + m + \frac{q}{q-1}\right) \frac{|H|}{|H^*|}, \left(\lceil \frac{\delta \cdot \theta}{n} \rceil + \frac{q^n}{q^n-1}\right) |H|\right\} & \text{otherwise.} \end{cases}$$

Then

$$C(G) \leq C(G/U) + \alpha_U.$$

We conclude this section with the theorem of the first author mentioned in the introduction.

THEOREM 5: [15, Main Theorem] *There exists an absolute constant $C$ such that $C(G) \leq C\sqrt{|G|}$ for any finite group $G$.*

## 3. Irreducible linear groups with few elements fixing a non-zero vector

Let $V$ be a finite dimensional vector space over an arbitrary field. In this section, our aim is to characterise the groups $H \leq GL(V)$, such that the set of elements which fix at least one non-zero vector in $V$ has cardinality bounded above by an absolute constant. For ease of notation, we will write

$$H^* = H^*(V) := \{h \in H : v^h = v \text{ for some } v \in V \backslash \{0\}\}$$

for such a subgroup $H$. Our main result reads as follows.

PROPOSITION 6: *Let $V$ be a vector space of dimension $n$ over a field $F$, and fix a constant $c > 0$. Suppose that $H$ is an irreducible subgroup of $GL(V)$ with the property that $|H^*| \leq c$. Then there exists positive integers $m$ and $k$ such that $n = mk$, and $H \leq R \wr \operatorname{Sym}(k)$, where either $|R|$ has order bounded above by a function of $|H^*|$, or $R \cong \Gamma_1(F_m)$ for some extension field $F_m$ of $F$ of degree $m$.*

Proposition 6 will follows almost immediately from our next result. Recall that if $F$ is a field, then an irreducible subgroup $H$ of a linear group $GL_n(F)$ is called *weakly quasiprimitive* if every characteristic subgroup of $G$ is homogeneous.

PROPOSITION 7: *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $F$ is a field, $n$ is a positive integer, and $H \leq GL_n(F)$ is finite and weakly quasiprimitive, then either $|H| \leq f(|H^*|)$, or $H$ is a subgroup of $\Gamma L_1(F_n)$, for some extension field $F_n$ of $F$ of degree $n$.*

*Proof.* If $n = 1$, then $\Gamma L_n(F) = GL_n(F)$. Thus, we may assume that $n > 1$. Fix a subgroup $H$ of $GL_n(F)$. We want to prove that if $H$ is not a subgroup of $\Gamma L_1(F_n)$ for some extension field $F_n$ of $F$ of degree $n$, then $|H|$ is bounded in terms of $|H^*|$.

Suppose first that every characteristic abelian subgroup of $H$ is contained in $Z(GL_n(F))$. Let $L$ be the generalised Fitting subgroup of $H$. Our aim is to prove that $|L|$ is bounded above in terms of $|H^*|$. Since $L$ is self-centralising, this will show that $|H|$ is bounded above in terms of $|H^*|$, which will give us what we need.

To this end, extend the field $F$ so that $F$ is a splitting field for all subgroups of $L$. Then $L$ may longer be homogeneous, but its irreducible constituents are algebraic conjugates of each other, so $L$ acts faithfully on them. Let $W$ be such a constituent, and let $r_i$, $m_i$, $s_i$, $t_i$, $S_i$ and $T_i$ be as in [8, Lemma 2.14]. In particular, the $r_i$ are prime numbers and $L$ is a central product of the collection of groups $O_{r_i}(G)$, $T_i$, where $T_i$ is a central product of $t_i$ copies of a quasisimple group $S_i$. By [8, Lemmas 2.15, 2.16 and 2.17], $W$ decomposes as a tensor product

$$W = W_Z \otimes W_{r_1} \otimes \ldots \otimes W_{r_a} \otimes W_{s_1} \otimes \ldots \otimes W_{s_b},$$

where $W_Z$ is a 1-dimensional module for $Z$; $W_{r_i}$ is an irreducible module for $O_{r_i}(G)$ of dimension $r_i^{m_i}$; and $W_{s_i}$ is an irreducible module for $T_i$ of dimension $s_i^{t_i}$. In particular, $[O_{r_i}(H), W_{r_j}] = [T_i, W_{s_j}] = 1$ for $i \neq j$, and $[O_{r_i}(H), W_{s_j}] = [T_i, W_{r_j}] = 1$, for all $i, j$. Hence, if $a + b > 1$, then $|L|$ is bounded above in terms of $|H^*|$, as needed. So we may assume that either $L = Z(G) \circ O_r(H)$, for some prime $r$, or $L = Z(G) \circ T$ is a central product of $t$ copies of a quasisimple group $S$. If $Z(G) \not\leq O_r(H)$ in the first case, or $Z(G) \not\leq T$ in the second case, then the same argument as above gives that $|L|$ is bounded in terms of $|H^*|$.

So we may assume that either $L = O_r(H)$, for some prime $r$, or $L = T$ is a central product of $t$ copies of a quasisimple group $S$. Hence, $W$ is a tensor product of $m$ [respectively $t$] copies of an irreducible module for an extraspecial group of order $r^3$ [resp. quasisimple group]. Thus, by arguing as in the paragraph above, we can immediately reduce to the case $m = 1$ [resp. $t = 1$].

Suppose first that $L = O_r(H) = M \rtimes \langle x \rangle$ is extraspecial of order $r^3$, for a prime $r$, where $M$ is cyclic of order $r^2$ if $L$ has exponent $r^2$, and $M$ is elementary abelian of order $r^2$ otherwise. Then, being an absolutely irreducible module for $L$ of dimension $r$, $W$ is isomorphic to $U \uparrow_M^L$, where $U$ is a one dimensional

module for $M$ in which $Z(L)$ acts non-trivially. Hence, we may write $W = \bigoplus_{i=0}^{r-1} U \otimes x^i$. It follows that for each non-zero vector $u \in U$, $x^j$ fixes the non-zero vector $u \otimes 1 + u \otimes x + \ldots + u \otimes x^{r-1}$. Thus, $r \leq |H^*|$, from which it follows that $|L| = r^3$ is bounded above in terms of $|H^*|$, as needed.

Finally, assume that $L$ is quasisimple. Since $L$ acts on $L^*$ by conjugation, we may assume that $L^* \leq Z$ (otherwise $L \leq \mathrm{Sym}(L^*)$, which would imply that $|L|$ is bounded above in terms of $|H^*|)$. However, since $Z = Z(H) \leq Z(GL_n(F))$, $Z$ acts on $V$ by scalar multiplication. Hence, $Z \cap H^* = 1$. It follows that $L^* = 1$, and hence that $L$ is a Frobenius complement in the group $V \rtimes L$. Since $L$ is perfect, it now follows from Zassenhaus' Theorem that $L \cong SL_2(5)$. Whence, $|L|$ is bounded, and this proves our claim.

Finally, assume that $H$ has a characteristic abelian subgroup not contained in $Z(GL_n(F))$, and let $M \leq H$ be maximal with this property. Then by [16, Lemma 1.10], $M$ is contained in $Z(GL_{\frac{n}{m}}(F_m))$ for some $m$ dividing $n$, and some extension field $F_m$ of $F$ of degree $m$. Hence, $H_1 := C_H(M)$ is a subgroup of $GL_{\frac{n}{m}}(F_m)$ with the property that every characteristic abelian subgroup of $H_1$ is contained in $Z(GL_{\frac{n}{m}}(F_m))$. Furthermore, $H_1$ is weakly quasiprimitive, since it is characteristic in $H$. Also, the group $H/H_1$ is naturally embedded in $\mathrm{Gal}(F_m/F)$, its action induced by a vector space isomorphism $F_m^{\frac{n}{m}} \to F^n$. Since $H_1^*(F_m^{\frac{n}{m}}) = H_1^*(F^n)$, it follows from the arguments above that either $|H_1|$ is bounded in terms of $|H^*|$; or $n = 1$. If $|H_1|$ is bounded in terms of $|H^*|$, then so is $|H|$, since $H_1$ is self-centralising and normal in $H$. If $n = 1$, then $H_1 \leq GL_1(F_n)$, so $H \leq \Gamma L_1(F_n)$, since $H/H_1$ acts on $M = Z(H_1)$ via the Galois group, as described above. This completes the proof. ■

Finally, we prove Proposition 6.

*Proof of Proposition 6.* If $H$ is primitive, then the result follows immediately from Proposition 7. Thus, we may assume that $H$ is not primitive. Then $V$ may be decomposed into a system $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$ of imprimitivity for $H$. Let $\Gamma := \{W_1, \ldots, W_k\}$, let $S := H^\Gamma$ denote the induced (transitive) action of $H$ on $\Gamma$, and let $R := \mathrm{Stab}_H(W_1)^{W_1}$ denote the induced action of $\mathrm{Stab}_H(W_1)$ on $W_1$. Then $H$ is isomorphic to a subgroup of the wreath product $R \wr S$.

Finally, since $\mathrm{Stab}_H(W)$ induces $R$ on $W$, we have $|R^*(W_1)| \leq |H^*(V)|$. Hence, Proposition 7 implies that either $R \leq \Gamma L_1(F_m)$, for some extension $F_m$ of $F$ of degree $m$, or $|R|$ is bounded above by a function of $|H^*|$. This completes the proof. ■

## 4. The proof of Theorem 1

We begin our preparations towards the proof of Theorem 1 with a lemma concerning the cohomology of an irreducible linear group which has a bounded number of elements fixing a non-zero vector.

LEMMA 8: *There exists an absolute constant $c$ such that if $V$ is a vector space of dimension $n$ over a field $F$ of characteristic $p > 0$, and $H$ is an irreducible subgroup of $GL(V)$ with the property that $|H| > \sqrt{|V|}$, then $2^m \leq c|H^*|^4$, where $m := \dim_F \mathrm{H}^1(H, V)$ and $F := \mathrm{End}_H V$.*

*Proof.* Clearly we may assume that $m > 0$. Then, it is proven in [15, Lemma 9] that

(1) $H$ has a unique minimal normal subgroup $N$, which is non-abelian.
(2) If $S$ is a component of $H$, then $C_H(S) \subseteq H^*$.
(3) If $W$ is an irreducible $N$-submodule of $V$ not centralised by $S$, then $m \leq \dim_F \mathrm{H}^1(S, W)$.

Write $N = S_1 \times \ldots \times S_t \cong S^t$, and view $H$ as a subgroup in the wreath product $\mathrm{Aut}(N) = \mathrm{Aut}(S) \wr K$, where $K$ denote the induced action of $H$ on the components in $N$. Suppose first that $t > 1$. Then (2) implies that $S_i \subseteq H^*$ for all $i$. Hence, $|H^*| \geq 1 + t(|S| - 1)$. Also, $|H^*| \geq C_H(S_1) \geq |H \cap B|| \mathrm{Stab}_K(1)| = |H \cap B| \frac{|K|}{t}$, where $B := \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_t)$. Note that $|H| \leq |H \cap B|| \mathrm{Aut}(S)||K|$. It follows that $|H| \leq |H^*|t| \mathrm{Aut}(S)| \leq |H^*|t(|S| - 1)^2 \leq |H^*|^3$.

Next, it is shown by Guralnick and Hoffman in [7, Theorem 1] that $m \leq \frac{n}{2}$. Since we also have $|H| > \sqrt{|V|}$, it follows that

$$m \leq \frac{n}{2} \leq \log \sqrt{|V|} < \log |H| \leq \log |H^*|^3.$$

Thus, we may assume that $H \leq \mathrm{Aut}(S)$ is almost simple. Before distinguishing cases, we make some remarks. First, $p = \mathrm{char}\, F$ divides $|H|$, since $\mathrm{H}^1(H, V) \neq 0$. Furthermore, $|H^*| \geq |H|_p$, since every element of a Sylow $p$-subgroup of $H$ fixes a non-zero vector in $V$. Finally, note that we may assume that $S$ is not sporadic, since there are a bounded number of such groups having an irreducible module with non-zero cohomology.

Thus, we have two cases.

(a) $S \cong \mathrm{Alt}(k)$. In this case, we have $\frac{n}{2} \leq \log \sqrt{|V|} \leq \log |H| \leq k \log k$, as long as $k > 6$. Hence, by [15, Proof of Proposition 10], we have $m \leq 4 \log k$ and $|H|_p > \frac{k}{2}$, if $k$ is large enough. Hence $2^m \leq k^4 \leq 16|H^*|^4$ in this case. If

$k$ is bounded, then $m$ is also bounded, since $m \leq \frac{n}{2} \leq \log|H|$. Hence, the result also follows in this case.

(b) $S \cong^{\epsilon} X_k(r)$ is a group of Lie type. Write $R_F(S)$ for the smallest degree of a non-trivial irreducible representation of $S$ over the field $F$. If char $F$ is different to the defining characteristic for $S$, then we have $p^{\frac{R_F(S)}{2}} > |\operatorname{Aut}(S)|$ for $|S|$ large enough (see [13, 18, 20]). Since $\sqrt{|V|} \leq |H|$, we conclude that either $|S|$ is bounded, or char $F$ coincides with the defining characteristic of $S$. In the latter case, we have $|H|_p > |S|^{\frac{1}{3}}$ by [11, Proposition 3.5]. Also, $|S| \geq |\operatorname{Aut}(S)|^{\frac{4}{5}}$ by [14, Proposition 4.4]. Hence,

$$|H^*| > |S|^{\frac{1}{3}} \geq |\operatorname{Aut}(S)|^{\frac{4}{15}} > |H|^{\frac{1}{4}} \geq 2^{\frac{m}{4}}.$$

Thus, either $|S|$ is bounded, or $2^m \leq |H^*|^4$. This gives us what we need.

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Next, we prove a reduction lemma.

LEMMA 9: *Fix a constant $\alpha > 0$. There exists absolute constants $b = b(\alpha)$, $c = c(\alpha)$ and $c_i = c_i(\alpha)$, $1 \leq i \leq 4$, depending only on $\alpha$, such that: If $G$ is a finite group with trivial Frattini subgroup with the property that $C(G) > \alpha\sqrt{|G|}$, and $U$ is as in Lemma 2, then one of the following holds.*

(i) *$U$ is non-abelian and $|G| \leq b$.*

(ii) *$U$ is abelian and $|U| \leq c$.*

(iii) *$U$ is abelian and $G$ has a factor group $\overline{G}$ such that*

(a) *$\overline{G} \cong V \rtimes H$, with $V \cong U$ an abelian chief factor of $G$, and $H \leq GL(V)$;*

(b) *$|H^*(V)| \leq c_1$;*

(c) *$\dim_{\operatorname{End}_H V} \operatorname{H}^1(H, V) \leq c_2$; and*

(d) *$c_3|V| \leq |H| \leq c_4|V|$.*

*Proof.* Adopt in its entirety the notation of Proposition 4, so that $U$, $V$ and $R = R_G(V)$ are as in Lemma 2. We first consider the case where $V$ is non-abelian. Then by Proposition 3 we have

$$\alpha\sqrt{|G|} < C(G/U) + \lceil b_3(\log|G|)^2 \rceil + \frac{b_1}{b_2}\sqrt{|G|^3}\log|G|(1 - b_2/\log|G|)^{\lceil b_3(\log|G|)^2 \rceil},$$

where $b_1$, $b_2$ and $b_3$ are the absolute constants from Proposition 3. Since $C(G/U) \leq C\sqrt{|G/U|}$, it follows that $\sqrt{|G|} \leq \alpha'\lceil b_3(\log|G|)^2 \rceil + \frac{b_1}{b_2}\sqrt{|G|^3}\log|G|(1 - b_2/\log|G|)^{\lceil b_3(\log|G|)^2 \rceil}$, for some constant $\alpha'$ depending only on $\alpha$. Hence, since the square root of $|G|$ divided by the right hand side of the above equation

tends to $\infty$ as $|G|$ tends to infinity, we must have that $|G|$ is bounded above by a constant $b = b(\alpha)$ depending only on $\alpha$.

Thus, we may assume that $U$ is abelian. Then by Proposition 4 and Theorem 5, there exists an absolute constant $C$ such that

$$\alpha\sqrt{|G|} \leq C(G) \leq C(G/U) + \alpha_U \leq c\sqrt{\frac{|G|}{|U|}} + \alpha_U.$$

In particular, using the definition of $\alpha_U$ from Proposition 4, we conclude that

$$(4.1) \qquad \alpha \leq \frac{c}{\sqrt{|U|}} + (\delta \cdot \theta + m + 2)\frac{\sqrt{|H|}}{\sqrt{|V|^{\delta}|H^*|}}, \text{ and}$$

$$(4.2) \qquad \alpha \leq \frac{c}{\sqrt{|U|}} + \left(\left\lceil\frac{\delta \cdot \theta}{n}\right\rceil + 2\right)\frac{\sqrt{|H|}}{\sqrt{|V|^{\delta}}}.$$

We claim first that $\delta = 1$. Indeed, assume otherwise, and note that $\frac{|H|}{|H^*|} \leq |H|/|H_v| \leq |V|$, for any non-zero $v \in V$. Hence, since $m \leq \frac{n}{2}$, we conclude from (4.1) that

$$(4.3) \qquad |V|^{\frac{\delta-1}{2}} \leq C_1(n+\delta),$$

where $C_1 = C_1(\alpha)$ depending only on $\alpha$. Now, since $|U| = |V|^{\delta} = q^{n\delta}$, we conclude that there exists a constant $c = c(\alpha)$ such that if $|U| > c$ and $\delta > 1$ then $|V|^{\frac{\delta-1}{2}} > C_1(n+\delta)$.

Hence, we may assume that $\delta = 1$. We will first prove that the properties (b) and (c) of Part (iii) of thge statement of the lemma hold in the factor group $\overline{G} := G/R_G(V)$. If $|H| \leq \frac{|V|}{n^2}$, then (4.1) [respectively (4.2)] implies that $|H^*|$ [resp. $n$] is bounded above by a constant depending only on $\alpha$. Properties (b) and (c) then follow immediately.

So we may assume that $|H| > \frac{|V|}{n^2}$. We then use (4.1) and the fact that $|H|/|H_v| \leq |V|$ to deduce that $|H^*| \leq C_2(1 + m^2)$, where $C_2 = C_2(\alpha)$ is a constant depending only on $\alpha$. Since $|H| > \sqrt{|V|}$, if follows from Lemma 8 that $|H^*| \leq C_3(1 + \log|H^*|^2)$, where $C_3 = C_3(\alpha)$ is a constant depending only on $\alpha$. It follows that $|H^*|$, and hence $m$, are bounded above by constants depending only on $\alpha$. This proves that Properties (b) and (c) hold.

Finally, the existence of $c_3$ follows immediately from (4.2), while the existence of $c_4$ follows from (4.1) and the bound $|H|/|H^*| \leq |V|$. This proves that Property (d) holds, and completes the proof. ∎

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* Let $C$ be the constant from Theorem 5; let $f$ be the function from Proposition 7; let $b_1$, $b_2$ and $b_3$ be the constants from Proposition 3; and let $b = b(\alpha)$ and $c = c(\alpha)$ be the constants from Lemma 9. Also, let $c_i$, $1 \leq i \leq 4$, be the functions of $\alpha$ from Lemma 9. Note that we may assume that $f$, $c_1$, $c_2$ and $c_4$ are increasing functions, while $c_3$ is decreasing. Hence, we may also assume that $g$ satisfies $g(\alpha_1 \alpha_2) \geq g(\alpha_1)\alpha_2$, for $g \in \{f, c_1\}$. For ease of notation, we will sometimes write $c_i$ in place of $c_i(\alpha)$.

Set $b_4 := \max\{b, \lceil b_3 (\log b)^2 \rceil + \frac{b_1}{b_2}\sqrt{b^3}\log b(1 - b_2/\log b)^{\lceil b_3(\log b)^2\rceil}\}$; $\alpha' := \max\{\alpha, C\}$; $c_5 := \max\{c, \frac{1}{c_3(\alpha')}f(\lfloor c_1(\alpha') \rfloor)^{\frac{c_1(\alpha')}{c_3(\alpha')}}\lfloor \frac{c_1(\alpha')}{c_3(\alpha')} \rfloor!\}$; and $c_6 := (2 + c_2)c_5$. Then define

$$\delta(\alpha) := \min\{f(\lfloor c_1(\beta) \rfloor) : 0 < \beta \leq \alpha'\} \text{ and}$$
$$k(\alpha) := \frac{c_1(\alpha')}{c_3(\alpha')}.$$

Finally, set $\beta := c_3$ and $\gamma := c_4$. Note that by construction $k$ is an increasing function of $\alpha$, and that

(4.4) $$\delta(\beta\sqrt{u}) \geq \delta(\beta)\sqrt{u} \geq \delta(\alpha)\sqrt{u},$$

whenever $\beta \leq \alpha$.

We will now prove by induction on $|G|$ that $G$ has a factor group $\overline{G}$ such that

(i) $\overline{G} \cong V \rtimes H$, with $V \cong \mathbb{F}_q^k$, and $H \leq \Gamma L_1(q) \wr \mathrm{Sym}(k)$, with $q$ a prime power and $k \leq k(\alpha)$;
(ii) $|\overline{G}| \geq \delta(\alpha)\sqrt{|G|}$; and
(iii) $\beta(\alpha)|V| \leq |H| \leq \gamma(\alpha)|V|$.

Suppose first that $\mathrm{Frat}(G) = 1$, and let $U$, $V$ and $R = R_V(G)$ be as in Lemma 2. We would like to reduce to the case where $|G| > b$ if $V$ is non-abelian, and $|U| > c_5$ if $V$ is abelian. We first deal with the non-abelian case. So assume that $V$ is non-abelian and that $|G| \leq b$. In this case, we have

$$\alpha\sqrt{|G|} < C(G/U) + b_4 \leq (1 + b_4)C(G/U),$$

by Proposition 3. In particular, it follows that $C(G/U) > \alpha_1\sqrt{|G/U|}$, where

$$\alpha_1 := \frac{\alpha\sqrt{|U|}}{1 + b_4}.$$

Note that $\gamma(\alpha_1) \leq \gamma(\alpha)$, since $\alpha_1 \leq \alpha$, and $\gamma$ is an increasing function. Similarly, $k(\alpha_1) \leq k(\alpha)$ and $\beta(\alpha) \leq \beta(\alpha_1)$. Furthermore, $\delta(\alpha_1) \geq \delta(\alpha)\sqrt{|U|}$ by (4.4). The inductive hypothesis now implies that $G$, and hence $G/U$, has a factor group $\overline{G}$ with the desired properties.

Next, assume that $V$ is abelian, and that $|U| \leq c$. Then since $\alpha_U \leq c_6$, Proposition 4 yields $C(G/U) > \alpha_2 \sqrt{|G/U|}$, where

$$\alpha_2 := \frac{\alpha\sqrt{|U|}}{1 + c_6}.$$

As above, it now follows from the inductive hypothesis and the definitions of $\delta(\alpha)$ and $k(\alpha)$ that $G$ has a factor group $\overline{G}$ with the desired properties.

Thus, we may assume that $|G| > b$ if $U$ is non-abelian, and $|U| > c_5 \geq c$ otherwise. However, Lemma 9 then implies that $U$ must be abelian, and that $G$ has a factor group $\overline{G}$ such that

(a) $\overline{G} \cong V \rtimes H$, with $V \cong U$ an abelian chief factor of $G$, and $H \leq GL(V)$;
(b) $|H^*(V)| \leq c_1(\alpha)$;
(c) $\dim_{\mathrm{End}_H V} \mathrm{H}^1(H, V) \leq c_2(\alpha)$; and
(d) $c_3(\alpha)|V| \leq |H| \leq c_4(\alpha)|V|$.

Furthermore, Lemma 6 guarantees the existence of positive integers $m$ and $k$, and a transitive permutation group $S$ of degree $k$, such that $n = mk$ and $H \leq R \wr S$, with either $|R| \leq f(c_1)$, or $R \leq \Gamma L_1(p^m)$. Hence, we just need to prove that $k \leq k(\alpha)$. Indeed, if this is true then we must have $R \leq \Gamma L_1(p^m)$, since otherwise $|V| \leq \frac{1}{c_3(\alpha)}|H| \leq \frac{1}{c_3(\alpha)} f(c_1(\alpha))^{\frac{c_1(\alpha)}{c_3(\alpha)}} \lfloor \frac{c_1(\alpha)}{c_3(\alpha)} \rfloor!$, contradicting $|U| > c_5$.

Now, note that (b) and (d) above imply that the number of orbits of $H$ in its action on $V$ is bounded above by $1 + \frac{c_1}{c_3}$. Hence, the number of orbits of $X := GL_m(p) \wr \mathrm{Sym}(k)$ is bounded above by $1 + \frac{c_1}{c_3}$. Then since $GL_m(p)$ has 2 orbits in its action on the natural module $(\mathbb{F}_p)^m$, it follows that the number of orbits of $X$ on $V$ is precisely the number of orbits of $\mathrm{Sym}(k)$ in its action on the $k$-fold cartesian power $\{0, 1\}^k$ by permutation of coordinates. This number is precisely $k + 1$. Hence, we have $k + 1 \leq 1 + \frac{c_1}{c_3}$, and this completes the proof in the case $\mathrm{Frat}(G) = 1$.

Finally, assume that $\mathrm{Frat}(G) > 1$. Then $C(G/\mathrm{Frat}(G)) = C(G) > \beta\sqrt{|G/\mathrm{Frat}(G)|}$, where $\beta := \alpha\sqrt{|\mathrm{Frat}(G)|}$. Now, since $\alpha\sqrt{|G|} < C(G/\mathrm{Frat}(G)) \leq C\sqrt{|G/\mathrm{Frat}(G)|}$, we have $|\mathrm{Frat}(G)| \leq (\frac{C}{\alpha})^2$. Hence, $\beta \leq C$. The result now follows from the inductive hypothesis and the definitions of $\delta(\alpha)$ and $k(\alpha)$. ∎

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