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AN UPPER BOUND ON THE CHEBOTAREV INVARIANT OF A FINITE GROUP*

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ABSTRACT

A subset $\{g_1, \dots, g_d\}$ of a finite group G invariably generates G if the set $\{g_1^{x_1}, \dots, g_d^{x_d}\}$ generates G for every choice of $x_i \in G$. The Chebotarev invariant $C(G)$ of G is the expected value of the random variable n that is minimal subject to the requirement that n randomly chosen elements of G invariably generate G. The first author recently showed that $C(G) \leq \beta\sqrt{|G|}$ for some absolute constant β. In this paper we show that, when G is soluble, then β is at most $5/3$. We also show that this is best possible. Furthermore, we show that, in general, for each $\epsilon > 0$ there exists a constant c_ϵ such that $C(G) \leq (1+\epsilon)\sqrt{|G|} + c_\epsilon$.

1. Introduction

Following [8] and [5], we say that a subset $\{g_1, g_2, \dots, g_d\}$ of a group G *invariably generates* G if $\{g_1^{x_1}, g_2^{x_2}, \dots, g_d^{x_d}\}$ generates G for every d-tuple $(x_1, x_2 \dots, x_d) \in G^d$. The Chebotarev invariant $C(G)$ of G is the expected value of the random variable n that is minimal subject to the requirement that n randomly chosen elements of G invariably generate G.

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In [9], Kowalski and Zywina conjectured that $C(G) = O(\sqrt{|G|})$ for every finite group G. Progress on the conjecture was first made in [8], where it was shown that $C(G) = O(\sqrt{|G|}\log|G|)$ (here, and throughout this paper, "log" means log to base 2). The conjecture was confirmed by the first author in [10]; more precisely, [10, Theorem 1] states that *there exists an absolute constant* β *such that* $C(G) \leq \beta\sqrt{|G|}$ *whenever* G *is a finite group.*

In this paper, we use a different approach to the problem. In doing so, we show that one can take $\beta = 5/3$ when G is soluble, and that this is best possible. Furthermore, we show that for each $\epsilon > 0$, there exists a constant c_ϵ such that $C(G) \leq (1+\epsilon)\sqrt{|G|} + c_\epsilon$. From [9, Proposition 4.1], one can see that this is also (asymptotically) best possible.

Our main result is as follows

THEOREM 1: *Let* G *be a finite group.*

(i) For any $\epsilon > 0$, *there exists a constant* c_ϵ *such that* $C(G) \leq (1+\epsilon)\sqrt{|G|}+c_\epsilon$;
(ii) If G *is a finite soluble group, then* $C(G) \leq \frac{5}{3}\sqrt{|G|}$, *with equality if and only if* $G = C_2 \times C_2$.

We also derive an upper bound on $C(G)$, for a finite soluble group G, in terms of the set of *crowns* for G. Before stating this result, we require the following notation: Let G be a finite soluble group. Given an irreducible G-module V which is G-isomorphic to a complemented chief factor of G, let $\delta_V(G)$ be the number of complemented factors in a chief series of G which are G-isomorphic to V. Then set $\theta_V(G) = 0$ if $\delta_V(G) = 1$, and $\theta_V(G) = 1$ otherwise. Also, let $q_V(G) := |\operatorname{End}_G(V)|$, let $n_V(G) := \dim_{\operatorname{End}_G(V)} V$, and let $H_V(G) := G/C_G(V)$ (we will suppress the G in this notation when the group is clear from the context). Also, let $\sigma := 2.118456563\ldots$ be the constant appearing in [11, Corollary 2]. The afore mentioned upper bound can now be stated as follows.

THEOREM 2: *Let* G *be a finite soluble group, and let* A *[respectively* B*] be a set of representatives for the irreducible* G*-modules which are* G*-isomorphic to a non-central [resp. central] complemented chief factor of* G*. Then*

$$C(G) \leq \sum_{V\in A} \min\left\{(\delta_V\cdot\theta_V + c_V)|V|, \left(\left\lceil\frac{\delta_V\cdot\theta_V}{n_V}\right\rceil + \frac{q_V^{n_V}}{q_V^{n_V}-1}\right)|H_V|\right\} + \max_{V\in B}\delta_V + \sigma$$

where $c_V := q_V/(q_V-1) \leq 2$.

The layout of the paper is as follows. In Section 2 we recall the notion of a *crown* in a finite group. In Section 3 we prove Theorem 2 and deduce a number of consequences, while Section 4 is reserved for the proof of Theorem 1 Part (i). Finally, we prove Theorem 1 Part (ii) in Section 5.

2. Crowns in finite groups

In Section 2, we recall the notion and the main properties of crowns in finite groups. Let L be a monolithic primitive group and let A be its unique minimal normal subgroup. For each positive integer k, let L^k be the k-fold direct product of L. The crown-based power of L of size k is the subgroup L_k of L^k defined by

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \bmod A\}.$$

Equivalently, $L_k = A^k \operatorname{Diag} L^k$.

Following [7], we say that two irreducible G-groups V_1 and V_2 are G-equivalent and we put $V_1 \sim_G V_2$, if there are isomorphisms $\phi : V_1 \to V_2$ and $\Phi : V_1 \rtimes G \to V_2 \rtimes G$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & V_1 \arrow[r] \arrow[d, "\phi"] & V_1 \rtimes G \arrow[r] \arrow[d, "\Phi"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & V_2 \arrow[r] & V_2 \rtimes G \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

Note that two G-isomorphic G-groups are G-equivalent. In the particular case where V_1 and V_2 are abelian the converse is true: if V_1 and V_2 are abelian and G-equivalent, then V_1 and V_2 are also G-isomorphic. It is proved (see for example [7, Proposition 1.4]) that two chief factors V_1 and V_2 of G are G-equivalent if and only if either they are G-isomorphic between them or there exists a maximal subgroup M of G such that $G/\operatorname{Core}_G(M)$ has two minimal normal subgroups N_1 and N_2 G-isomorphic to V_1 and V_2 respectively. For example, the minimal normal subgroups of a crown-based power L_k are all L_k-equivalent.

Let $V = X/Y$ be a chief factor of G. A complement U to V in G is a subgroup U of G such that $UV = G$ and $U \cap X = Y$. We say that $V = X/Y$ is a Frattini chief factor if X/Y is contained in the Frattini subgroup of G/Y; this is equivalent to saying that V is abelian and there is no complement to V in G. The number $\delta_V(G)$ of non-Frattini chief factors G-equivalent to V in any chief series of G does not depend on the series. Now, we denote by L_V the monolithic primitive group associated to V, that is

$$L_V = \begin{cases} V \rtimes (G/C_G(V)) & \text{if } V \text{ is abelian,} \\ G/C_G(V) & \text{otherwise.} \end{cases}$$

If V is a non-Frattini chief factor of G, then L_V is a homomorphic image of G. More precisely, there exists a normal subgroup N of G such that $G/N \cong L_V$ and $\mathrm{soc}(G/N) \sim_G V$. Consider now all the normal subgroups N of G with the property that $G/N \cong L_V$ and $\mathrm{soc}(G/N) \sim_G V$: the intersection $R_G(V)$ of all these subgroups has the property that $G/R_G(V)$ is isomorphic to the crown-based power $(L_V)_{\delta_V(G)}$. The socle $I_G(V)/R_G(V)$ of $G/R_G(V)$ is called the V-crown of G and it is a direct product of $\delta_V(G)$ minimal normal subgroups G-equivalent to V.

Lemma 3: [1, Lemma 1.3.6] *Let G be a finite group with trivial Frattini subgroup. There exists a chief factor V of G and a non trivial normal subgroup U of G such that $I_G(V) = R_G(V) \times U$.*

Lemma 4: [4, Proposition 11] *Assume that G is a finite group with trivial Frattini subgroup and let $I_G(V), R_G(V), U$ be as in the statement of Lemma 3. If $KU = KR_G(V) = G$, then $K = G$.*

3. Crown-based powers with abelian socle

The aim of this section is to prove Theorem 2. For a finite group G and an irreducible G-group V, we write $\Omega_{G,V}$ for the set of maximal subgroups M of G such that either $\mathrm{soc}\,(G/\operatorname{Core}_G(M)) \sim_G V$ or $\mathrm{soc}\,(G/\operatorname{Core}_G(M)) \sim_G V \times V$. Also, for $M \in \Omega_{G,V}$, we write $\widetilde{M}$ for the union of the G-conjugates of M. We will also say that the elements $g_1, g_2, \ldots, g_k \in G$ *satisfy the V-property in G* if $g_1, g_2, \ldots, g_k \in \widetilde{M}$ for some $M \in \Omega_V$. Finally, let $P^*_{G,V}(k)$ denote the probability that k randomly chosen elements of G satisfy the V-property in G.

Suppose now that V is abelian, and consider the faithful irreducible linear group $H := G/C_G(V)$. We will denote by $\mathrm{Der}(H, V)$ the set of the derivations from H to V (i.e. the maps $\zeta : H \to V$ with the property that $\zeta(h_1h_2) = \zeta(h_1)^{h_2} + \zeta(h_2)$ for every $h_1, h_2 \in H$). If $v \in V$ then the map $\zeta_v : H \to V$ defined by $\zeta_v(h) = [h, v]$ is a derivation, called an *inner derivation* from H to V. The set $\mathrm{InnDer}(H, V) = \{\zeta_v \mid v \in V\}$ of the inner derivations from H to V is a subgroup of $\mathrm{Der}(V,H)$ and the factor group $\mathrm{H}^1(H,V) = \mathrm{Der}(H,V)/\,\mathrm{InnDer}(H,V)$ is the first cohomology group of H with coefficients in V.

Proposition 5: *Let H be a group acting faithfully and irreducibly on an elementary abelian p-group V. For a positive integer u, we consider the semidirect product $G = V^u \rtimes H$ where the action of H is diagonal on V^u; that is, H acts in the same away on each of the u direct factors. Assume also that $u = \delta_V(G)$. View V as a vector space over the field $F = End_H(V)$. Let $h_1, \dots, h_k \in H$, and $w_1, \dots, w_k \in V^u$, and write $w_i = (w_{i,1}, w_{i,2}, \dots, w_{i,u})$. Assume that $h_1w_1, h_2w_2, \dots, h_kw_k$ satisfy the V-property in G. Then for $1 \le j \le u$, the vectors*

$$r_j := (w_{1,j}, w_{2,j}, \dots, w_{k,j})$$

of V^k are linearly dependent modulo the subspace $W + D$, where

$$W := \{(y_1, y_2, \dots, y_k) : y_i \in [h_i, V] \text{ for } 1 \le i \le k\}, \text{ and}$$
$$D := \left\{(\zeta(h_1), \zeta(h_2), \dots, \zeta(h_k)) \in V^k : \zeta \in \mathrm{Der}(H,V)\right\}.$$

Proof. Let M be a maximal subgroup of G such that $M \in \Omega_V$, and $h_1w_1, \dots, h_kw_k \in \widetilde{M}$. Since $u = \delta_V(G)$, M cannot contain V^u, and hence $MV^u = G$. Thus, $M/M \cap V^u \cong H$, and hence there exists an integer $t \ge 0$ and elements $h_{k+1}w_{k+1}, \dots, h_{k+t}w_{k+t} \in M$ such that $h_1, \dots, h_k, h_{k+1}, \dots, h_{k+t}$ invariably generate H. But then, [10, Proposition 6] implies, in particular, that $r_1, \dots, r_u \in V^k$ are linearly dependent modulo $W + D$, as needed. ∎

Before proceeding to the proof of Theorem 2, we require the following easy result from probability theory.

Proposition 6: *Write $B(k,p)$ for the binomial random variable with k trials and probability $0 < p \le 1$. Fix $l \ge 0$. Then*

$$\sum_{k=l}^{\infty} P(B(k,p) = l) \le \frac{1}{p}.$$

Proof. Note first that

$$\binom{k}{l} x^{k-l} = \frac{1}{l!}\frac{d^l}{dx^l} x^k$$

where $\frac{d^l}{dx^l}x^k$ denotes the l-th derivative of x^k. Let $x = 1 - p$. By definition, $P(B(k,p) = l) = \binom{k}{l}(1-x)^l x^{k-l}$. Thus

$$\begin{aligned}\sum_{k=l}^{\infty} P(B(k,p)=l) &= (1-x)^l \sum_{k=l}^{\infty} \binom{k}{l} x^{k-l}\\ &= \frac{(1-x)^l}{l!}\sum_{k=l}^{\infty}\frac{d^l}{dx^l}x^k\\ &= \frac{(1-x)^l}{l!}\frac{d^l}{dx^l}\sum_{k=l}^{\infty}x^k\\ &\leq \frac{(1-x)^l}{l!}\frac{d^l}{dx^l}\frac{1}{1-x}\\ &= \frac{(1-x)^l}{l!}\frac{l!}{(1-x)^{(l+1)}} = \frac{1}{1-x} = \frac{1}{p}\end{aligned}$$

as needed. (Note that the third equality above follows since the series $\sum_{k=l}^{\infty} x^k$ is convergent.) ∎

We shall also require the following. We remark that since $P^*_{G,V}(k) \leq \sum_{\widetilde{M}\in\Omega_V}\left(\frac{|\widetilde{M}|}{|G|}\right)^k$ and $\frac{|\widetilde{M}|}{|G|} < 1$, $\sum_{k=0}^{\infty} P^*_{G,V}(k)$ converges.

Proposition 7: *Let G be a finite group, and let A [respectively B] be a set of representatives for the irreducible G-groups which are G-equivalent to a non-central [resp. central] non-Frattini chief factor of G. Then*

(1) $C(G) \leq \sum_{V\in A}\sum_{k=0}^{\infty} P^*_{G,V}(k) + \max_{V\in B}\delta_V + \sigma$, and;
(2) If $\mathrm{Frat}(G) = 1$ and U and V are as in Lemma 3, then $C(G) \leq C(G/U) + \sum_{k=0}^{\infty} P^*_{G,V}(k)$.

Proof. By definition, $C(G) = \sum_{k=0}^{\infty}(1 - P_I(G,k))$, where $P_I(G,k)$ denotes the probability that k randomly chosen elements of G invariably generate G. Let $P_{G,G/G'}(k)$ denote the probability that k randomly chosen elements $g_1, \ldots, g_k$ of G satisfy $\langle G'g_1, \ldots, G'g_k\rangle = G$. Then it is easy to see that

$$1 - P_I(G,k) \leq 1 - P_{G,G/G'}(k) + \sum_{V\in A} P^*_{G,V}(k). \tag{3.1}$$

Clearly $P_{G,G/G'}(k)$ is the probability that a random k-tuple of elements from G/G' generates G/G'. Hence, $C(G/G') = \sum_{k=0}^{\infty}(1 - P_{G,G/G'}(k))$ is at most $d(G/G')+\sigma$ by [11, Corollary 2] (here, for a group X, $d(X)$ denotes the minimal number of elements required to generate X). Since $d(G/G') \le \max_{V\in B}\delta_V$, it follows from (3.1) that $C(G) \le \max_{V\in B}\delta_V + \sigma + \sum_{V\in A}\sum_{k=0}^{\infty} P^*_{G,V}(k)$, and Part (i) follows.

Assume that $\mathrm{Frat}(G)=1$, and let U and V be as in Lemma 3. Then

$$1 - P_I(G,k) \le 1 - P_I(G/U,k) + \sum_W P^*_{G,W}(k) \tag{3.2}$$

where the sum in the second term goes over all complemented chief factors W of G not containing U. Now, if M is a maximal subgroup of G not containing U, then M contains $R_G(V)$, by Lemma 4. Hence, $\mathrm{Core}_G(M)$ contains $R_G(V)$, so $M \in \Omega_{G,V}$. Since $C(G) = \sum_{k=0}^{\infty}(1-P_I(G,k))$, Part (ii) now follows immediately from (3.2), and this completes the proof. ∎

The proof of Theorem 2 will follow as a corollary of the proof of the next proposition. For a finite group G, and an abelian chief factor V of G, set $H_V = H_V(G) := G/C_G(V)$, $m = m_V = m_V(G) := \dim_{\mathrm{End}_G(V)} \mathrm{H}^1(H_V,V)$, and write $p = p_V = p_V(G)$ for the probability that a randomly chosen element h of H_V fixes a non zero vector in V. Also, let $\delta_V = \delta_V(G)$ be the number of complemented factors in a chief series of G which are G-isomorphic to V, and set $\theta_V = \theta_V(G) = 0$ if $\delta_V = 1$, and $\theta_V = 1$ otherwise. Finally, let $q_V = q_V(G) := |\mathrm{End}_G(V)|$ and $n_V = n_V(G) := \dim_{\mathrm{End}_G(V)} V$.

Proposition 8: *Let G be a finite group with trivial Frattini subgroup, and let U, V and $R = R_G(V)$ be as in Lemma 3. If V is nonabelian, then set $\alpha_U := \sum_{k=0}^{\infty} P^*_{G,V}(k)$. If V is abelian, then write $q = q_V$, $n = n_V$ and $H = H_V$, $p = p_V$ and $m = m_V$. Also, set $\delta = \delta_V$ and define $\theta = 0$ if $\delta = 1$, $\theta = 1$ otherwise, and set*

$$\alpha_U := \begin{cases} \sum_{0\le i\le \delta-1} \frac{q^\delta}{q^\delta - q^i} \le \delta + \frac{q}{(q-1)^2} & \text{if } H = 1,\\ \min\left\{ \left(\delta\cdot\theta + m + \frac{q}{q-1}\right)\frac{1}{p}, \left(\lceil \frac{\delta\cdot\theta}{n}\rceil + \frac{q^n}{q^n-1}\right)|H| \right\} & \text{otherwise.} \end{cases}$$

Then

$$C(G) \le C(G/U) + \alpha_U.$$

Proof. By Proposition 7 Part (ii), we have

$$C(G) \le C(G/U) + \sum_{k=0}^{\infty} P^*_{G,V}(k). \tag{3.3}$$

Thus, we just need to prove that $\sum_{k=0}^{\infty} P^*_{G,V}(k) \leq \alpha_U$. Therefore, we may assume that V is abelian. Writing bars to denote reduction modulo $R_G(V)$, note that if M is a maximal subgroup of G with $M \in \Omega_{G,V}$, then $R_G(V) \leq M$ and $\overline{M} \in \Omega_{\overline{G},V}$. Hence, $P^*_{G,V}(k) \leq P^*_{\overline{G},V}(k)$, so we may assume that $R_G(V) = 1$. Thus, $G \cong V^\delta \rtimes H$, where H acts faithfully and irreducibly on V, and diagonally on V^δ.

Suppose first that $|H| = 1$. Then $G = V^\delta \cong (C_r)^\delta$, for some prime r, and $P^*_{G,V}(k)$ is the probability that k randomly chosen elements of G fail to generate G. Hence, $\sum_{k=0}^{\infty} P^*_{G,V}(k)$ is the expected number of random elements to generate $(C_r)^\delta$, which is well known to be

$$\sum_{i=0}^{\delta-1} \frac{r^\delta}{r^\delta - r^i}.$$

See, for instance, [11, top of page 193].

So we may assume that $|H| > 1$. Let $F = \mathrm{End}_H V$, so that $|F| = q$, $\dim_F V = n$, and $|V| = q^n$. Fix elements x_1, x_2 ,..., x_k in G, and for $i \in \{1, \ldots, k\}$, let $x_i = w_i h_i$ with $w_i \in V^\delta$ and $h_i \in H$. For $t \in \{1, \ldots, \delta\}$ let

$$r_t = (\pi_t(w_1), \ldots, \pi_t(w_k)) \in V^k.$$

where π_t denotes projection onto the t-th direct factor of V^δ. Moreover let

$$W := \{(u_1, u_2, \ldots, u_k) : u_i \in [h_i, V] \text{ for } 1 \leq i \leq k\}, \text{ and}$$
$$D := \left\{(\zeta(h_1), \zeta(h_2), \ldots, \zeta(h_k)) \in V^k : \zeta \in \mathrm{Der}(H, V)\right\}.$$

By Proposition 5, $P^*_{G,V}(k)$ is at most the probability that $r_1, \ldots, r_\delta$ are linearly dependent modulo $W + D$. Also, for an f-tuple $J := (j_1, j_2, \ldots, j_f)$ of distinct elements j_i of $\{1, \ldots, k\}$, set

$$r_{t,J} := (\pi_t(w_{j_1}), \pi_t(w_{j_2}), \ldots, \pi_t(w_{j_f})) \in V^f$$

for $t \in \{1, \ldots, \delta\}$, and set

$$W_J := \left\{(u_{j_1}, u_{j_2}, \ldots, u_{j_f}) \in V^f : u_i \in [h_{j_i}, V] \text{ for } 1 \leq i \leq f\right\}, \text{ and}$$
$$D_J := \left\{(\zeta(h_{j_1}), \zeta(h_{j_2}), \ldots, \zeta(h_{j_f})) \in V^f : \zeta \in \mathrm{Der}(H, V)\right\}.$$

Notice that: $(*)$ If J is fixed and $r_1, \ldots, r_\delta$ are F-linearly dependent modulo $W + D$, then the vectors $r_{1,J}, \ldots, r_{\delta,J}$ of V^f are F-linearly dependent modulo $W_J + D_J$.

We will prove first that

$$\sum_{k=0}^{\infty} P^*_{G,V}(k) \le (\delta\cdot\theta + m + c_V)\frac{1}{p}, \tag{3.4}$$

where c_V is as in the statement of Theorem 2. To this end, let Δ_l be the subset of H^k consisting of the k-tuples $(h_1,\dots,h_k)$ with the property that $C_V(h_i)\neq 0$ for precisely l different choices of $i\in\{1,\dots,k\}$. If $(h_1,\dots,h_k)\in\Delta_l$, then, by [10, Lemma 7], $W+D$ is a subspace of $V^k\cong F^{nk}$ of codimension at least $l-m$: so the probability that $r_1,\dots,r_\delta$ are F-linearly dependent modulo $W+D$ is at most

$$\begin{aligned} p_l &= 1-\left(\frac{q^{nk}-q^{nk-l+m}}{q^{nk}}\right)\cdots\left(\frac{q^{nk}-q^{nk-l+m+\delta-1}}{q^{nk}}\right)\\ &= 1-\left(1-\frac{1}{q^{l-m}}\right)\cdots\left(1-\frac{q^{\delta-1}}{q^{l-m}}\right)\\ &\le \min\left\{1,\left(\frac{q^{\delta}-1}{q-1}\right)\frac{1}{q^{l-m}}\right\}\le \min\left\{1,1/q^{l-m-\delta\cdot\theta}\right\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \sum_{k=0}^{\infty} P^*_{G,V}(k) &\le \sum_{k=0}^{\infty}\sum_{l=0}^{k} P(B(k,p)=l)\min\left\{1,q^{\delta\cdot\theta+m-l}\right\}\\ &\le \sum_{k=0}^{\infty} P(B(k,p)<\delta\cdot\theta+m)+\sum_{k=0}^{\infty}\sum_{l=\delta\cdot\theta+m}^{k} P(B(k,p)=l)q^{\delta\cdot\theta+m-l}\\ &\le \sum_{k=0}^{\infty} P(B(k,p)<\delta\cdot\theta+m)+\sum_{l=0}^{\infty}q^{-l}\sum_{k=l+\delta\cdot\theta+m}^{\infty} P(B(k,p)=l+\delta\cdot\theta+m)\\ &\le \frac{\delta\cdot\theta+m+c_V}{p} \end{aligned}$$

where $c_V=\frac{q}{q-1}$. Note that the last step above follows from Proposition 6.

Thus, all that remains is to show that

$$\sum_{k=0}^{\infty} P^*_{G,V}(k) \le \left(\left\lceil\frac{\delta\cdot\theta}{n}\right\rceil+\frac{q^n}{q^n-1}\right)|H|. \tag{3.5}$$

For this, we define Ω_l to be the subset of H^k consisting of the k-tuples $(h_1,\dots,h_k)$ with the property that $h_i=1$ for precisely l different choices of $i\in\{1,\dots,k\}$. Suppose that $(h_1,\dots,h_k)\in\Omega_l$, and set $J:=(j_1,j_2,\dots,j_l)$, where $j_1<j_2<\dots<j_l$ and $\{j_1,j_2,\dots,j_l\}=\{i\mid 1\le i\le k, h_i=1\}$. Then, by $(*)$, the probability p'_l that r_1, r_2, …, r_δ are F-linearly dependent modulo $W+D$ is at

most the probability that the vectors $r_{1,J}$, $r_{2,J}$, $\ldots$, $r_{\delta,J} \in V^l$ are F-linearly dependent modulo $W_J + D_J$. But $W_J + D_J = 0$, by the definition of J. Thus we have

$$\begin{aligned}
p'_l &\leq 1 - \left(\frac{q^{nl}-1}{q^{nl}}\right)\cdots\left(\frac{q^{nl}-q^{nl-\delta-1}}{q^{nl}}\right) \\
&= 1 - \left(1-\frac{1}{q^{nl}}\right)\cdots\left(1-\frac{q^{\delta-1}}{q^{nl}}\right) \leq \min\left\{1, \left(\frac{q^{\delta}-1}{q-1}\right)\frac{1}{q^{nl}}\right\} \leq \min\left\{1, \frac{1}{q^{nl-\delta\cdot\theta}}\right\}.
\end{aligned}$$

Hence, if $\alpha := \lceil \frac{\delta\cdot\theta}{n} \rceil$, and $p' = 1/|H|$ is the probability that a randomly chosen element of H is the identity, then we have

$$\begin{aligned}
\sum_{k=0}^{\infty} P^*_{G,V}(k) &\leq \sum_{k=0}^{\infty} P(B(k,p') < \alpha) + \sum_{k=0}^{\infty}\sum_{l=\alpha}^{k} P(B(k,p') = l) q^{\delta\cdot\theta - nl} \\
&\leq \sum_{k=0}^{\infty} P(B(k,p') < \alpha) + \sum_{l=0}^{\infty} q^{-nl-n\alpha+\delta\cdot\theta} \sum_{k=l+\alpha}^{\infty} P(B(k,p') = l+\alpha) \\
&\leq \sum_{k=0}^{\infty} P(B(k,p') < \alpha) + \sum_{l=0}^{\infty} q^{-nl} \sum_{k=l+\alpha}^{\infty} P(B(k,p') = l+\alpha) \\
&\leq \frac{1}{p'}\left(\alpha + \frac{q^n}{q^n-1}\right)
\end{aligned}$$

Note that the last step above again follows from Proposition 6. Since $p' = 1/|H|$, (3.5) follows, whence the result. ∎

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Proposition 7 Part (i), we have

$$C(G) \leq \max_{V\in B} \delta_V + \sigma + \sum_{k=0}^{\infty}\sum_{V\in A} P^*_{G,V}(k)$$

Thus, it will suffice to prove that

$$\sum_{k=0}^{\infty} P^*_{G,V}(k) \leq \min\left\{(\delta_V + c_V)q_V^{n_V}, \left(\left\lceil\frac{\delta_V}{n_V}\right\rceil + \frac{q_V^{n_v}}{q_V^{n_V}-1}\right)|H_V|\right\} \tag{3.6}$$

for each non-central complemented chief factor V of G.

However, since $\mathrm{H}^1(H,V) = 0$ by [12, Lemma 1], and since $p_V \leq |H_v|/|H| \leq 1/|V|$ (for any non-zero vector $v \in V$), this follows immediately from the proof of Proposition 8. ∎

Corollary 9: *Let G be a finite soluble group, and let A and B be as in Theorem 2. Then*

$$C(G) \leq d(G) \sum_{V \in A} \left(1 + \frac{q_V^{n_V}|H_V|}{q_V^{n_V}-1}\right) + \sigma.$$

Proof. For $V \in A \cup B$, set $\gamma_V := \lceil \delta_V / n_V \rceil$, and $p'_V = 1/|H_V|$. Note also that $n_V = |H_V| = 1$ when $V \in B$. Arguing as in the last paragraph of the proof of Proposition 8, we have

$$\begin{aligned}
C(G) &\leq \sum_{V \in A} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \min\{q_V^{-n_V l + \delta_V}, 1\} P(B(k, p'_V) = l) + \max_{V \in B} \delta_V + \sigma \\
&\leq \sum_{V \in A} \sum_{k=0}^{\infty} P(B(k, p'_V) < \gamma_V) + \sum_{V \in A} \sum_{l=0}^{\infty} q_V^{-n_V l} \sum_{k=l+\gamma_V}^{\infty} P(B(k, p'_V) = l + \gamma_V) + \\
&\quad \max_{V \in B} \delta_V + \sigma \\
&\leq \sum_{V \in A} \gamma_V / p'_V + \sum_{V \in A} \frac{{q_V}^{n_V}}{p'_V({q_V}^{n_V} - 1)} + \max_{V \in B} \delta_V + \sigma \\
&\leq \left(\max_{V \in A \cup B} \gamma_V\right) \sum_{V \in A} \left(1 + \frac{{q_V}^{n_V}}{{q_V}^{n_V - 1}}\right) |H_V| + \sigma.
\end{aligned}$$

We remark that the third inequality above follows from Proposition 6. Finally, [3, Theorem 1.4 and paragraph after the proof of Theorem 2.7] imply that $d(G) = \max_{V \in A \cup B} \left\{1 + a_V + \left\lfloor \frac{\delta_V - 1}{n_V} \right\rfloor\right\}$, where $a_V = 0$ if $V \in B$, and $a_V = 1$ otherwise. In particular, $d(G) \geq \max_{V \in A \cup B} \gamma_V$, and the result follows. ∎

4. Proof of Theorem 1 Part (i)

Before proceeding to the proof of Part (i) of Theorem 1, we require the following result, which follows immediately from the arguments used in [10, Proof of Proposition 10].

Proposition 10: [10, Proof of Proposition 10] *Let H be a finite group acting faithfully and irreducibly on an elementary abelian group V, and denote by p the probability that a randomly chosen element h of H centralises a non zero vector of V. Also, write $m := \dim_{\mathrm{End}_H(V)} \mathrm{H}^1(H, V)$. Assume that $\mathrm{H}^1(H, V)$ is nontrivial and that $|H| \geq |V|$. Then there exists an absolute constant C such that $p|H| \geq 2(m+1)^2$ if $|H| \geq C$.*

Proof of Theorem 1 Part (i). Since $C(G) = C(G/\operatorname{Frat}(G))$, we may assume that $\operatorname{Frat}(G) = 1$. Thus, Proposition 8 applies: adopting the same notation as used therein, we have

$$C(G) \leq C(G/U) + \alpha_U. \tag{4.1}$$

Using (4.1), the proof of the theorem reduces to proving that

$$\alpha_U \leq (1 + \beta_U)\sqrt{|G|} \tag{4.2}$$

where $\beta_U \to 0$ as $|U| \to \infty$. Indeed, suppose that (4.2) holds, fix $\epsilon > 0$, and suppose that Theorem 1 holds for groups of order less than $|G|$. Then since $|U| > 1$, there exists a constant c_ϵ such that $C(G/U) \leq (1+\epsilon)\sqrt{|G/U|} + c_\epsilon$. Hence, by (4.1) and (4.2) we have $C(G) \leq (1 + \beta_U + \frac{1+\epsilon}{\sqrt{|U|}})\sqrt{|G|} + c_\epsilon$. It is now clear that by choosing $|U|$ to be large enough, we have $C(G) \leq (1+\epsilon)\sqrt{|G|} + c_\epsilon$, as needed.

Assume first that U is nonabelian. By [10, Proof of Lemma 13], there exist absolute constants c_1 and c_2 such that

$$P^*_{G,V}(k) \leq \min\left\{1, c_1\sqrt{|G|^3}(1 - c_2/\log|G|)^k\right\}.$$

Also, there exists a constant c_3 such that if $k \geq c_3(\log|G|)^2$, then $c_1\sqrt{|G|^3}(1 - c_2/\log|G|)^k$ tends to 0 as $|G|$ tends to ∞. It follows that

$$\begin{aligned}
\alpha_U &= \sum_{k=0}^{\infty} P^*_{G,V}(k) \\
&\leq \lceil c_3(\log|G|)^2\rceil + c_1\sqrt{|G|^3}(1 - c_2/\log|G|)^{\lceil c_3(\log|G|)^2\rceil}\sum_{k=0}^{\infty}(1 - c_2/\log|G|)^k \\
&= \lceil c_3(\log|G|)^2\rceil + \frac{c_1}{c_2}\sqrt{|G|^3}\log|G|(1 - c_2/\log|G|)^{\lceil c_3(\log|G|)^2\rceil}
\end{aligned}$$

and (4.2) holds.

So we may assume that U is abelian, and hence $|G| \geq |V|^\delta|H|$. The inequality (4.2) then follows easily from the definition of α_U, except when $\delta = 1$ and $|H| \geq |V|$. Indeed, if $|H| \leq |V|$ and $\delta = 1$, then $\frac{q^n}{q^n-1} \to 1$ as $|U| = q^n \to \infty$; if $|H| \leq |V|$ and $\delta > 1$, then

$$\left(\left\lceil\frac{\delta}{n}\right\rceil + \frac{q^n}{q^n - 1}\right)|H| \leq \frac{\left\lceil\frac{\delta}{n}\right\rceil + \frac{q^n}{q^n-1}}{|V|^{\frac{\delta-1}{2}}}\sqrt{|G|}$$

which clearly gives us what we need, since $|U| = |V|^\delta$ is tending to ∞. The other cases are similar.

So assume that $\delta = 1$ and $|H| \geq |V|$. We distinguish two cases:

(1) $m \neq 0$ and $|V| \leq |H| \leq (m+1)^2|V|$. Denote by p the probability that a randomly chosen element h of H centralizes a non-zero vector of V: By Proposition 10, there exists an absolute constant C such that $p|H| \geq 2(m+1)^2$ if $|H| \geq C$. Thus,

$$\alpha_U \leq \left(m + \frac{q}{q-1}\right)\frac{1}{p} \leq (m+2)\frac{|H|}{2(m+1)^2} \leq \frac{|H|}{m+1} \leq \sqrt{|H||V|}$$

if $|H| \geq C$, from which (4.2) follows.

(2) $|H| \geq |V|(m+1)^2$. We remark first that, for any fixed nonzero vector v in V, we have $p \geq \frac{|H_v|}{|H|}$, where H_v denotes the stabiliser of v in H. If H is not a transitive linear group, then there is an orbit Ω for the action of H on $V \setminus \{0\}$ with $|\Omega| \leq q^n/2$. Choose $v \in \Omega$: we have

$$\frac{1}{p} \leq \frac{|H|}{|H_v|} \leq \frac{q^n}{2},$$

hence

$$\alpha_U \leq \frac{m+2}{p} \leq (m+1)q^n \leq \sqrt{|H||V|}.$$

We remain with the case when H is a transitive linear group. There are four infinite families:

(a) $H \leq \Gamma L(1, q^n)$;
(b) $SL(a, r) \trianglelefteq H$, where $r^a = q^n$;
(c) $Sp(2a, r) \trianglelefteq H$, where $a \geq 2$ and $r^{2a} = q^n$;
(d) $G_2(r) \trianglelefteq H$, where q is even, and $q^n = r^6$.

Furthermore, H and m are exhibited in [2, Table 7.3]: in each case, we have $m \leq 1$. Furthermore, we have $|H| = (q^n - 1)\rho$, where ρ is the order of a point stabiliser. Hence, if $\rho \geq 9$, then

$$\alpha_U \leq \frac{m+2}{p} \leq \frac{3}{p} \leq 3|V| \leq \sqrt{|H||V|}.$$

So we may assume that $\rho \leq 8$. Suppose first that (a) holds. Then H is soluble, so $m = 0$. Also, $\rho = |H_v| \leq 8$ implies that $n \leq 8$. Hence, as $q^n \leq q^8$ approaches ∞, $\frac{q}{q-1}$ approaches 1, and (4.2) follows since

$$\alpha_U \leq \frac{q}{q-1}|V| \leq \frac{q}{q-1}\sqrt{|H||V|}.$$

So we may assume that (a) does not hold. In particular, if (b) or (c) holds then $a \geq 2$. It follows (in either of the cases (b), (c) or (d)) that if q^n is large enough, then $|H| \geq 9q^n$, and so

$$\alpha_U \leq \frac{(m+2)}{p} \leq 3q^n \leq \sqrt{|H||V|}.$$

This gives us what we need, and completes the proof. ∎

5. Proof of Theorem 1 Part (ii)

In this section, we prove Part (ii) of Theorem 1 in a number of steps. The first is as follows:

Lemma 11: *Let G be a finite soluble group with trivial Frattini subgroup, and let U and V be as in Lemma 3. Assume that V is abelian and non-central in G, and let $H = H_V$. Then*

$$\frac{\alpha_U}{|G|^{1/2}} < \frac{5}{3}\left(\frac{|U|^{1/2}-1}{|U|^{1/2}}\right)$$

except when $|H| < |V|$ and one of the following cases occur:

(1) $\delta = 2$, $q^n = 4$ and $|R_G(V)| = 1$.
(2) $\delta = 2$, $q^n = 3$ and $|R_G(V)| \leq 2$.
(3) $\delta = 1$, $4 \leq q^n \leq 7$ and $|R_G(V)| = 1$.
(4) $\delta = 1$, $q^n = 3$ and $|R_G(V)| \leq 3$.

Proof. Note that $m = 0$ since H is soluble. We distinguish the following cases: Case 1) $|H| < |V|$ and $\delta \neq 1$. Since, $|G| = \lambda|H||V|^\delta$ for some positive integer λ it suffices to prove

$$\frac{3\left(\delta + \frac{q^n}{q^n-1}\right)\left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1}\right)|H|}{5\lambda^{1/2}|H|^{1/2}q^{n\delta/2}} \leq \frac{3}{5\lambda^{1/2}}\left(\delta + \frac{q^n}{q^n-1}\right)\left(\frac{(q^n-1)^{1/2}}{q^{n\delta/2}-1}\right) < 1. \tag{5.1}$$

If $\delta \geq 3$ then

$$\frac{3}{5\lambda^{1/2}}\left(\delta + \frac{q^n}{q^n-1}\right)\left(\frac{(q^n-1)^{1/2}}{q^{n\delta/2}-1}\right) \leq \frac{3}{5\lambda^{1/2}}\left(3 + \frac{q^n}{q^n-1}\right)\left(\frac{(q^n-1)^{1/2}}{q^{3n/2}-1}\right) < 1.$$

Suppose $\delta = 2$. If $q^n \geq 5$, then

$$\frac{3}{5\lambda^{1/2}}\left(\delta + \frac{q^n}{q^n-1}\right)\left(\frac{(q^n-1)^{1/2}}{q^{n\delta/2}-1}\right) \leq \frac{3}{5\lambda^{1/2}}\left(2 + \frac{q^n}{q^n-1}\right)\left(\frac{(q^n-1)^{1/2}}{q^n-1}\right) < 1.$$

Suppose $\delta = 2$ and $q^n = 4$. We have $|H| = 3$ so if $\lambda \neq 1$, then

$$\frac{3\left(\delta+\frac{q^n}{q^n-1}\right)\left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1}\right)|H|}{5\lambda^{1/2}|H|^{1/2}q^{n\delta/2}} \leq \frac{2\cdot 3^{1/2}}{3\cdot\lambda^{1/2}} < 1$$

Suppose $\delta = 2$ and $q^n = 3$. We have $|H| = 2$ so if $\lambda > 2$, then

$$\frac{3\left(\delta+\frac{q^n}{q^n-1}\right)\left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1}\right)|H|}{5\lambda^{1/2}|H|^{1/2}q^{n\delta/2}} \leq \frac{21\cdot 2^{1/2}}{20\cdot\lambda^{1/2}} < 1.$$

Case 2) $|H| \geq |V|$ (and consequently $n \neq 1$) and $\delta \neq 1$. It suffices to prove that

$$\frac{3\left(\delta+\frac{q}{q-1}\right)\left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1}\right)q^n}{5|H|^{1/2}q^{n\delta/2}} < 1. \tag{5.2}$$

Suppose $q^n \neq 4$.

$$\frac{3\left(\delta+\frac{q}{q-1}\right)\left(\frac{q^{n\delta/2}}{q^{n\delta/2}-1}\right)q^n}{5|H|^{1/2}q^{n\delta/2}} \leq \frac{3\left(2+\frac{q}{q-1}\right)\left(\frac{q^n}{q^n-1}\right)}{5q^{n/2}} < 1.$$

Suppose $q^n = 4$. We have $H = \mathrm{GL}(2,2) \cong \mathrm{Sym}(3)$, and consequently $|H| = 6$ and $p = 2/3$. Hence

$$\frac{\alpha_U}{|G|^{1/2}}\frac{5}{3}\left(\frac{|U|^{1/2}}{|U|^{1/2}-1}\right) \leq \frac{(\delta+2)\cdot\frac{1}{p}\cdot\frac{3}{5}\cdot\frac{4}{3}}{|H|^{1/2}\cdot 2^{\delta}} \leq \frac{6}{5\sqrt{6}} < 1.$$

Case 3) $|H| < |V|$ and $\delta = 1$. Since, $|G| = \lambda|H||V|^{\delta}$ for some positive integer λ it suffices to prove

$$\frac{3\left(\frac{q^n}{q^n-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right)|H|^{1/2}}{5\lambda^{1/2}q^{n/2}} < 1. \tag{5.3}$$

If $q^n \geq 8$, or $7 \geq q^n \geq 4$ and $\lambda \neq 1$, or $q^n = 3$ and $\lambda > 3$, then

$$\frac{3\left(\frac{q^n}{q^n-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right)|H|^{1/2}}{5\cdot\lambda^{1/2}\cdot q^{n/2}} \leq \frac{3\left(\frac{q^n}{q^n-1}\right)\left(\frac{(q^n-1)^{1/2}}{q^{n/2}-1}\right)}{5\cdot\lambda^{1/2}} = \frac{3\left(\frac{q^n}{(q^n-1)^{1/2}(q^{n/2}-1)}\right)}{5\cdot\lambda^{1/2}} < 1.$$

Case 4) $|H| \geq |V|$ (and consequently $n \neq 1$) and $\delta = 1$. It suffices to prove that

$$\frac{3\left(\frac{q}{q-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5|H|^{1/2}q^{n/2}p} < 1. \tag{5.4}$$

If H is not a transitive linear group, then $|H|^{1/2}q^{n/2}p \geq 2$, so it suffices to have

$$\left(\frac{q}{q-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right) \leq \frac{10}{3},$$

which is true if $(q,n) \neq (2,2)$. On the other hand, we may exclude the case $(q,n) = (2,2)$: indeed the only soluble irreducible subgroup of $\mathrm{GL}(2,2)$ with order ≥ 4 is $\mathrm{GL}(2,2)$, which is transitive on the nonzero vectors.

If H is a transitive linear group, then $|H| = (q^n-1)\rho$, with ρ the order of the stabilizer in H of a nonzero vector and

$$\frac{3\left(\frac{q}{q-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5|H|^{1/2}q^{n/2}p} \leq \frac{3\left(\frac{q}{q-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5\sqrt{\rho}},$$

so it suffices to have

$$\left(\frac{q}{q-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right) \leq \frac{5\sqrt{\rho}}{3},$$

which is true if $q \geq 3$ and if $(q,n,\rho) \notin \{(2,4,2), (2,3,2), (2,3,3), (2,2,2))\}$. We may exclude the case $(q,n,\rho) = (2,3,2)$ (there is no transitive linear subgroup of $\mathrm{GL}(3,2)$ of order 14). If $(q,n,\rho) = (2,4,2)$, then $H = \mathrm{GL}(1,16) \rtimes C_2$, hence $p = 6/30$ so $|H|^{1/2}q^{n/2}p \geq 2$ and (5.4) is true. If $(q,n,\rho) = (2,3,3)$ then $H = \Gamma\mathrm{L}(1,8)$ and consequently $p = 15/21$ and

$$\frac{3\left(\frac{q}{q-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5|H|^{1/2}q^{n/2}p} = \frac{3\cdot 2\cdot\sqrt{8}\cdot 21}{5\cdot(\sqrt{8}-1)\cdot 15\cdot\sqrt{21}\sqrt{8}} < 1.$$

If $(q,n,\rho) = (2,2,2)$ then $H = \mathrm{GL}(2,2)$ and consequently $p = 2/3$ and

$$\frac{3\left(\frac{q}{q-1}\right)\left(\frac{q^{n/2}}{q^{n/2}-1}\right)}{5|H|^{1/2}q^{n/2}p} = \frac{3\cdot 2\cdot 2\cdot 3}{5\cdot 2\cdot 2\cdot\sqrt{6}} < 1. \qquad \blacksquare$$

Lemma 12: *If G is one of the exceptional cases in the statement of Lemma 11, then $C(G) < \frac{5}{3}\sqrt{|G|}$.*

Proof. This follows easily by direct computation. We use MAGMA, and the code from [9, Appendix, page 36] to compute $C(G)$ explicitly whenever G is a group satisfying the conditions of one of the exceptional cases of Lemma 11. $\blacksquare$

The next step is to deal with the case of a central chief factor.

Lemma 13: *If $G \cong C_p^\delta$, then $C(G) \leq \frac{5}{3}\sqrt{|G|}$, with equality if and only if $G = C_2 \times C_2$.*

Proof. If $p \neq 3$ or $p = 2$ and $\delta > 3$, then

$$C(G) = \sum_{0 \leq i \leq \delta - 1} \frac{p^\delta}{p^\delta - p^i} \leq \delta + \frac{p}{(p-1)^2} < \frac{5 \cdot p^{\delta/2}}{3} = \frac{5 \cdot \sqrt{|G|}}{3}.$$

If $(p, \delta) = (2, 1)$ then

$$\frac{C(G)}{\sqrt{G}} = \frac{2}{\sqrt{2}} = \sqrt{2};$$

if $(p, \delta) = (2, 2)$ then

$$\frac{C(G)}{\sqrt{G}} = \frac{\frac{4}{2} + \frac{4}{3}}{2} = \frac{5}{3};$$

if $(p, \delta) = (2, 3)$ then

$$\frac{C(G)}{\sqrt{G}} = \frac{\frac{8}{4} + \frac{8}{6} + \frac{8}{7}}{\sqrt{8}} \sim 1.5826. \qquad \blacksquare$$

Proof of Part (ii) of Theorem 1. We prove the claim by induction on the order of $|G|$. If $\operatorname{Frat}(G) \neq 1$, then the conclusion follows immediately since $C(G) = C(G/\operatorname{Frat}(G))$. Otherwise G contains a normal subgroup U as in Lemma 4. If $G = U \cong C_p^\delta$, then the conclusion follows from Lemma 13. Otherwise, Lemma 11, together with the inductive hypothesis gives

$$C(G) \leq C(G/U) + \alpha_U < \frac{5\sqrt{|G|}}{3\sqrt{|U|}} + \frac{5(\sqrt{|U|} - 1)\sqrt{|G|}}{3\sqrt{|U|}} = \frac{5}{3}\sqrt{|G|}$$

as claimed. $\blacksquare$

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