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# SINGULARITY OF RANDOM SYMMETRIC MATRICES REVISITED 

MARCELO CAMPOS, MATTHEW JENSSEN, MARCUS MICHELEN, AND JULIAN SAHASRABUDHE


#### Abstract

Let $M_{n}$ be drawn uniformly from all $\pm 1$ symmetric $n \times n$ matrices. We show that the probability that $M_{n}$ is singular is at $\operatorname{most} \exp \left(-c(n \log n)^{1 / 2}\right)$, which represents a natural barrier in recent approaches to this problem. In addition to improving on the best-known previous bound of Campos, Mattos, Morris and Morrison of $\exp \left(-c n^{1 / 2}\right)$ on the singularity probability, our method is different and considerably simpler: we prove a "rough" inverse Littlewood-Offord theorem by a simple combinatorial iteration.


## 1. Introduction

Let $A_{n}$ denote a random $n \times n$ matrix drawn uniformly from all matrices with $\{-1,1\}$ coefficients. It is an old problem, of uncertain origin ${ }^{1}$, to determine the probability that $A_{n}$ is singular. While a few moments of consideration reveals a natural lower bound of $(1+o(1)) n^{2} 2^{-n+1}$, which comes from the probability that two rows or columns are equal up to sign, it is widely believed that in fact

$$
\begin{equation*}
\mathbb{P}\left(\operatorname{det} A_{n}=0\right)=(1+o(1)) n^{2} 2^{-n+1} \tag{1}
\end{equation*}
$$

This singularity probability was first shown to tend to zero in 1967 by Komlós [10], who obtained the bound $\mathbb{P}\left(\operatorname{det}\left(A_{n}\right)=0\right)=O\left(n^{-1 / 2}\right)$. The first exponential upper bound was established by Kahn, Komlós, and Szemerédi [9] in 1995 with subsequent improvements on the exponent by Tao and $\mathrm{Vu}[17,18]$ and Bourgain, Vu and Wood [1]. In 2018, Tikhomirov [20] settled this conjecture up to lower order terms by showing $\mathbb{P}\left(\operatorname{det}\left(A_{n}\right)=0\right)=(1 / 2+o(1))^{n}$. Very recently, a closely related problem was resolved by Jain, Sah and Sawhney [8], who showed that the analogue of (1) holds when the entries of $A_{n}$ are i.i.d. discrete variables of finite support that are not uniform on their support. The conjecture (1) remains open for matrices with mean-zero $\{-1,1\}$ entries.

The focus of this paper is on the analogous question for symmetric random matrices. In particular, let $M_{n}$ denote a uniformly drawn matrix among all $n \times n$ symmetric matrices with entries in $\{-1,1\}$. In this setting it is also widely believed that $\mathbb{P}\left(\operatorname{det} M_{n}=0\right)=\Theta\left(n^{2} 2^{-n}\right)$ as in the asymmetric case $[2,3,22]$ although here much less is known. For instance, the fact that

[^0]$\mathbb{P}\left(\operatorname{det} M_{n}=0\right)=o(1)$, was only resolved in 2005 by Costello, Tao and Vu [3]. Subsequent superpolynomial upper bounds of the form $n^{-C}$ for all $C$ and $\exp \left(-n^{c}\right)$ were proven respectively by Nguyen [12] and Vershynin [21] by different techniques: Nguyen used an inverse Littlewood-Offord theorem for quadratic forms based on previous work by Nguyen and Vu [11, 13], while Vershynin used a more geometric approach pioneered by Rudelson and Vershynin [14, 15, 16].

A combinatorial approach developed by Ferber, Jain, Luh and Samotij [5] was applied by Ferber and Jain [4] in 2018 to prove that $\mathbb{P}\left(\operatorname{det} M_{n}=0\right) \leq \exp \left(-c n^{1 / 4}(\log n)^{1 / 2}\right)$. Another combinatorial approach was taken by Campos, Mattos, Morris and Morrison [2] who achieved the bound $\mathbb{P}\left(\operatorname{det} M_{n}=0\right) \leq \exp \left(-c n^{1 / 2}\right)$. Their argument centers around an inverse Littlewood-Offord theorem inspired by the method of hypergraph containers.

The proofs of $[2,4,21]$ all follow the same general shape: divide all potential vectors $v$ for which we could have $M_{n} v=0$ into "structured" and "unstructured" vectors, show that the unstructured vectors do not contribute, and union bound over the structured vectors. The main difficulty (and novelty) in these proofs arises in a careful understanding of the contribution of the structured vectors.

While we have this method to thank for the recent successes on this problem, an important limitation was pointed out in [2, Section 2.2] who argued that this method could not provide any improvement to the singularity probability beyond $\exp (-c \sqrt{n \log n})$, provided the randomness in the matrix is not "reused". Here we show that this natural barrier is attainable.

Theorem 1. Let $M_{n}$ be drawn uniformly from all $n \times n$ symmetric matrices with entries in $\{-1,1\}$. Then for $c=2^{-13}$ and $n$ sufficiently large

$$
\mathbb{P}\left(\operatorname{det}\left(M_{n}\right)=0\right) \leq \exp (-c \sqrt{n \log n})
$$

Indeed, our proof of Theorem 1 follows the shape of [2, 4, 21] and improves upon these results primarily by proving an improved and considerably simpler "rough" inverse Littlewood-Offord theorem. This theorem parallels Theorem 2.1 in [2] and improves upon it by replacing the use of Fourier analysis in [2] with a simple combinatorial algorithm. This proof additionally gives us more information in our inverse theorem, which allows for a simplified application to the proof of Theorem 1.

To state our rough inverse theorem, we need a few notions. For a vector $v \in \mathbb{Z}_{p}^{n}$ and $\mu \in[0,1]$, we define the random variable $X_{\mu}(v):=\varepsilon_{1} v_{1}+\cdots+\varepsilon_{n} v_{n}$, where $\varepsilon_{i} \in\{-1,0,1\}$ are i.i.d. and $\mathbb{P}\left(\varepsilon_{i}=1\right)=\mathbb{P}\left(\varepsilon_{i}=-1\right)=\mu / 2$. Also define $\rho_{\mu}(v)=\max _{x} \mathbb{P}\left(X_{\mu}(v)=x\right)$ and ${ }^{2}$ let $|v|$ denote the number of non-zero entries of $v$. Finally for $T \subseteq[n]$, let $v_{T}:=\left(v_{i}\right)_{i \in T}$.

[^1]We now introduce a simple concept that is key to our rough inverse Littlewood-Offord theorem. For a vector $w=\left(w_{1}, \ldots, w_{d}\right)$ we define the neighbourhood of $w$ (relative to $\mu$ ) as

$$
\begin{equation*}
N_{\mu}(w):=\left\{x \in \mathbb{Z}_{p}: \mathbb{P}\left(X_{\mu}(w)=x\right)>2^{-1} \mathbb{P}\left(X_{\mu}(w)=0\right)\right\} \tag{2}
\end{equation*}
$$

which is the set of places where our random walk is "likely" to terminate, relative to 0 . The following result, which is our "rough" Littlewood-Offord theorem, says that if $v \in \mathbb{Z}_{p}^{n}$ and $\rho_{\mu}(v)$ is large then there is a small subvector $x$ of $v$ so that $v_{i} \in N_{\mu}(x)$ for many $i \in[n]$.

Theorem 2. Let $\mu \in(0,1 / 4], k, n \in \mathbb{N}$, $p$ be prime and $v \in \mathbb{Z}_{p}^{n}$. Set $d=\frac{2}{\mu} \log \rho_{\mu}(v)^{-1}$, suppose that $|v| \geq k d$ and $\rho_{\mu}(v) \geq \frac{2}{p}$. Then there exists $T \subseteq[n]$ with $|T| \leq d$ so that if we set $w=v_{T}$ then $v_{i} \in N_{\mu}(w)$ for all but at most $k d$ values of $i \in[n]$ and

$$
\left|N_{\mu}(w)\right| \leq \frac{256}{(\mu k)^{1 / 5}} \cdot \frac{1}{\rho_{\mu}(v)} .
$$

In practice we will not apply Theorem 2 directly, but rather in two parts (Lemmas 7 and 8). The first can be found in Section 6 and uses Fourier analysis in the style of Halász [6], whose influential techniques pervade the literature. The second is a novel (and simple) iterative application of a greedy algorithm. This can be found in Section 2 along with the proof of Theorem 2.

In what follows we discuss the proof of Theorem 1. In addition to illustrating the method of [4, 2] in a little more detail, we hope the reader will get some feeling for why Theorem 2 is so integral to the problem.
1.1. Discussion of proof. The event ' $M_{n}$ is singular' can, somewhat daftly, be expressed as $\bigcup_{v \in \mathbb{R}^{n} \backslash\{0\}}\{M v=0\}$. To reduce the size of this unwieldy union, we notice that it is sufficient to consider all non-zero $v \in \mathbb{Z}^{n}$ and then reduce modulo $p$, for a prime $p \approx \exp \left(c(n \log n)^{1 / 2}\right)$. Since the probability that $M v$ is zero is certainly bounded by the probability $M v$ is zero modulo $p$, it is enough for us to upper bound the probability of the event $\bigcup_{v \in \mathbb{Z}_{p}^{n} \backslash\{0\}}\{M v=0\}$, where all operations are taken over the field of $p$ elements.

Having reduced our event to a union of a finite number of sets, it is temping to greedily apply the union bound to the events $\{M v=0\}$, for non-zero $v \in \mathbb{Z}_{p}^{n}$. Unfortunately in our case, a small wrinkle arises with vectors for which $\rho(v) \approx 1 / p$; that is, very close to the "mixing" threshold. To get around this, we again follow [4, 2] and use a lemma that allows us to safely exclude all $v$ with $\rho(v)<c n / p$ from our union bound, at the cost of working with a slightly different event which, in practice, adds little difficulty to our task. In particular, to prove Theorem 1, it will be enough to establish the following.

Theorem 3. Let $c=1 / 800, n \in \mathbb{N}$ sufficiently large and $p \leq \exp (c \sqrt{n \log n})$ prime. Then for $\beta=\Theta(n / p)$ we have

$$
\begin{equation*}
\sum_{v: \rho(v) \geq \beta} \max _{w \in \mathbb{Z}_{p}^{n}} \mathbb{P}(M v=w) \leq e^{-c n} \tag{3}
\end{equation*}
$$

To bound the sum on the left hand side of (3), we invoke our inverse Littlewood-Offord result (Theorem 2, in the form of Lemmas 4 and 7).

We will in fact first sketch a proof of Theorem 3 under the stricter assumption that $p \leq \exp (c \sqrt{n})$ and then show how to recover the missing $\sqrt{\log n}$ factor. We do this for two reasons. Firstly, the proof under the stricter assumption on $p$ already contains the key ideas, and so we feel this is the clearest way to present the argument. Secondly, the reader can extract a very short proof of the bound $\mathbb{P}\left(\operatorname{det}\left(M_{n}\right)=0\right) \leq \exp (-c \sqrt{n})$ if they so desire.

In Section 5, we provide the short derivation of Theorem 1 from Theorem 3 and [2, Lemma 2.1]. Remark: Simultaneously to our work, Jain, Sah and Sawhney [7] obtained an upper bound on the singularity probability of the form $\exp \left(-c n^{1 / 2}(\log n)^{1 / 4}\right)$ and a bound on the lower tail of the least singular value for symmetric random matrices with subgaussian entries.

## 2. An inverse Littlewood-Offord lemma

In this section we present one of the key ideas of this paper, namely a greedy algorithm which furnishes us with a simple yet powerful inverse Littlewood-Offord result.

To go further, we introduce a little notation. Let $\mathbb{Z}_{p}^{*}$ denote the set of all vectors of finite dimension with entries in $\mathbb{Z}_{p}$. For $v=\left(v_{1}, \ldots, v_{k}\right), w=\left(w_{1}, \ldots, w_{l}\right) \in \mathbb{Z}_{p}^{*}$, let $v w:=\left(v_{1}, \ldots, v_{k}, w_{1}, \ldots, w_{l}\right)$ denote the concatenation of $v$ and $w$ and let $v^{k}$ denote the concatenation of $k$ copies of $v$. For $v \in \mathbb{Z}_{p}^{n}$ and $T \subseteq[n]$, let $v_{T}:=\left(v_{i}\right)_{i \in T}$ and say that $w$ is a subvector of $v$ if $w=v_{T}$ for some $T \subseteq[n]$. We also define $|v|$ to the be size of the support of $v$, the number of non-zero coordinates.

Unless specified otherwise, take $\mu=1 / 4$ for definiteness. We recall the key definition introduced in (2). For $w \in \mathbb{Z}_{p}^{*}$, we define the neighbourhood of $w$ as

$$
N(w):=\left\{x \in \mathbb{Z}_{p}: \mathbb{P}\left(X_{\mu}(w)=x\right)>2^{-1} \mathbb{P}\left(X_{\mu}(w)=0\right)\right\}
$$

This is motivated by the fact that for $\mu \in[0,1 / 2]$, the walk $X_{\mu}$ is most likely to be found at 0 (see e.g. [19, Corollary 7.12]), i.e.

$$
\begin{equation*}
\rho_{\mu}(w)=\mathbb{P}\left(X_{\mu}(w)=0\right) . \tag{4}
\end{equation*}
$$

Hence, we may think of $N(w)$ as the set of all values of the random walk $X_{\mu}(w)$, which are at least half as likely as the most likely value. We can also easily control the size of $N(w)$ in terms of $\rho_{\mu}(w)$.

Indeed,

$$
1 \geq \sum_{x \in N(w)} \mathbb{P}\left(X_{\mu}(w)=x\right)>\frac{1}{2}|N(w)| \cdot \mathbb{P}\left(X_{\mu}(w)=0\right)=\frac{1}{2}|N(w)| \rho_{\mu}(w)
$$

and so

$$
\begin{equation*}
|N(w)| \leq \frac{2}{\rho_{\mu}(w)} \tag{5}
\end{equation*}
$$

We now turn to our greedy algorithm which, given a vector $v \in \mathbb{Z}_{p}^{*}$, returns a short subvector $w$ of $v$ such that each coordinate of $v$ is contained in $N(w)$. The following simple lemma can be interpreted as an inverse Littlewood-Offord result in its own right, and is almost as good as Theorem 2, however it only gives a bound of $|N(w)| \leq 1 / \rho_{\mu}(w) \leq 1 / \rho_{\mu}(v)$, which is lacking the crucial factor of $k^{-1 / 5}$. For this lemma we use the monotonicity of $\rho_{\mu}$ [19, Corollary 7.12]: if $w, v \in \mathbb{Z}_{p}^{*}$ where $w$ is a subvector of $v$, then

$$
\begin{equation*}
\rho_{\mu}(v) \leq \rho_{\mu}(w) \tag{6}
\end{equation*}
$$

Lemma 4. For $\mu \in(0,1 / 4]$ and $n \in \mathbb{N}$, let $v \in \mathbb{Z}_{p}^{n}$. Then there exists $T \subseteq[n]$, such that $v_{i} \in N\left(v_{T}\right)$ for all $i \notin T, \rho_{\mu}\left(v_{T}\right) \leq(1-\mu / 2)^{|T|}$ and so

$$
|T| \leq \frac{2}{\mu} \log \frac{1}{\rho_{\mu}(v)}
$$

Proof. We build a sequence of sets $T_{1}, \ldots, T_{d} \subseteq[n]$ with $\left|T_{i}\right|=i$ via the following greedy process. Let $T_{1}=\{1\}$. Given $T_{t} \subseteq[n]$ with $\left|T_{t}\right|=t$ for $t \geq 1$, let $v_{T_{t}}=\left(x_{1}, \ldots, x_{t}\right)$. Pick $i \in[n] \backslash T_{t}$ such that

$$
\begin{equation*}
\rho_{\mu}\left(x_{1} \ldots x_{t} v_{i}\right) \leq(1-\mu / 2) \rho_{\mu}\left(x_{1} \ldots x_{t}\right) \tag{7}
\end{equation*}
$$

If no such $i$ exists we terminate the process and set $T=T_{t}$. Suppose this process runs for $d$ steps producing $T \subseteq[n]$ such that $v_{T}=\left(x_{1}, \ldots, x_{d}\right)$. By the termination condition, we have that for $i \in[n] \backslash T$

$$
\rho_{\mu}\left(x_{1} \ldots x_{d} v_{i}\right)>(1-\mu / 2) \rho_{\mu}\left(x_{1} \ldots x_{d}\right)
$$

Conditioning on the coefficient of $v_{i}$ and using that $\mathbb{P}\left(X_{\mu}\left(x_{1} \ldots x_{d}\right)=v_{i}\right)=\mathbb{P}\left(X_{\mu}\left(x_{1} \ldots x_{d}\right)=-v_{i}\right)$ by symmetry, we can rewrite the left hand side to obtain

$$
\mu \mathbb{P}\left(X_{\mu}\left(x_{1} \ldots x_{d}\right)=v_{i}\right)+(1-\mu) \rho_{\mu}\left(x_{1} \ldots x_{d}\right)>(1-\mu / 2) \rho_{\mu}\left(x_{1} \ldots x_{d}\right)
$$

Rearranging shows $v_{i} \in N\left(v_{T}\right)$. For the bound on $d=|T|$, observe that by (6), inequality (7) and the fact that $\rho_{\mu}\left(x_{1}\right)=(1-\mu)$ we have

$$
\rho_{\mu}(v) \leq \rho_{\mu}\left(x_{1} \ldots x_{d}\right) \leq(1-\mu / 2)^{d} \leq e^{-\mu d / 2}
$$

## 3. A weak version of Theorem 3

In this section we sketch how a weak version of Theorem 3 follows from Lemma 4. This section is not needed for the proof of Theorem 1, but is included to illustrate some of the key ideas. Moreover, the bound $\mathbb{P}\left(\operatorname{det}\left(M_{n}\right)=0\right) \leq \exp (-c \sqrt{n})$ follows easily from Theorem 6 and Lemma 9 below and so we obtain a particularly short proof of this result.

First we need the following strengthening of the monotonicity property (6). This type of lemma abounds in the literature, first appearing in [9]. Since the proof is a simplification of the Fourier arguments in the proof Lemma 12 below, we omit the details.

Lemma 5. For $\alpha>0$ there is a $\nu, K>0$ so that for all $v$ with $\rho_{\mu}(v)=\Omega(n / p)$ and $|v| \geq K$ we have

$$
\rho_{\mu}(v) \leq \alpha \rho_{\nu}(v) .
$$

Theorem 6. There exists $c>0$ such that for $n \in \mathbb{N}$ sufficiently large, $p \leq \exp (c \sqrt{n})$ prime and $\beta=\Theta(n / p)$ we have

$$
\begin{equation*}
\sum_{v: \rho(v) \geq \beta} \max _{w \in \mathbb{Z}_{p}^{n}} \mathbb{P}(M v=w)=e^{-\Omega(n)} \tag{8}
\end{equation*}
$$

Proof. Let $\alpha=2^{-16}$, let $\nu \in(0,1 / 4]$ and $K>0$ be given by Lemma 5 and set $d=\frac{2}{\nu} \log p$. Let $c>0$ be a constant taken sufficiently small so that the bounds in the following proof hold. We will use Lemma 4 to count the number of possible $v$ with $\rho(v) \geq \beta$ : for each such $v$, there must be a subset $T \subset[n]$ so that $v_{i} \in N_{\nu}\left(v_{T}\right)$ for all $i \notin T$. More formally, let $\mathcal{V}=\left\{v \in \mathbb{Z}_{p}^{n} \backslash\{0\}: \rho(v) \geq \beta\right\}$ and for $v \in \mathcal{V}$, let $f(v):=\left(T, v_{T}\right)$ where $T \subset[n]$ is the set obtained by applying Lemma 4 to $v$ (with $\mu=\nu$ ). Let $\mathcal{S}:=f(\mathcal{V})$.

For a given $s=(T, u) \in \mathcal{S}$ we have $|T| \leq d$ and $u \in \mathbb{Z}_{p}^{|T|}$. We may therefore bound

$$
\begin{equation*}
|\mathcal{S}| \leq 2^{n} \cdot p^{d} \leq 2^{\left(1+2 c^{2} / \nu\right) n} \tag{9}
\end{equation*}
$$

Further, for a given $s=(T, u) \in \mathcal{S}$, we note that for every $v \in f^{-1}(s)$ we must have $v_{i} \in N_{\nu}(u)$ for every $i \notin T$ and so

$$
\begin{equation*}
\left|f^{-1}(s)\right| \leq\left|N_{\nu}(u)\right|^{n-|T|} \leq\left(\frac{2}{\rho_{\nu}(u)}\right)^{n-|T|} \tag{10}
\end{equation*}
$$

For each $v \in f^{-1}(s)$, let $T \subset S$ where $|S|=\min \{|T|+K,|v|\}$ (and $v_{i} \neq 0$ for all $i \in S$ ). We may then bound

$$
\begin{equation*}
\mathbb{P}(M v=w) \leq \max _{w^{\prime}} \mathbb{P}\left(M_{S^{c} \times S}\left(v_{S}\right)=w^{\prime}\right) \leq \rho\left(v_{S}\right)^{n-|S|} \tag{11}
\end{equation*}
$$

where for the second inequality we used that the entries of $M_{S^{c} \times S}$ are i.i.d. (see Figure 1). Now if $|T|+K \leq|v|$ we apply Lemma 5 to get $\rho\left(v_{S}\right) \leq c \rho_{\nu}\left(v_{S}\right) \leq c \rho_{\nu}(u)$. Then applying (10) with (11)


Figure 1
for each $s=(T, u)$ yields

$$
\sum_{\substack{v \in f^{-1}(s) \\|v| \geq|T|+K}} \max _{w \in \mathbb{Z}_{p}^{n}} \mathbb{P}(M v=w) \leq\left|f^{-1}(s)\right| \rho(u)^{n-|S|} \leq\left(\frac{2}{\rho_{\nu}(u)}\right)^{n-|T|}\left(c \rho_{\nu}(u)\right)^{n-|T|-K} \leq 2^{-4 n},
$$

where for the final inequality we used that $\rho_{\nu}(u) \geq 1 / p$, and so $\rho_{\nu}(u)^{K} \geq 2^{-n}$.
Combining with (9) shows that

$$
\sum_{s \in \mathcal{S}} \sum_{\substack{v \in f^{-1}(s) \\|v| \geq|T|+K}} \max _{w \in \mathbb{Z}_{p}^{n}} \mathbb{P}(M v=w) \leq 2^{-n} .
$$

On the other hand if $|T|=t$ and $|v| \leq t+K$ we use that $\rho\left(v_{S}\right) \leq \rho_{\nu}\left(v_{S}\right) \leq \rho_{\nu}(u) \leq(1-\nu / 2)^{t}$ (where the final inequality follows from Lemma 4). In this case there are $p^{t+K}\binom{n}{\leq t+K}$ choices for $v$. Combining this with (11) we have

$$
\sum_{s \in \mathcal{S}} \sum_{\substack{v \in f^{-1}(s) \\|v| \leq|T|+K}} \max _{w \in \mathbb{Z}_{p}^{n}} \mathbb{P}(M v=w) \leq \sum_{t=1}^{d} p^{t}\binom{n}{\leq t+K}(1-\nu / 2)^{t(n-d-K)} \leq(1-\nu)^{n / 8} .
$$

## 4. The greedy algorithm iterated

In this section we show that we can strengthen Lemma 4 by applying it iteratively. This will be key to regaining this crucial $k^{-1 / 5}$ in Theorem 2, and will ultimately give our $\sqrt{\log n}$ gain in the exponent of the singularity probability.

For this lemma we need the following property of $\rho_{\mu}$, which can be found in [19, Corollary 7.12].
Let $w_{1}, \ldots, w_{k} \in \mathbb{Z}_{p}^{*}$ and $\mu \in(0,1 / 2)$ then

$$
\begin{equation*}
\rho_{\mu}\left(w_{1} \cdots w_{k}\right) \leq \max _{j \in[k]} \rho_{\mu}\left(w_{j}^{k}\right) . \tag{12}
\end{equation*}
$$

Lemma 7. Let $\mu \in(0,1 / 4], n \in \mathbb{N}$ and $v \in \mathbb{Z}_{p}^{n}$. Set $d=\frac{2}{\mu} \log \rho_{\mu}(v)^{-1}$ and let $k \in \mathbb{N}$ be such that $k d \leq n$. Then there exists $T \subseteq S \subseteq[n]$ with $|T| \leq d,|S| \leq k d$ such that $v_{i} \in N\left(v_{T}\right)$ for all $i \notin S$ and $\rho_{\mu}\left(v_{S}\right) \leq \rho_{\mu}\left(v_{T}^{k}\right)$.

Proof. We will define a sequence of sets $[n]=A_{1} \supseteq \cdots \supseteq A_{k} \supseteq A_{k+1}$. Given $v_{A_{j}}$, we choose $T_{j} \subseteq[n]$ with $v_{T_{j}}=\left(x_{1}, \ldots, x_{d(j)}\right)$ given by Lemma 4 applied to $v_{A_{j}}$ and let

$$
A_{j+1}=A_{j} \backslash T_{j} \quad \text { and } \quad S=\bigcup_{j=1}^{k} T_{j} .
$$

By Lemma 4, we have that $v_{i} \in N\left(v_{T_{j}}\right)$ for all $i \in A_{j+1}$. In particular, since $S^{c} \subseteq A_{j}$ for all $1 \leq j \leq k+1, v_{i} \in N\left(v_{T_{j}}\right)$ for all $i \notin S$ and $1 \leq j \leq k$. Note also that $\left|T_{j}\right| \leq d$ for all $1 \leq j \leq k$.

Let $T$ be the $T_{j}$ for which $\rho_{\mu}\left(v_{T_{j}}^{k}\right)$ is maximized. The first claim of the lemma follows from the above. For the second claim note that, by (12) we have

$$
\rho_{\mu}\left(v_{S}\right) \leq \max _{1 \leq j \leq k} \rho_{\mu}\left(v_{T_{j}}^{k}\right)=\rho_{\mu}\left(v_{T}^{k}\right) .
$$

To conclude the proof of our Theorem 2-and to understand the strength of Lemma 7-we introduce our main Fourier ingredient, the proof of which is found in Section 6.

Lemma 8. Let $\mu \in(0,1 / 4], k \in \mathbb{N}$ and $v \in \mathbb{Z}_{p}^{*}$ such that $|v| \neq 0$. Then

$$
\rho_{\mu}\left(v^{k}\right) \leq 64(\mu k)^{-1 / 5} \rho_{\mu}(v)+p^{-1} .
$$

Proof of Theorem 2. Let $k, n \in \mathbb{N}$ and $v \in \mathbb{Z}_{p}^{n}$ be as in the theorem statement. By Lemma 7, there exists $T \subseteq S \subseteq[n]$ with $|T| \leq d,|S| \leq k d$ such that $v_{i} \in N\left(v_{T}\right)$ for all $i \notin S$ and $\rho_{\mu}\left(v_{S}\right) \leq \rho_{\mu}\left(v_{T}^{k}\right)$. Moreover, since $|v| \geq k d$, the support of $v_{T}$ is non-zero. Applying Lemma 8 we conclude that

$$
\rho_{\mu}\left(v_{S}\right) \leq \rho_{\mu}\left(v_{T}^{k}\right) \leq 64(\mu k)^{-1 / 5} \rho_{\mu}\left(v_{T}\right)+p^{-1} .
$$

By (5) and (6) we then have

$$
\left|N_{\mu}\left(v_{T}\right)\right| \leq \frac{2}{\rho_{\mu}\left(v_{T}\right)} \leq \frac{128}{(\mu k)^{1 / 5}\left(\rho_{\mu}\left(v_{S}\right)-p^{-1}\right)} \leq \frac{256}{(\mu k)^{1 / 5} \rho_{\mu}(v)},
$$

where on the final bound we use that $\rho_{\mu}(v) \geq \frac{2}{p}$.

## 5. Proof of Theorem 1

In this section we prove our main theorem, Theorem 1. We first show how Theorem 1 follows quickly from Theorem 3 and then we switch our focus to proving Theorem 3.

Define

$$
q_{n}(\beta):=\max _{w \in \mathbb{Z}_{p}^{n}} \mathbb{P}\left(\exists v \in \mathbb{Z}_{p}^{n} \backslash\{0\}: M v=w \text { and } \rho(v) \geq \beta\right)
$$

and note the following lemma from [2] (their Lemma 2.1).
Lemma 9. Let $n \in \mathbb{N}$ and $p>2$ be a prime. Then for every $\beta>0$

$$
\mathbb{P}\left(\operatorname{det}\left(M_{n}\right)=0\right) \leq n \sum_{m=n-1}^{2 n-3}\left(\beta^{1 / 8}+\frac{q_{m}(\beta)}{\beta}\right)
$$

Proof of Theorem 1 assuming Theorem 3. Pick a prime $p=t \exp (c \sqrt{n \log n})$ with $c=1 / 800$ and $t \in[1 / 2,1]$. Letting $\beta=\Theta(n / p)$, we apply the union bound and Theorem 3 to conclude that for $n-1 \leq m \leq 2 n-3$, we have

$$
q_{m}(\beta) \leq \sum_{v: \rho(v) \geq \beta} \max _{w \in \mathbb{Z}_{p}^{m}} \mathbb{P}(M v=w) \leq e^{-c m} .
$$

Thus, we apply Lemma 9, to obtain

$$
\mathbb{P}\left(\operatorname{det}\left(M_{n}\right)=0\right) \leq e^{-c(1+o(1)) \sqrt{n \log n} / 8}+e^{-c n(1+o(1))} \leq e^{-c \sqrt{n \log n} / 9},
$$

for $n$ sufficiently large.
With this reduction firmly in-hand, we turn to prove Theorem 3.
Proof of Theorem 3. Throughout we assume that $n$ is sufficiently large so that all inequalities in the proof hold, we let $k=n^{1 / 4}, d=\frac{2}{\mu} \log p \leq \frac{2}{\mu} \sqrt{n \log n}$ and define $\mathcal{V}:=\left\{v \in \mathbb{Z}_{p}^{n} \backslash\{0\}: \rho_{\mu}(v) \geq \beta\right\}$. Our task is to bound

$$
\begin{equation*}
Q_{n}(\beta):=\sum_{v \in \mathcal{V}} \max _{w} \mathbb{P}\left(M_{n} v=w\right) . \tag{13}
\end{equation*}
$$

We start our analysis of (13) by partitioning this sum by way of a function $f: \mathcal{V} \rightarrow \mathcal{S}$. To define $f$, let $v \in \mathbb{Z}_{p}^{n}$ and apply Lemma 7 to obtain $S, T \subseteq[n]$. We then apply Lemma 4 to $v_{S}$ to obtain a further set $T^{\prime} \subseteq[n]$. We then define $f(v)=\left(S, T, T^{\prime}, v_{T}, v_{T^{\prime}}\right)$ and put $\mathcal{S}:=f(\mathcal{V})$. We thus partition our sum (13) as

$$
\begin{equation*}
Q_{n}(\beta)=\sum_{s \in \mathcal{S}} \sum_{v \in f^{-1}(s)} \max _{w} \mathbb{P}\left(M_{n} v=w\right) . \tag{14}
\end{equation*}
$$

Note that if $s=\left(S, T, T^{\prime}, u_{1}, u_{2}\right) \in \mathcal{S}$, then

$$
\begin{equation*}
|S| \leq k d, \quad\left|u_{1}\right|,\left|u_{2}\right| \leq d, \quad \rho_{\mu}\left(u_{1}\right) \geq \beta, \quad \rho_{\mu}\left(u_{2}\right) \leq(1-\mu / 2)^{\left|u_{2}\right|} \quad \text { and } \quad u_{2} \neq 0 \tag{15}
\end{equation*}
$$

by Lemmas 4 and 7 together with (6), and note that we have the bound

$$
\begin{equation*}
|\mathcal{S}| \leq 8^{n} p^{2 d} \tag{16}
\end{equation*}
$$

since there are $8^{n}$ choices for $S, T, T^{\prime}$ and at most $p^{2 d}$ choices for $u_{1}, u_{2}$.
We now turn to bounding a given term in the sum (14), based on which piece of the partition it is in. Let $s=\left(S, T, T^{\prime}, u_{1}, u_{2}\right) \in \mathcal{S}$ and $v \in f^{-1}(s)$. For any $w \in \mathbb{Z}_{p}^{n}$, we bound $\mathbb{P}\left(M_{n} v=w\right)$ by first revealing the rows indexed by $S^{c}$ and then revealing the rows indexed by $S \backslash T^{\prime}$,

$$
\mathbb{P}(M v=w) \leq \mathbb{P}\left(M_{\left(S \backslash T^{\prime}\right) \times[n]} \cdot v=w_{S \backslash T^{\prime}} \mid M_{S^{c} \times[n]} \cdot v=w_{S^{c}}\right) \cdot \mathbb{P}\left(M_{S^{c} \times[n]} \cdot v=w_{S^{c}}\right)
$$

Looking only on the off-diagonal blocks $\left(S \backslash T^{\prime}\right) \times T^{\prime}$ and $S^{c} \times S$ and considering the "worst case" vectors for these blocks, we have

$$
\mathbb{P}(M v=w) \leq \max _{u} \mathbb{P}\left(M_{\left(S \backslash T^{\prime}\right) \times T^{\prime}} \cdot v_{T^{\prime}}=u\right) \cdot \max _{u} \mathbb{P}\left(M_{S^{c} \times S} \cdot v_{S}=u\right) .
$$

The crucial point here is that these events can be written as an intersection of independent events concerning the rows. That is

$$
\begin{equation*}
\mathbb{P}(M v=w) \leq \rho\left(v_{T^{\prime}}\right)^{|S|-\left|T^{\prime}\right|} \rho\left(v_{S}\right)^{n-|S|} \leq \rho_{\mu}\left(v_{T^{\prime}}\right)^{|S|-\left|T^{\prime}\right|} \rho_{\mu}\left(v_{S}\right)^{n-|S|}, \tag{17}
\end{equation*}
$$

where this last inequality follows from the monotonicity of $\rho$ in the parameter $\mu$, noted at (6).
We now bound the size of a piece of our partition $\left|f^{-1}(s)\right|$. By (5) together with Lemmas 4 and 7 , the number of choices for $v_{S^{c}}$ and $v_{S \backslash T^{\prime}}$ are (respectively) at most

$$
\left|N\left(u_{1}\right)\right|^{n-|S|} \leq\left(\frac{2}{\rho_{\mu}\left(u_{1}\right)}\right)^{n-|S|}, \quad\left|N\left(u_{2}\right)\right|^{|S|-\left|T^{\prime}\right|} \leq\left(\frac{2}{\rho_{\mu}\left(u_{2}\right)}\right)^{|S|-\left|T^{\prime}\right|}
$$

so that

$$
\begin{equation*}
\left|f^{-1}(s)\right| \leq\left(\frac{2}{\rho_{\mu}\left(u_{1}\right)}\right)^{n-|S|}\left(\frac{2}{\rho_{\mu}\left(u_{2}\right)}\right)^{|S|-\left|T^{\prime}\right|} \tag{18}
\end{equation*}
$$

By (17) and the fact that $|S| \leq k d=o(n)$ (by our choice of parameters), we have

$$
\begin{equation*}
\sum_{v \in f^{-1}(s)} \max _{w} \mathbb{P}\left(M_{n} \cdot v=w\right) \leq 2^{n}\left(\frac{\rho_{\mu}\left(u_{1}^{k}\right)}{\rho_{\mu}\left(u_{1}\right)}\right)^{n-|S|} \leq 2^{n}\left(\frac{\rho_{\mu}\left(u_{1}^{k}\right)}{\rho_{\mu}\left(u_{1}\right)}\right)^{24 n / 25} \tag{19}
\end{equation*}
$$

We consider first the case where $\left|u_{1}\right| \neq 0$; then we may apply Lemma 8 to obtain the bound

$$
\rho_{\mu}\left(u_{1}^{k}\right) \leq 64(\mu k)^{-1 / 5} \rho_{\mu}\left(u_{1}\right)+\frac{1}{p} .
$$

By the bound $\rho_{\mu}\left(u_{1}\right) \geq \beta=\Theta(n / p)$, we then have

$$
\frac{\rho_{\mu}\left(u_{1}^{k}\right)}{\rho_{\mu}\left(u_{1}\right)} \leq 64(\mu k)^{-1 / 5}+\Theta\left(n^{-1}\right) \leq n^{-1 / 24} .
$$

Combining this with (16) and (19) shows that

$$
\begin{align*}
\sum_{\substack{s \in \mathcal{S},\left|u_{1}\right| \neq 0}} \sum_{v \in f^{-1}(s)} \max _{w} \mathbb{P}\left(M_{n} v=w\right) & \leq|\mathcal{S}| \cdot n^{-n / 25} \leq 8^{n} p^{2 d} n^{-n / 25} \\
& \leq 8^{n} \exp \left(\frac{4 c}{\mu} n \log n-\frac{1}{25} n \log n\right) \leq e^{-n}, \tag{20}
\end{align*}
$$

provided $c \leq \mu / 200$. Now if $\left|u_{1}\right|=0$ then there are at most

$$
\left|f^{-1}(s)\right| \leq\left(\frac{2}{\rho_{\mu}\left(u_{2}\right)}\right)^{|S|-\left|T^{\prime}\right|}
$$

choices for $v$. Notice that $\rho_{\mu}\left(v_{S}\right) \leq \rho_{\mu}\left(u_{2}\right)$ and so

$$
\sum_{v \in f^{-1}(s)} \max _{w} \mathbb{P}\left(M_{n} v=w\right) \leq \rho_{\mu}\left(u_{2}\right)^{n-\left|T^{\prime}\right|}\left(\frac{1}{\rho_{\mu}\left(u_{2}\right)}\right)^{|S|-\left|T^{\prime}\right|} \leq \rho_{\mu}\left(u_{2}\right)^{n / 2} \leq(1-\mu / 2)^{n\left|u_{2}\right| / 2}
$$

where for the final inequality we used (15). On the other hand, by (15), the number of choices for $s=\left(S, T, T^{\prime}, u_{1}, u_{2}\right)$ such that $\left|u_{1}\right|=0,\left|u_{2}\right|=t$ is at most

$$
\binom{n}{\leq k d}^{3} p^{t} \leq \exp (c t \cdot \sqrt{n \log n}+3 k d \log n)
$$

Putting our bounds together, we have

$$
\sum_{\substack{s \in \mathcal{S},\left|u_{1}\right|=0,\left|u_{2}\right|=t}} \sum_{v \in f^{-1}(s)} \max _{w} \mathbb{P}\left(M_{n} v=w\right) \leq \exp (c t \cdot \sqrt{n \log n}+3 k d \log n-n \mu t / 4) \leq e^{-n \mu t / 5}
$$

Summing over all $t \geq 1$ (recalling that $u_{2} \neq 0$ ) and using (20), we conclude that

$$
Q_{n}(\beta)=\sum_{s \in \mathcal{S}} \sum_{v \in f^{-1}(s)} \max _{w} \mathbb{P}\left(M_{n} v=w\right) \leq e^{-\mu n / 6}
$$

as desired.

## 6. Proof of Lemma 8

In this section, we pin down one final loose end, the proof of Lemma 8, which is our main Fourier lemma. For $v \in \mathbb{Z}_{p}^{n}$, and $\mu \in[0,1]$ we note a standard Fourier expression for $\rho_{\mu}(v)$. Define

$$
\begin{equation*}
f_{\mu, v}(\xi):=\prod_{i=1}^{n}\left((1-\mu)+\mu c_{p}\left(v_{i} \xi\right)\right) \tag{21}
\end{equation*}
$$

where we let $c_{p}(x)=\cos (2 \pi x / p)$. We then have

$$
\begin{equation*}
\rho_{\mu}(v)=\mathbb{E}_{\xi \in \mathbb{Z}_{p}} f_{\mu, v}(\xi) . \tag{22}
\end{equation*}
$$

Clearly $\left|f_{\mu, v}(\xi)\right| \leq 1$ and for $\mu \leq 1 / 2$ each of the terms in the product $f_{\mu, v}(\xi)$ is non-negative. In this case it is natural to work with $\log f_{\mu, v}$. For this, we let $\|x\|_{\mathbb{T}}$ denote the distance from $x \in \mathbb{R}$ to the nearest integer and note the following bounds. For $\mu \in[0,1 / 4]$ we have

$$
\begin{equation*}
\mu\|x / p\|_{\mathbb{T}}^{2} \leq-\log \left(1-\mu+\mu c_{p}(x)\right) \leq 32 \mu\|x / p\|_{\mathbb{T}}^{2} \tag{23}
\end{equation*}
$$

which are elementary ${ }^{3}$ and can be found in (7.1) in [19].
For the following lemma, one of the main results of this section, we need the well-known CauchyDavenport inequality which tells us that for $A, B \subseteq \mathbb{Z}_{p}$ we have $|A+B| \geq \min \{|A|+|B|-1, p\}$. Here, as usual, $A+B:=\{a+b: a \in A, b \in B\}$.

A first step towards Lemma 8 is to prove it in the case when $\rho_{\mu}(v)$ is not too large.
Lemma 10. Let $\mu \in(0,1 / 4], v \in \mathbb{Z}_{p}^{*}$ and $k \in \mathbb{N}$. Then

$$
\rho_{\mu}\left(v^{k}\right) \leq\left(\rho_{\mu}(v)^{\frac{k-1}{k}}+\frac{8}{\sqrt{\mu k}}\right) \rho_{\mu}(v)+p^{-1} .
$$

To prove this lemma, we adopt some temporary notation. Let $F=f_{\mu, v^{k}}$ and $G=f_{\mu, v}$, be as defined in (21) and note that $G=F^{1 / k}$. We note also that $F$ is non-negative since $\mu \leq 1 / 4$. Let $\ell:=\frac{1}{8}(\mu k)^{1 / 2}$. For all $\alpha \in(0,1)$, we consider the level sets

$$
A_{\alpha}:=\left\{\xi \in \mathbb{Z}_{p}: F(\xi)>\alpha\right\} \quad B_{\alpha}:=\left\{\xi \in \mathbb{Z}_{p}: G(\xi)>\alpha\right\} .
$$

Claim 11. For $\alpha \in(0,1)$, we have $\ell \cdot A_{\alpha} \subseteq B_{\alpha}$.
Proof. To see this, assume $\xi_{1}, \ldots, \xi_{\ell} \in A_{\alpha}$ and so $G\left(\xi_{i}\right)=\left(F\left(\xi_{i}\right)\right)^{1 / k}>\alpha^{1 / k}$ for each $i \in[\ell]$. Taking logs of both sides and applying (23) gives, for each $i \in[\ell]$,

$$
\begin{equation*}
\mu \sum_{j=1}^{n}\left\|\xi_{i} v_{j}\right\|_{\mathbb{T}}^{2} \leq-\log G\left(\xi_{i}\right) \leq k^{-1} \log \alpha^{-1} \tag{24}
\end{equation*}
$$

Thus, using the triangle inequality along with (24) gives

$$
\left(\sum_{j=1}^{n}\left\|\left(\xi_{1}+\cdots+\xi_{\ell}\right) v_{j}\right\|_{\mathbb{T}}^{2}\right)^{1 / 2} \leq \sum_{i=1}^{\ell}\left(\sum_{j=1}^{n}\left\|\xi_{i} v_{j}\right\|_{\mathbb{T}}^{2}\right)^{1 / 2} \leq \ell\left(\frac{\log \alpha^{-1}}{\mu k}\right)^{1 / 2}
$$

It then follows from the upper bound in (23) that

$$
-\log G\left(\xi_{1}+\cdots+\xi_{\ell}\right) \leq 32 \sum_{j=1}^{n}\left\|\left(\xi_{1}+\cdots+\xi_{\ell}\right) v_{j}\right\|_{\mathbb{T}}^{2} \leq 32 \frac{\ell^{2}}{\mu k} \log \alpha^{-1}
$$

Thus, using our choice of $\ell=\frac{1}{8}(\mu k)^{1 / 2}$, we have $G\left(\xi_{1}+\cdots+\xi_{\ell}\right)>\alpha$, and so $\xi_{1}+\cdots+\xi_{\ell} \in B_{\alpha}$.

[^2]Proof of Lemma 10. Letting $g:=\mathbb{E}_{\xi} G=\rho_{\mu}(v)$, we want to show that $\mathbb{E}_{\xi} F \leq\left(g^{(k-1) / k}+\frac{8}{\sqrt{\mu k}}\right) g+$ $p^{-1}$. We do this in two ranges. First we recall that $F^{1 / k}=G$ and so

$$
\mathbb{E}_{\xi}[F \mathbf{1}(F \leq g)] \leq \mathbb{E}_{\xi}\left[G \cdot g^{(k-1) / k}\right]=g^{\frac{2 k-1}{k}}
$$

Next we treat the $\xi$ for which $F(\xi)>g$. First note that by Markov's inequality $\left|B_{\alpha}\right|<p$, for all $\alpha>g$. It follows from Claim 11 and the Cauchy-Davenport inequality that $\left|A_{\alpha}\right| \leq \ell^{-1}\left|B_{\alpha}\right|+1$ for all $\alpha>g$. Thus,

$$
\mathbb{E}_{\xi}[F \mathbf{1}(F>g)]=\int_{g}^{1}\left|A_{t}\right| p^{-1} d t \leq \ell^{-1} \int_{g}^{1}\left|B_{t}\right| p^{-1} d t+1 / p \leq g / \ell+1 / p
$$

Putting our bounds together we have

$$
\rho_{\mu}\left(v^{k}\right) \leq\left(g^{(k-1) / k}+\frac{8}{\sqrt{\mu k}}\right) g+1 / p=\left(\rho_{\mu}(v)^{(k-1) / k}+\frac{8}{\sqrt{\mu k}}\right) \rho_{\mu}(v)+p^{-1},
$$

as desired.
To complete our proof of Lemma 8, we need the following classical result:
Lemma 12. If $v \in \mathbb{Z}_{p}^{*}$ with $v \neq 0$ then $\rho_{\mu}(v) \leq \frac{64}{\sqrt{\mu|v|}}+p^{-1}$.
Letting $d=|v|$, this lemma may be deduced by bounding $\rho_{\mu}(v) \leq \rho_{\mu}\left(v_{j}^{d}\right)$ for some $j$ by (12), noting that $\rho_{\mu}\left(v_{j}^{d}\right)=\rho_{\mu}\left(1^{d}\right)$ and bounding the latter either directly or using a standard local central limit theorem. Alternatively, a stronger statement may be found in [2, Lemma 2.3].

Proof of Lemma 8. If $\rho_{\mu}(v) \leq(\mu k)^{-1 / 4}$ then Lemma 10 tells us that $\rho_{\mu}\left(v^{k}\right) \leq 64(\mu k)^{-1 / 5} \rho_{\mu}(v)+$ $p^{-1}$, as desired. On the other hand, if $\rho_{\mu}(v)>(\mu k)^{-1 / 4}$,

$$
\rho_{\mu}\left(v^{k}\right)=\frac{64}{\sqrt{\mu k|v|}}+1 / p \leq 64(\mu k)^{-1 / 4} \rho_{\mu}(v)+1 / p
$$

thus completing the proof.

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[^0]:    The first named author is partially supported by CNPq.
    ${ }^{1}$ See [9] for a short discussion on the history of this conjecture

[^1]:    ${ }^{2}$ We will also write $\rho_{1}(v)=\rho(v)$.

[^2]:    ${ }^{3}$ For these explicit constants, note the bounds $a \leq-\log (1-a) \leq(3 / 2) a$ for $a \in[0,1 / 4]$ and $x^{2} \leq 1-\cos (2 \pi x) \leq 20 x^{2}$ for $|x| \leq 1 / 2$.

