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# Analogues of Khintchine's theorem for random attractors 

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#### Abstract

In this paper we study random iterated function systems. Our main result gives sufficient conditions for an analogue of a well known theorem due to Khintchine from Diophantine approximation to hold almost surely for stochastically self-similar and self-affine random iterated function systems.


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Key words and phrases: Random iterated function systems, Diophantine approximation, self-similar systems, self-affine systems, Khintchine's theorem.

## 1 Introduction

Khintchine's theorem is an important result in number theory which demonstrates that the Lebesgue measure of certain limsup sets defined using the rationals is determined by the convergence/divergence of naturally occurring volume sums. Inspired by this result, the first author studied fractal analogues of Khintchine's theorem where the role of the rationals is played by a natural set of points that is generated by the underlying iterated function system $[2,3,4,5]$. The results of [5] demonstrate that for many parameterised families of overlapping iterated function systems, we typically observe Khintchine like behaviour. The results of [5] also demonstrate that by viewing overlapping iterated function systems through the lens of Diophantine approximation, we obtain a new meaningful framework for classifying iterated function systems.

In this article we consider analogues of Khintchine's theorem for random models of attractors. In particular, we investigate Khintchine type results for random recursive fractal sets, a natural class of randomly generated sets commonly used as a model for self-similar and self-affine sets. Our main result shows that under appropriate hypothesis for our random recursive model, we will almost surely observe Khintchine like behaviour. To state our main result in full it is necessary to properly formalise our model and introduce several other notions. To motivate what follows we include the following easier to state theorem that is a consequence of Theorem 3.4.

Theorem 1.1. Let $t_{1}$ and $t_{2}$ be two distinct real numbers. Fix $r_{1}$ and $r_{2}$ satisfying $0 \leq r_{1}<$ $r_{2}<1$ and let $\eta$ be the normalised Lebesgue measure on $\left[r_{1}, r_{2}\right]$. Let $\boldsymbol{r}:=\left(r_{\mathbf{a}}\right)_{\mathbf{a} \in \cup_{n=1}^{\infty}\{1,2\}^{n}}$ be a sequence of real numbers enumerated by the finite words with digits in $\{1,2\}$, such that each $r_{\mathbf{a}}$ is chosen independently from $\left[r_{1}, r_{2}\right]$ according to the law $\eta$. For each $\boldsymbol{r}$ we define a projection map $\Pi_{r}:\{1,2\}^{\mathbb{N}} \rightarrow \mathbb{R}^{d}$ given by

$$
\Pi_{r}(\mathbf{b}):=\sum_{k=1}^{\infty} t_{b_{k}} \cdot \prod_{j=1}^{k-1} r_{b_{1} \ldots b_{j}} .
$$

Let $\mathbf{b} \in\{1,2\}^{\mathbb{N}}$ and assume $\log 2>-\int_{r_{1}}^{r_{2}} \log r d r$. Then for almost every $\boldsymbol{r}$ the set

$$
\left\{x \in \mathbb{R}^{d}:\left|x-\Pi_{r}\left(a_{1} \ldots a_{m} \mathbf{b}\right)\right| \leq \frac{1}{2^{m} \cdot m} \text { for i.m. } a_{1} \ldots a_{m} \in \cup_{n=1}^{\infty} \mathcal{A}^{n}\right\}
$$

has positive Lebesgue measure.
Here and throughout we write i.m. as a shorthand for infinitely many. We emphasise that our main result also covers higher dimensional random iterated function systems that may contain affine maps.

## 2 Background

In this section we recall some background results from fractal geometry and Diophantine approximation. We also detail our motivating problem in the deterministic setting and provide some background on random models for iterated function systems.

### 2.1 Fractal Geometry

Given a finite set of contractions $\Phi=\left\{\phi_{i}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}\right\}_{i \in \mathcal{A}}$ there exists, by a well know result due to Hutchinson [19], a unique non-empty compact set $X \subseteq \mathbb{R}^{d}$ satisfying

$$
X=\bigcup_{i \in \mathcal{A}} \phi_{i}(X) .
$$

This set $X$ is called the attractor of $\Phi$ and $\Phi$ is commonly referred to as an iterated function system or IFS for short. When each element of the IFS is an affine map we refer to the attractor as a self-affine set. Similarly, when each element of the IFS is a similarity, i.e. there exists $r_{i} \in(0,1)$ such that $\left|\phi_{i}(x)-\phi_{i}(y)\right|=r_{i}|x-y|$ for all $x, y \in \mathbb{R}^{d}$, we say that the attractor is a self-similar set. For a self-similar IFS $\Phi$ we define the similarity dimension $\operatorname{dim}_{S}(\Phi)$ to be the unique solution to $\sum_{i \in \mathcal{A}} r_{i}^{s}=1$. The similarity dimension is always an upper bound for the Hausdorff dimension of $X$. For self-affine sets there is a similar upper bound for the Hausdorff dimension defined in terms of the affinity dimension; see [15] for its definition. The additional structure of affine maps or similarities makes questions on the attractor more tractable and the two classes are the most studied types of attractor.

A classical problem from fractal geometry is to determine the metric and topological properties of self-similar sets and self-affine sets; see [12, 13]. To make progress with this problem one often studies the pushforwards of dynamically interesting measures onto the attractor. This approach has resulted in many significant breakthroughs; see e.g. $[6,17,18,22,31,35,36]$ and the references therein. Many important conjectures in this area can be summarised by the statement: either an IFS contains an exact overlap, ${ }^{1}$ or the corresponding attractor and

[^0]the dynamically interesting measures supported upon it exhibit the expected behaviour. These conjectures have been verified in certain special cases, see [17, 32, 33, 35, 38]. Part of the motivation behind this paper is to obtain a deeper classification of iterated function systems that goes beyond the exact overlap versus no exact overlap dichotomy.

### 2.2 Diophantine approximation

Given a function $\Psi: \mathbb{N} \rightarrow[0, \infty)$, we can define a limsup set in terms of neighbourhoods of the rationals. Let

$$
J(\Psi):=\left\{x \in \mathbb{R}:\left|x-\frac{p}{q}\right| \leq \Psi(q) \text { for i.m. }(p, q) \in \mathbb{Z} \times \mathbb{N}\right\}
$$

The Borel-Cantelli lemma implies that if $\sum_{q=1}^{\infty} q \cdot \Psi(q)<\infty$, then $J(\Psi)$ has zero Lebesgue measure. Interestingly, a theorem due to Khintchine shows that a partial converse to this statement holds.

Theorem 2.1 ([21]). If $\Psi: \mathbb{N} \rightarrow[0, \infty)$ is decreasing and

$$
\sum_{q=1}^{\infty} q \cdot \Psi(q)=\infty
$$

then Lebesgue almost every $x \in \mathbb{R}$ is contained in $J(\Psi)$.
An example due to Duffin and Schaeffer [11] shows that one cannot remove the monotonicity assumptions from Theorem 2.1. This lead to the famous Duffin and Schaeffer conjecture that was recently proved by Koukoulopoulos and Maynard [24].

Results analogous to Khintchine's theorem which show that the measure of a limsup set is determined by the convergence/divergence of some naturally occurring volume sum are present throughout Diophantine approximation and metric number theory (see [8]). For our purposes, the important aspect of the above is that by studying the metric properties of the sets $J(\Psi)$ for those $\Psi$ satisfying $\sum_{q=1}^{\infty} q \cdot \Psi(q)=\infty$, one obtains a quantitative description of how the rational numbers are distributed within the real numbers. In particular, the example due to Duffin and Schaeffer of a $\Psi$ for which $\sum_{q=1}^{\infty} q \cdot \Psi(q)=\infty$, yet $J(\Psi)$ has zero Lebesgue measure, reveals certain subtleties in the geometry of the rational numbers.

### 2.3 Overlapping iterated function systems from the perspective of metric number theory

Khintchine's theorem provides a quantitative description of how the rationals are distributed within $\mathbb{R}$. The motivation behind the work discussed below comes from a desire to obtain an analogous quantitative description for how an iterated function system overlaps.

Given a finite set $\mathcal{A}$ we let $\mathcal{A}^{*}=\bigcup_{n=1}^{\infty} \mathcal{A}^{n}$ denote the corresponding set of finite words. Given an IFS $\left\{\phi_{i}\right\}_{i \in \mathcal{A}}$ and a word $\mathbf{a}=\left(a_{1} \ldots a_{n}\right) \in \mathcal{A}^{*}$, we let $\phi_{\mathbf{a}}:=\phi_{a_{1}} \circ \cdots \circ \phi_{a_{n}}$. We also let $|\mathbf{a}|$ denote the length of a word $\mathbf{a}$. Now suppose we have an IFS $\Phi$, a function $\Psi: \mathcal{A}^{*} \rightarrow[0, \infty)$, and $z \in X$, we define the following analogue of the set $J(\Psi)$ :

$$
W_{\Phi}(z, \Psi):=\left\{x \in \mathbb{R}^{d}:\left|x-\phi_{\mathbf{a}}(z)\right| \leq \Psi(\mathbf{a}) \text { for i.m. } \mathbf{a} \in \mathcal{A}^{*}\right\} .
$$

For the set $W_{\Phi}(z, \Psi)$ the role of the rationals is played by the images of $z$ obtained by repeatedly applying elements of the IFS. Proceeding via analogy with Theorem 2.1, it is reasonable to expect that there exists a divergence condition on volume sums which implies that being contained in
$W_{\Phi}(z, \Psi)$ holds almost surely with respect to some measure. One particular instance of this could be formalised as follows: Let $\mathcal{H}^{s}$ be the $s$-dimensional Hausdorff measure. Is it true that

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sum_{\mathbf{a} \in \mathcal{A}^{n}} \Psi(\mathbf{a})^{\operatorname{dim}_{H}(X)}=\infty \Rightarrow \mathcal{H}^{\operatorname{dim}_{H}(X)}\left(W_{\Phi}(z, \Psi)\right)=\mathcal{H}^{\operatorname{dim}_{H}(X)}(X) ? \tag{2.1}
\end{equation*}
$$

The existence of a general class of $\Psi$ for which (2.1) holds demonstrates how well the images of $z$ are spread out within $\mathbb{R}^{d}$. Studying those $\Psi$ for which (2.1) holds provides a quantitative description of how an IFS overlaps.

In a series of recent papers, the first author established that for many IFSs we do observe Khintchine like behaviour, i.e. (2.1) holds for some suitable class of $\Psi$, see [2, 3, 4, 5]. Related results had appeared previously in papers of Persson and Reeve [29, 30], and Levesley, Salp, and Velani [25]. In [2] it was shown that whenever $\Phi$ is an IFS consisting of similarities and satisfies the open set condition, then an appropriate analogue of Theorem 2.1 holds ${ }^{2}$. See [1] for some further related work. The more challenging and interesting case is when the underlying IFS satisfies $\operatorname{dim}_{S}(\Phi)>d$, or the equivalent inequality for the affinity dimension. Loosely speaking, when these inequalities are satisfied it is possible for a better rate of approximation to hold generically. It was shown in [5] that for many parameterised families of IFSs for which $\operatorname{dim}_{S}(\Phi)>d$ holds for each member of the family, or the equivalent inequality for affinity dimension, an analogue of Khintchine's theorem holds generically. To detail this analogue we introduce the following notation which we will use throughout.

Given a set $B \subset \mathbb{N}$, we define the the upper density of $B$ to be

$$
\bar{d}(B):=\limsup _{n \rightarrow \infty} \frac{\#\{1 \leq j \leq n: j \in B\}}{n} .
$$

Given $\varepsilon>0$, let

$$
G_{\varepsilon}:=\left\{g: \mathbb{N} \rightarrow[0, \infty): \sum_{n \in B} g(n)=\infty, \forall B \subseteq \mathbb{N} \text { s.t. } \bar{d}(B)>1-\varepsilon\right\} .
$$

We also define

$$
\begin{equation*}
G:=\bigcup_{\varepsilon \in(0,1)} G_{\varepsilon} \tag{2.2}
\end{equation*}
$$

For example, it can be shown that the function $g(n)=1 / n$ is contained in $G$.
Let $\lambda \in(0,1)$ and $O$ be a $d \times d$ orthogonal matrix. For any $\mathbf{t}=\left(t_{1}, \ldots, t_{\# \mathcal{A}}\right) \in \mathbb{R}^{\# \mathcal{A} \cdot d}$ we can define an iterated function system $\Phi_{\mathbf{t}}:=\left\{\phi_{i}(x)=\lambda \cdot O x+t_{i}\right\}_{i \in \mathcal{A}}$. We let $X_{\mathbf{t}}$ denote the corresponding attractor of $\Phi_{\mathrm{t}}$. The following theorem follows from [5, Theorem 2.6, Corollary $2.7]$ and demonstrates that Khintchine like behaviour typically occurs within this family.

Theorem 2.2. Fix $\lambda \in(0,1 / 2)$ and $O$ a $d \times d$ orthogonal matrix. Suppose $\frac{\log \# \mathcal{A}}{-\log \lambda}>d$. Then for Lebesgue almost every $t \in \mathbb{R}^{\# \cdot \mathcal{A} \cdot d}$, for any $g \in G$ and $z \in X_{\mathbf{t}}$, the set

$$
\left\{x \in \mathbb{R}^{d}:\left|x-\phi_{\mathbf{a}}(z)\right| \leq\left(\frac{g(|\mathbf{a}|)}{(\# \mathcal{A})^{|\mathbf{a}|}}\right)^{1 / d} \text { for i.m. } \mathbf{a} \in \mathcal{A}^{*}\right\}
$$

has positive Lebesgue measure.
Suitable analogues of Theorem 2.2 hold with different rates of contraction and with similarities replaced by affine maps (see [5, Theorem 2.6]).

[^1]The utility of studying IFSs using ideas from Diophantine approximation is emphasised by an observation made in [5]. Consider the parameterised family of IFSs given by

$$
\Phi_{t}=\left\{\phi_{1}(x)=\frac{x}{2}, \phi_{2}(x)=\frac{x+1}{2}, \phi_{3}(x)=\frac{x+t}{2}, \phi_{4}(x)=\frac{x+1+t}{2}\right\}
$$

Here $t \in[0,1]$ and the attractor of $\Phi_{t}$ is $[0,1+t]$. For $t \in[0,1]$ and a probability vector $\mathbf{p}=\left(p_{1}, p_{2}, p_{3}, p_{4}\right)$ we let $\mu_{\mathbf{p}, t}$ be the self-similar measure corresponding to $\mathbf{p}$ and $\Phi_{t}$ (see [13] for the definition of a self-similar measure). It was shown in [5] that there exists $t, t^{\prime} \in[0,1]$ such that for any probability vector $\mathbf{p}$ we have $\operatorname{dim} \mu_{\mathbf{p}, t}=\operatorname{dim} \mu_{\mathbf{p}, t^{\prime}}=\min \left\{\frac{-\sum_{i=1}^{4} p_{i} \log p_{i}}{\log 2}, 1\right\}$, and the set of $\mathbf{p}$ for which it is known that $\mu_{\mathbf{p}, t}$ is absolutely continuous equals the set of $\mathbf{p}$ for which $\mu_{\mathbf{p}, t^{\prime}}$ is known to be absolutely continuous. However there exists $\Psi$ for which $W_{\Phi_{t}}(z, \Psi)$ has full measure within $[0,1+t]$ for all $z \in[0,1+t]$ and $W_{\Phi_{t^{\prime}}}(z, \Psi)$ has zero measure for all $z \in\left[0,1+t^{\prime}\right]$. In other words $\Phi_{t}$ and $\Phi_{t^{\prime}}$ are indistinguishable in terms of the properties of their self-similar measures, but their overlapping behaviours can be distinguished using the language of the sets $W_{\Phi}(z, \Psi)$.

### 2.4 Random models for iterated function systems

The main results of [5] hold for several families of parameterised IFSs. In the absence of a general result for parameterised families of IFSs, it is natural to study suitable random analogues that mirror the key properties a family exhibits. This approach benefits from small random perturbations that "smooth out" the parts that are intractable in a deterministic approach. This was employed in [22] by adding random translations to the deterministic linear parts to determine the almost sure dimensions of their random attractors, as well as finding conditions for absolute continuity. A complementary approach was taken in [27] which randomised the linear part while keeping the translates fixed. It further assumed that the linear parts were similarities and that the randomisation is uniform for all cylinders in that level of the construction (knows as random homogeneous or 1-variable attractor, see [37]). A similar model was considered in [20], where the authors determined the dimensions of random self-affine sets. In this paper we randomise the linear part at every stage using the random recursive model, where we allow the linear parts to be both self-similar and self-affine. Theorem 3.4 is the main result of this paper. It gives sufficient conditions for a random model to ensure that an analogue of Khintchine's theorem holds almost surely.

Notation. For two real valued functions $f$ and $g$ defined on some set $S$, we write $f \ll g$ or $f=\mathcal{O}(g)$ if there exists $C>0$ such that $|f(x)| \leq C \cdot g(x)$ for all $x \in S$. We write $f \asymp g$ if $f \ll g$ and $g \ll f$.

Let $\mathcal{A}$ be a finite set and $i \in \mathcal{A}$. Given a finite word $\mathbf{a} \in \mathcal{A}^{*}$ we let $|\mathbf{a}|_{i}:=\#\{1 \leq k \leq$ $\left.|\mathbf{a}|: a_{k}=i\right\}$ denote the number of occurrences of the digit $i$ in $\mathbf{a}$. Moreover, given two words $\mathbf{a}, \mathbf{b} \in \mathcal{A}^{*}$ we let $\mathbf{a} \wedge \mathbf{b}$ denote the maximal common prefix of $\mathbf{a}$ and $\mathbf{b}$, assuming such a prefix exists. If no such prefix exists, $\mathbf{a} \wedge \mathbf{b}$ is the empty word.

## 3 Our random model and statements of results

In this paper we will consider the stochastically self-similar (and self-affine) model which is is also known as the random recursive or $\infty$-variable model. It is one of the most important models of randomness in fractal geometry and was introduced, independently, by Falconer [14] and Graf [16] and has subsequently attracted a lot of attention, see e.g. $[26,34,37]$ and the references therein. To define this randomisation rigorously, we first define random functions $f_{\mathbf{a}}$ indexed by
$\mathbf{a} \in \mathcal{A}^{*}$. Each $f_{\mathbf{a}}$ is chosen independently from all other $\mathbf{b} \neq \mathbf{a}$ following a distribution that only depends on the last letter of $\mathbf{a}$.

Let $M_{d} \subset \mathbb{R}^{d^{2}}$ denote the set of invertible $d \times d$ matrices with real entries satisfying $\|A\|<1$, where $\|$.$\| denotes the usual operator norm. We write S_{d}$ for those elements of $M_{d}$ that are also similarities, i.e. are a scalar multiple of an orthogonal matrix. For each $i \in \mathcal{A}$ we let $\Omega_{i}$ be a subset of $M_{d}$ with operator norm uniformly bounded away from 1 . Moreover, for each $i \in A$ we let $\eta_{i}$ denote a Borel probability measure supported on $\Omega_{i}$. We define a product measure indexed by the elements of $\mathcal{A}^{*}$ such that the distribution corresponding to the coordinate $\mathbf{a} \in \mathcal{A}^{*}$ depends only on the last letter $l(\mathbf{a}):=a_{|\mathbf{a}|}$. That is, we set $\eta=\prod_{\mathbf{a} \in \mathcal{A}^{*}} \eta_{l(\mathbf{a})}$ as the product measure on the product space $\Omega=\prod_{\mathbf{a} \in \mathcal{A}^{*}} \Omega_{l(\mathbf{a})}$. Thus, a particular realisation $\omega \in \Omega$ is a collection of matrices in $M_{d}$ indexed by $\mathbf{a} \in \mathcal{A}^{*}$, where each entry is distributed according to its respective $\eta_{l(\mathbf{a})}$.

We will make the distinction between $\omega \in \Omega$ as a realisation chosen with law $\eta$, and the linear component it defines at a particular index by writing $A_{\omega, \mathbf{a}}(x):=(\omega)_{\mathbf{a}} \cdot x$ for the linear function given by the random matrix indexed by $\mathbf{a}$. We will often write $A_{\mathbf{a}}$ for $A_{\omega, \mathbf{a}}$ when the choice of $\omega$ is implicit. Note that $A_{\mathbf{a}}$ is distributed with law $\eta_{l(\mathbf{a})}$, the distribution corresponding to the last letter of $\mathbf{a}$. By definition, this function is independent from $A_{\mathbf{b}}$ for all $\mathbf{b} \in \mathcal{A}^{*}$ with $\mathbf{b} \neq \mathbf{a}$.

Let $\left\{t_{i}\right\}_{i \in \mathcal{A}}$ be a finite collection of distinct translation vectors in $\mathbb{R}^{d}$, that is $i \neq j \Rightarrow\left|t_{i}-t_{j}\right| \neq$ 0 . For every $\omega \in \Omega$ we define a random contraction $f_{\omega, \mathbf{a}}$ for every finite word $\mathbf{a} \in \mathcal{A}^{*}$ to be

$$
f_{\omega, \mathbf{a}}(x):=A_{\omega, \mathbf{a}}(x)+t_{l(\mathbf{a})} .
$$

We emphasise that although the distribution of $A_{\omega, \mathbf{a}}$ only depends on the last letter of $\mathbf{a}$, the exact realisation depends upon $\mathbf{a}$ and is independent of all $\mathbf{b} \neq \mathbf{a}$. We will often omit the realisation $\omega$ from $f_{\omega, \mathbf{a}}$ when it is clear from context. Given $\omega \in \Omega$ and $\left(a_{1} \ldots a_{n}\right) \in \mathcal{A}^{*}$ we denote the corresponding concatenation of matrices as follows:

$$
\widehat{A}_{\omega, a_{1} \ldots a_{n}}:=A_{\omega, a_{1}} \circ \cdots \circ A_{\omega, a_{1} \ldots a_{n}} .
$$

For a finite word $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ and $\omega \in \Omega$ we let

$$
\phi_{\omega, \mathbf{a}}(x):=f_{\omega, a_{1}} \circ \cdots \circ f_{\omega, a_{1} \ldots a_{n}}(x)
$$

Given $\omega \in \Omega$ we define the projection map $\Pi_{\omega}: \mathcal{A}^{\mathbb{N}} \rightarrow \mathbb{R}^{d}$ via the equation

$$
\Pi_{\omega}(\mathbf{a})=\lim _{k \rightarrow \infty} \phi_{\omega, a_{1} \ldots a_{k}}(\mathbf{0}) .
$$

Notice that $\mathbf{0}$ can be replaced with any element of $\mathbb{R}^{d}$. In addition, given a finite word $\mathbf{a} \in \mathcal{A}^{*}$ and $\omega \in \Omega$, we define the projection map $\Pi_{\omega, \mathbf{a}}: \mathcal{A}^{\mathbb{N}} \rightarrow \mathbb{R}^{d}$ to be

$$
\Pi_{\omega, \mathbf{a}}(\mathbf{b})=\lim _{k \rightarrow \infty} f_{\omega, \mathbf{a} b_{1}} \circ \cdots \circ f_{\omega, \mathbf{a} b_{1} \ldots b_{k}}(\mathbf{0})
$$

Notice that for any $\mathbf{a} \in \mathcal{A}^{*}$ and $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$ we have

$$
\Pi_{\omega}(\mathbf{a b})=\phi_{\omega, \mathbf{a}}\left(\Pi_{\omega, \mathbf{a}}(\mathbf{b})\right) .
$$

In what follows we refer to the tuple $\left(\left\{\Omega_{i}\right\}_{i \in \mathcal{A}},\left\{\eta_{i}\right\}_{i \in \mathcal{A}},\left\{t_{i}\right\}_{i \in \mathcal{A}}\right)$ as a random iterated function system or RIFS for short. For any $\omega \in \Omega$ its unique random attractor is defined to be

$$
F_{\omega}:=\bigcup_{\mathbf{a} \in \mathcal{A}^{\mathbb{N}}} \Pi_{\omega}(\mathbf{a}) .
$$

$F_{\omega}$ is a non-empty compact set for all $\omega \in \Omega$. By definition, the set $F_{\omega}$ is stochastically selfsimilar in the sense that

$$
F_{\omega} \equiv_{d} \bigcup_{i \in \mathcal{A}} f_{\kappa, i}\left(F_{\tau_{i}}\right)
$$

holds in distribution, where $\omega, \kappa, \tau_{1}, \ldots, \tau_{\# \mathcal{A}}$ are independently realisations in $(\Omega, \eta)$.
Given $\Psi: \mathcal{A}^{*} \rightarrow[0, \infty), \mathbf{b} \in \mathcal{A}^{\mathbb{N}}$, and $\omega \in \Omega$, our random analogue of the deterministic set $W_{\Phi}(z, \Psi)$ is defined to be

$$
W_{\omega}(\mathbf{b}, \Psi):=\left\{x \in \mathbb{R}^{d}:\left|x-\Pi_{\omega}(\mathbf{a b})\right| \leq \Psi(\mathbf{a}) \text { for i.m. } \mathbf{a} \in \mathcal{A}^{*}\right\} .
$$

### 3.1 An auxiliary family of sets

Directly studying the sets $W_{\omega}(\mathbf{b}, \Psi)$ for a general $\Psi$ is a challenging problem. Instead, we study properties of an auxiliary family that we can then use to deduce results about general $W_{\omega}(\mathbf{b}, \Psi)$. This auxiliary family is defined below using dynamically interesting measures on $\mathcal{A}^{\mathbb{N}}$. As such it is necessary to introduce some definitions describing important properties of these measures.

The cylinder set associated with a finite word $\mathbf{a}=a_{1} \ldots a_{n} \in \mathcal{A}^{*}$ is

$$
[\mathbf{a}]:=\left\{\mathbf{b} \in \mathcal{A}^{\mathbb{N}}: b_{k}=a_{k} \text { for all } 1 \leq k \leq n\right\} .
$$

Let $\sigma: \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}, \sigma\left(a_{1} a_{2} \ldots\right)=a_{2} a_{3} \ldots$ denote the usual left shift map. Given a probability measure $\mathfrak{m}$ supported on $\mathcal{A}^{\mathbb{N}}$, we say that $\mathfrak{m}$ is $\sigma$-invariant if $\mathfrak{m}([\mathbf{a}])=\mathfrak{m}\left(\sigma^{-1}([\mathbf{a}])\right)$ for all finite words $\mathbf{a} \in \mathcal{A}^{*}$. We call a probability measure $\mathfrak{m}$ ergodic if $\sigma^{-1}(A)=A$ implies $\mathfrak{m}(A)=0$ or $\mathfrak{m}(A)=1$. Given a $\sigma$-invariant, ergodic probability measure $\mathfrak{m}$, we define the measure theoretic entropy of $\mathfrak{m}$ to be

$$
h(\mathfrak{m}):=\lim _{k \rightarrow \infty} \frac{-\sum_{\mathbf{a} \in \mathcal{A}^{k}} \mathfrak{m}([\mathbf{a}]) \log \mathfrak{m}([\mathbf{a}])}{k} .
$$

Note that this limit always exists. We say that a probability measure $\mathfrak{m}$ is slowly decaying if

$$
c_{\mathfrak{m}}:=\operatorname{ess} \inf \inf _{k \in \mathbb{N}} \frac{\mathfrak{m}\left(\left[a_{1}, \ldots, a_{k+1}\right]\right)}{\mathfrak{m}\left(\left[a_{1}, \ldots, a_{k}\right]\right)}>0
$$

If $\mathfrak{m}$ is slowly decaying, then clearly for every $\mathbf{a}$ in the support of $\mathfrak{m}$ we have

$$
\frac{\mathfrak{m}\left(\left[a_{1}, \ldots, a_{k+1}\right]\right)}{\mathfrak{m}\left(\left[a_{1}, \ldots, a_{k}\right]\right)} \geq c_{\mathfrak{m}}
$$

for all $k \in \mathbb{N}$. Specific examples of slowly decaying measures include Bernoulli measures, and Gibbs measures for Hölder continuous potentials (see [10]). If $\mathfrak{m}$ is a slowly decaying probability measure with $c_{\mathfrak{m}}$ defined as above, then for each $n \in \mathbb{N}$ we define the level set

$$
\begin{equation*}
L_{\mathfrak{m}, n}:=\left\{\mathbf{a} \in \mathcal{A}^{*}: \mathfrak{m}\left(\left[a_{1}, \ldots, a_{|\mathbf{a}|}\right]\right) \leq c_{\mathfrak{m}}^{n}<\mathfrak{m}\left(\left[a_{1}, \ldots, a_{|\mathbf{a}|-1}\right]\right)\right\} \tag{3.1}
\end{equation*}
$$

The elements of $L_{\mathfrak{m}, n}$ are disjoint and the union of their cylinders has full $\mathfrak{m}$ measure. It follows from the slowly decaying property that cylinders corresponding to elements of $L_{\mathfrak{m}, n}$ have comparable measure up to a multiplicative constant. Note that when $\mathfrak{m}$ is the uniform $\left(\frac{1}{\# \mathcal{A}}, \ldots, \frac{1}{\# \mathcal{A}}\right)$-Bernoulli measure the set $L_{\mathfrak{m}, n}$ is simply $\mathcal{A}^{n}$.

Given $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$, a slowly decaying probability measure $\mathfrak{m}, \omega \in \Omega$, and $g: \mathbb{N} \rightarrow[0, \infty)$, we let

$$
U_{\omega}(\mathbf{b}, \mathfrak{m}, g):=\left\{x \in \mathbb{R}^{d}:\left|x-\Pi_{\omega}(\mathbf{a b})\right| \leq(\mathfrak{m}([\mathbf{a}]) g(n))^{1 / d} \text { for some } \mathbf{a} \in L_{\mathfrak{m}, n} \text { for i.m. } n\right\} .
$$

The sets $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ are the auxiliary sets that will allow us to deduce metric statements about certain $W_{\omega}(\mathbf{b}, \Psi)$ for particular choices of $\Psi$ (see Corollary 3.5 below). The property of those $\Psi$
that allows us to use the sets $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ is described in the following definition. Given a slowly decaying probability measure $\mathfrak{m}$ and $g: \mathbb{N} \rightarrow[0, \infty)$, we say that a function $\Psi$ is equivalent to $(\mathfrak{m}, g)$ if

$$
\Psi(\mathbf{a}) \asymp(\mathfrak{m}([\mathbf{a}]) g(n))^{1 / d}
$$

for all $\mathbf{a} \in L_{\mathfrak{m}, n}$. A natural class of $\Psi$ to consider are those for which $\Psi(\mathbf{a})$ only depends upon the length of $\mathbf{a}$. For such a $\Psi$ if we were to take $\mathfrak{m}$ to be the uniform Bernoulli measure then we can always find a function $g$ such that $\Psi(\mathbf{a})=(\mathfrak{m}([\mathbf{a}]) g(n))^{1 / d}$. As such our definition of equivalent allows us to study this natural class of functions as well as more exotic choices of $\Psi$.

If $\sum_{n=1}^{\infty} g(n)<\infty$ then it can be shown that $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ has zero Lebesgue measure for any choice of $\mathbf{b}, \mathfrak{m}$, and $\omega$. As such, to prove a Khintchine type theorem it is necessary to include a divergence assumption for the function $g$. In our results, the divergence assumption will be that $g$ is an element of $G$ (see (2.2)).

### 3.2 Statement of results

To state our main result we require the following definitions.
Definition 3.1. We say that our random iterated function system is non-singular if there exists $C>0$ such that for all $i \in \mathcal{A}, x \in \cup_{\omega \in \Omega} \Pi_{\omega}\left(\mathcal{A}^{\mathbb{N}}\right)$, and balls $B(y, r)$, we have

$$
\eta_{i}\left(A \in \Omega_{i}: A \cdot x \in B(y, r)\right) \leq C \cdot r^{d} .
$$

Definition 3.2. We say that a random iterated function system is distantly non-singular if there exists $C>0$ such that for all $i \in \mathcal{A}, x \in \bigcup_{\omega \in \Omega} \Pi_{\omega}\left(\mathcal{A}^{\mathbb{N}}\right)$, and $y \in \mathbb{R}^{d} \backslash B\left(0, \frac{\min _{i \neq j}\left|t_{i}-t_{j}\right|}{8}\right)$, we have

$$
\eta_{i}\left(A \in \Omega_{i}: A \cdot x \in B(y, r)\right) \leq C \cdot r^{d} .
$$

Note that the distantly non-singular condition only considers balls that are not "too near" the origin, whereas the non-singular condition considers any ball. Thus, being distantly non-singular is a weaker condition than being non-singular. We will use the distantly non-singular condition when dealing with similarities, and the non-singular condition when dealing with affinities. In the latter case we will use an equivalent formulation that is provided by the following lemma. It allows us to consider more general sets then just balls.

Lemma 3.3. $A$ random iterated function system is non-singular if and only if there exists $C>0$ such that for all $i \in \mathcal{A}, x \in \cup_{\omega \in \Omega} \Pi_{\omega}\left(\mathcal{A}^{\mathbb{N}}\right)$, and Borel set $E \subset \mathbb{R}^{d}$, we have

$$
\eta_{i}\left(A \in \Omega_{i}: A \cdot x \in E\right) \leq C \cdot \operatorname{Vol}(E) .
$$

We omit the proof of Lemma 3.3 which follows from a simple covering argument.
Given a RIFS $\left(\left\{\Omega_{i}\right\}_{i \in \mathcal{A}},\left\{\eta_{i}\right\}_{i \in \mathcal{A}},\left\{t_{i}\right\}_{i \in \mathcal{A}}\right)$ and a probability measure $\mathfrak{m}$ on $\mathcal{A}^{\mathbb{N}}$ we associate the quantities

$$
\lambda^{\prime}\left(\eta_{i}\right):=-\int_{\Omega_{i}} \log (|\operatorname{Det}(A)|) d \eta_{i}(A)
$$

and

$$
\lambda(\eta, \mathfrak{m}):=\sum_{i \in \mathcal{A}} \mathfrak{m}([i]) \cdot \lambda^{\prime}\left(\eta_{i}\right) .
$$

We call $\lambda(\eta, \mathfrak{m})$ the Lyapunov exponent of our RIFS with respect to $\mathfrak{m}$. We will make the running assumption throughout this paper that $\lambda^{\prime}\left(\eta_{i}\right) \in \mathbb{R}$ and that the logarithmic moment condition

$$
\begin{equation*}
\log \int_{\Omega_{i}} \exp (s \log |\operatorname{Det}(A)|) d \eta_{i}(A)=\log \int_{\Omega_{i}}|\operatorname{Det}(A)|^{s} d \eta_{i}(A)<\infty \tag{3.2}
\end{equation*}
$$

is satisfied for all $i \in \mathcal{A}$ and $s \in \mathbb{R}$ with $|s|$ sufficiently small. This assumption is made solely for the purpose of using Cramér's theorem on large deviations in the proof of Theorem 3.4, and other suitable generalisations may be made. In particular, this assumption is trivially satisfied if there exists $c>0$ such that $|\operatorname{Det}(A)| \geq c>0$. We also note that the moment condition directly implies $\lambda^{\prime}\left(\eta_{i}\right) \in \mathbb{R}$.

We are now in a position to state our main result.
Theorem 3.4. Let $\left(\left\{\Omega_{i}\right\}_{i \in \mathcal{A}},\left\{\eta_{i}\right\}_{i \in \mathcal{A}},\left\{t_{i}\right\}_{i \in \mathcal{A}}\right)$ be a RIFS and assume one of the following:
A. Assume $\Omega_{i} \subset S_{d}$ for all $i \in \mathcal{A}$ and that the RIFS is distantly non-singular.
B. Assume $\Omega_{i} \subset M_{d}$ for all $i \in \mathcal{A}$ and that the RIFS is non-singular.

Suppose $\mathfrak{m}$ is a slowly decaying $\sigma$-invariant ergodic probability measure such that $\frac{h(\mathfrak{m})}{\lambda(\eta, \mathfrak{m})}>1$. Then the following statements hold:

1. For any $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$, for $\eta$-almost every $\omega \in \Omega$, for any $g \in G$ the set $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ has positive Lebesgue measure.
2. For any $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$, for $\eta$-almost every $\omega \in \Omega$, for any $\Psi: \mathcal{A}^{*} \rightarrow[0, \infty)$ the set $W_{\omega}(\mathbf{b}, \Psi)$ has positive Lebesgue measure if there exists $g \in G$ such that $\Psi$ is equivalent to ( $\mathfrak{m}, g$ ).

When restricting to Bernoulli probability measures, the second statement from Theorem 3.4 implies the following corollary.

Corollary 3.5. Let $\left(\left\{\Omega_{i}\right\}_{i \in \mathcal{A}},\left\{\eta_{i}\right\}_{i \in \mathcal{A}},\left\{t_{i}\right\}_{i \in \mathcal{A}}\right)$ be an RIFS and assume one of the following.
A. Assume $\Omega_{i} \subset S_{d}$ for all $i \in \mathcal{A}$ and that the RIFS is distantly non-singular.
B. Assume $\Omega_{i} \subset M_{d}$ for all $i \in \mathcal{A}$ and that the RIFS is non-singular.

Let $\left(p_{i}\right)_{i \in \mathcal{A}}$ be a probability vector satisfying $\frac{-\sum_{i \in \mathcal{A}} p_{i} \log p_{i}}{\sum_{i \in \mathcal{A}} p_{i} \cdot \lambda^{\prime}\left(\eta_{i}\right)}>1$. Then for any $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$, for $\eta$-almost every $\omega \in \Omega$, the set

$$
\left\{x \in \mathbb{R}^{d}:\left|x-\Pi_{\omega}(\mathbf{a b})\right| \leq\left(\frac{\prod_{k=1}^{|\mathbf{a}|} p_{a_{k}}}{|\mathbf{a}|}\right)^{1 / d} \text { for i.m. } \mathbf{a} \in \mathcal{A}^{*}\right\}
$$

has positive Lebesgue measure.
By the compactness of $F_{\omega}$ it follows that $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ is a subset of $F_{\omega}$ whenever $\mathfrak{m}([\mathbf{a}]) g(|\mathbf{a}|) \rightarrow$ 0 as $|\mathbf{a}| \rightarrow \infty$. Therefore Theorem 3.4 immediately implies the following result which can be seen to generalise the work of Peres, Simon, and Solomyak [27] for 1-variable RIFS in $\mathbb{R}$ and the work of Koivusalo [23].

Corollary 3.6. Let $\left(\left\{\Omega_{i}\right\}_{i \in \mathcal{A}},\left\{\eta_{i}\right\}_{i \in \mathcal{A}},\left\{t_{i}\right\}_{i \in \mathcal{A}}\right)$ be a RIFS and assume one of the following:
A. Assume $\Omega_{i} \subset S_{d}$ for all $i \in \mathcal{A}$ and that the RIFS is distantly non-singular.
B. Assume $\Omega_{i} \subset M_{d}$ for all $i \in \mathcal{A}$ and that the RIFS is non-singular.

If there exists a slowly decaying $\sigma$-invariant ergodic probability measure $\mathfrak{m}$ satisfying $\frac{h(\mathfrak{m})}{\lambda(\eta, \mathfrak{m})}>1$, then for $\eta$ almost every $\omega \in \Omega$ the set $F_{\omega}$ has positive Lebesgue measure.

The condition $\frac{h(\mathfrak{m})}{\lambda(\eta, \mathfrak{m})}>1$ appearing in the above statements is natural and similar conditions appear throughout the fractal literature. We emphasise that there exist RIFS and $\mathfrak{m}$ for which $\frac{h(\mathfrak{m})}{\lambda(\eta, \mathfrak{m})}<1$ implies that $F_{\omega}$ has zero Lebesgue measure almost surely, and therefore $U_{\omega}(b, \mathfrak{m}, g)$ has zero Lebesgue measure for many natural choices of $g$.

The rest of the paper is organised as follows. In Section 4 we prove several technical results that will enable us to prove Theorem 3.4 in Section 5. In Section 6 we demonstrate how Corollary 3.5 follows from Theorem 3.4. In Section 7 we detail some examples of RIFSs that satisfy either assumption A or assumption B from the statement of our results. Finally in Section 8 we make some concluding remarks.

## 4 Technical results

In this section we prove a number of technical results that will enable us to prove Theorem 3.4. In the first subsection we prove Proposition 4.2. This proposition allows us to assert that for $\eta$ almost every $\omega \in \Omega$, for $n$ sufficiently large there exists a large subset $\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$ contained in $L_{\mathfrak{m}, n}$ for which each element satisfies good determinant bounds and good measure decay bounds that are described in terms of a parameters $\varepsilon_{1}>0$. In the second subsection we prove Lemma 4.3. This lemma provides a good upper bound for the probability that two projections are close to each other. In the final subsection we recall some general results from [5] and [9] which can be used to ensure that a limsup set has positive Lebesgue measure.

### 4.1 Constructing $\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}$

Given a RIFS and a slowly decaying $\sigma$-invariant ergodic probability measure $\mathfrak{m}$, recall that the Lyapunov exponent of our RIFS with respect to $\mathfrak{m}$ is

$$
\lambda(\eta, \mathfrak{m})=\sum_{i \in \mathcal{A}} \mathfrak{m}([i]) \lambda^{\prime}\left(\eta_{i}\right)=-\sum_{i \in \mathcal{A}} \mathfrak{m}([i]) \int_{\Omega_{i}} \log |\operatorname{Det}(A)| d \eta_{i}(A),
$$

and the entropy of $\mathfrak{m}$ is given by

$$
h(\mathfrak{m})=\lim _{n \rightarrow \infty} \frac{-\sum_{\mathbf{a} \in \mathcal{A}^{n}} \mathfrak{m}([\mathbf{a}]) \log \mathfrak{m}([\mathbf{a}])}{n} .
$$

The Shannon-McMillan-Breiman theorem tells us that for $\mathfrak{m}$-almost every $\mathbf{a} \in \mathcal{A}^{\mathbb{N}}$ we have

$$
\lim _{n \rightarrow \infty} \frac{-\log \mathfrak{m}\left(\left[a_{1}, \ldots, a_{n}\right]\right)}{n}=h(\mathfrak{m}) .
$$

We will combine this statement with Egorov's theorem to obtain uniform estimates on the measures of cylinders. The first step in our proof of Proposition 4.2 is the following proposition which states that for $\eta$ almost every $\omega$ there is a large subset of $\mathcal{A}^{\mathbb{N}}$ on which we have good determinant bounds.

Proposition 4.1. Fix a RIFS and a $\sigma$-invariant ergodic probability measure $\mathfrak{m}$. Then for any $\varepsilon_{1}>0$, there exists $C=C\left(\mathfrak{m}, \eta, \varepsilon_{1}\right)>0$ such that for $\eta$ almost every $\omega \in \Omega$, there exists $N=N(\omega) \in \mathbb{N}$ such that ${ }^{3}$

$$
\mathfrak{m}\left(\mathbf{a} \in \mathcal{A}^{\mathbb{N}}:\left|\operatorname{Det}\left(\widehat{A}_{\omega, a_{1} \ldots a_{n}}\right)\right| \in\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C}, C e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right) \text { for all } n \geq N\right)>\frac{13}{16}
$$

[^2]Proof. We fix a RIFS, a $\sigma$-invariant ergodic probability measure $\mathfrak{m}$, and let $\epsilon_{1}>0$. Let $\varepsilon_{2}=$ $\varepsilon_{2}\left(\varepsilon_{1}\right)>0$ be sufficiently small such that

$$
\begin{equation*}
\varepsilon_{2}\left(1+\sum_{i \in \mathcal{A}} \lambda^{\prime}\left(\eta_{i}\right)\right)<\varepsilon_{1} \tag{4.1}
\end{equation*}
$$

By an application of the Birkhoff Ergodic theorem and Egorov's theorem, there exists $C_{1}=$ $C_{1}\left(\mathfrak{m}, \varepsilon_{2}\right)>1$ such that if we let

$$
\Sigma_{\mathfrak{m}}:=\left\{\mathbf{a} \in \mathcal{A}^{\mathbb{N}}: \frac{e^{n\left(\mathfrak{m}([i])-\varepsilon_{2}\right)}}{C_{1}} \leq e^{\left|\left(a_{k}\right)_{k=1}^{n}\right|_{i}} \leq C_{1} e^{n\left(\mathfrak{m}([i])+\varepsilon_{2}\right)} \text { for all } i \in \mathcal{A}, n \in \mathbb{N}\right\}
$$

then

$$
\begin{equation*}
\mathfrak{m}\left(\Sigma_{\mathfrak{m}}\right)>\frac{15}{16} \tag{4.2}
\end{equation*}
$$

For each $n \in \mathbb{N}$, let

$$
\Sigma_{\mathfrak{m}, n}:=\left\{\mathbf{a} \in \mathcal{A}^{n}:\left[a_{1} \ldots a_{n}\right] \cap \Sigma_{\mathfrak{m}} \neq \emptyset\right\}
$$

be the words of length $n$ with "good" digit frequencies.
We split the remainder of our proof into two parts. In the first part we obtain an exponential upper bound for the probability that for a specific $\mathbf{a} \in \Sigma_{\mathfrak{m}, n}$ the determinant of $\widehat{A}_{\omega, \mathbf{a}}$ behaves poorly. In the second part we use this bound to show that for almost every $\omega \in \Omega$, there exists a large subset of $\mathcal{A}^{\mathbb{N}}$ upon which the determinant behaves well.

Part 1: $\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right|$ is regular with high probability. Let us temporarily fix some element $\mathbf{a}=a_{1} \ldots a_{n} \in \Sigma_{\mathfrak{m}, n}$. We want to obtain a good upper bound for the probability that

$$
\left|\operatorname{Det}\left(\widehat{A}_{\omega, a_{1} \ldots a_{n}}\right)\right| \notin\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{2}}, C_{2} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)
$$

for some $C_{2}>0$. Since the determinant is multiplicative, and $A_{\omega, a_{1} \ldots a_{i}}$ is independent of $A_{\omega, a_{1} \ldots a_{j}}$ for $i \neq j$, we can break up the determinant into the $\# \mathcal{A}$ different contributions coming from each of the probability measures $\eta_{i}$. This means that there exists $\# \mathcal{A}$ words $\mathbf{b}_{1}, \ldots, \mathbf{b}_{\# \mathcal{A}}$ consisting solely of the digits $1, \ldots, \# \mathcal{A}$ respectively, such that

$$
\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right| \equiv_{d} \prod_{i \in \mathcal{A}}\left|\operatorname{Det}\left(\widehat{A}_{\mathbf{b}_{i}}\right)\right|
$$

and

$$
\left|\mathbf{b}_{i}\right|_{i}=\left|\mathbf{b}_{i}\right|=\left|\left(a_{1} \ldots a_{n}\right)\right|_{i} \text { for all } i \in \mathcal{A}
$$

Moreover, for each element of the word $\mathbf{b}_{i}$ the corresponding matrix is chosen independently with respect to the probability measure $\eta_{i}$. Therefore it follows from Cramér's theorem on large deviations and our assumption (3.2), that for each $i \in \mathcal{A}$ there exists $\rho_{i}=\rho_{i}\left(\varepsilon_{1}, \eta_{i}\right) \in(0,1)$ and $C_{3}=C_{3}\left(\varepsilon_{1}, \eta_{i}\right)>0$ such that

$$
\eta\left(\omega:\left|\operatorname{Det}\left(\widehat{A}_{\mathbf{b}_{i}}\right)\right| \notin\left(e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)+\varepsilon_{2}\right)}, e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)-\varepsilon_{2}\right)}\right)\right) \leq C_{3} \rho_{i}^{\left|\mathbf{b}_{i}\right|}
$$

Given the finiteness of $\mathcal{A}$ and the fact that $\mathbf{a} \in \Sigma_{\mathfrak{m}, n}$, which implies a lower bound for $\left|\mathbf{b}_{i}\right|$ in terms of a constant times $n$, one can derive a uniform exponential bound in $n$. In particular, there exists $\rho=\rho\left(\varepsilon_{1}, \eta, \mathfrak{m}\right) \in(0,1)$ and $C_{4}=C_{4}\left(\varepsilon_{1}, \eta, \mathfrak{m}\right)$ such that

$$
\eta\left(\omega:\left|\operatorname{Det}\left(\widehat{A}_{\mathbf{b}_{i}}\right)\right| \notin\left(e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)+\varepsilon_{2}\right)}, e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)-\varepsilon_{2}\right)}\right)\right) \leq C_{4} \rho^{n} \text { for all } i \in \mathcal{A} .
$$

For each $\mathbf{a} \in \Sigma_{\mathfrak{m}, n}$ consider the event

$$
E_{\mathbf{a}}=\left\{\omega \in \Omega: \prod_{i \in \mathcal{A}}\left|\operatorname{Det}\left(\widehat{A}_{\mathbf{b}_{i}}\right)\right| \notin\left(\prod_{i \in \mathcal{A}} e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)+\varepsilon_{2}\right)}, \prod_{i \in \mathcal{A}} e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)-\varepsilon_{2}\right)}\right)\right\} .
$$

Clearly, if $\omega \in E_{\mathbf{a}}$ then $\left|\operatorname{Det}\left(\widehat{A}_{\mathbf{b}_{i}}\right)\right| \notin\left(e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)+\varepsilon_{2}\right)}, e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)-\varepsilon_{2}\right)}\right)$ for some $i$ and therefore

$$
\eta\left(E_{\mathbf{a}}\right) \leq \# A \cdot C_{4} \rho^{n} .
$$

Let $C_{2}=C_{2}(\eta)>0$ be such that

$$
\begin{equation*}
C_{2} \geq \max \left\{e^{-\# \mathcal{A} \log C_{1} \sum_{i} \lambda^{\prime}\left(\eta_{i}\right)}, e^{\# \mathcal{A} \log C_{1} \sum_{i} \lambda^{\prime}\left(\eta_{i}\right)}\right\} \tag{4.3}
\end{equation*}
$$

Manipulating the lower bound we obtain that for each $\mathbf{a} \in \Sigma_{\mathfrak{m}, n}$ we have

$$
\begin{align*}
\prod_{i \in \mathcal{A}} e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)+\varepsilon_{2}\right)} & =e^{-n \varepsilon_{2}} \prod_{i \in \mathcal{A}} e^{-|\mathbf{a}|_{i} \lambda^{\prime}\left(\eta_{i}\right)} \\
& \geq e^{-n \varepsilon_{2}} \prod_{i \in \mathcal{A}} e^{-\lambda^{\prime}\left(\eta_{i}\right)\left(n\left(\mathfrak{m}([i])-\varepsilon_{2}\right)-\log C_{1}\right)} \\
& \geq e^{\# \mathcal{A} \log C_{1} \cdot \sum_{i} \lambda^{\prime}\left(\eta_{i}\right)} \exp \left(-n\left(\varepsilon_{2}-\sum_{i \in \mathcal{A}} \lambda^{\prime}\left(\eta_{i}\right)\left(\mathfrak{m}([i])-\varepsilon_{2}\right)\right)\right) \\
& \geq C_{2}^{-1} \exp \left(-n\left(\varepsilon_{2}+\lambda(\eta, \mathfrak{m})+\varepsilon_{2} \sum_{i \in \mathcal{A}} \lambda^{\prime}\left(\eta_{i}\right)\right)\right)  \tag{4.3}\\
& \geq C_{2}^{-1} e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)} \tag{4.1}
\end{align*}
$$

The following upper bound for each $\mathbf{a} \in \Sigma_{\mathfrak{m}, n}$ is proved similarly

$$
\prod_{i=1}^{\# \mathcal{A}} e^{-\left|\mathbf{b}_{i}\right|\left(\lambda^{\prime}\left(\eta_{i}\right)-\varepsilon_{2}\right)} C_{2} \leq e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}
$$

We conclude that for each $\mathbf{a} \in \Sigma_{\mathfrak{m}, n}$ the event

$$
E_{\mathbf{a}}^{\prime}=\left\{\omega \in \Omega:\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right| \notin\left(C_{2}^{-1} e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}, C_{2} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)\right\}
$$

satisfies $E_{\mathbf{a}}^{\prime} \subset E_{\mathbf{a}}$ and so $\eta\left(E_{\mathbf{a}}^{\prime}\right) \leq \eta\left(E_{\mathbf{a}}\right) \leq \# \mathcal{A} C_{4} \rho^{n}$. In summary, we have shown that

$$
\begin{equation*}
\eta\left(\omega \in \Omega:\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right| \notin\left(C_{2}^{-1} e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}, C_{2} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)\right) \leq \# \mathcal{A} C_{4} \rho^{n} \tag{4.4}
\end{equation*}
$$

for all $\mathbf{a} \in \Sigma_{\mathfrak{m}, n}$.
Part 2: Constructing a large subset of $\mathcal{A}^{N}$ on which the determinant is regular. Let $\varepsilon_{3}>0$ and $\theta \in(0,1)$ be such that

$$
\begin{equation*}
\rho \theta^{-1}<1 \text { and } \theta^{\prime}:=e^{2 \varepsilon_{3}} \theta<1 . \tag{4.5}
\end{equation*}
$$

Combining the Shannon-McMillan-Breiman theorem, Egorov's theorem, and (4.2), we may assert that there exists $C_{5}=C_{5}\left(\mathfrak{m}, \varepsilon_{3}\right)>0$ such that if we let

$$
\Sigma_{\mathfrak{m}}^{*}:=\Sigma_{\mathfrak{m}} \cap\left\{\mathbf{a} \in \mathcal{A}^{\mathbb{N}}: \frac{e^{-n\left(h(\mathfrak{m})+\varepsilon_{3}\right)}}{C_{5}} \leq \mathfrak{m}\left(\left[a_{1} \ldots a_{n}\right]\right) \leq C_{5} e^{-n\left(h(\mathfrak{m})-\varepsilon_{3}\right)} \text { for all } n \in \mathbb{N}\right\}
$$

be the set of sequences with "good" digit frequency and "good" measure decay, then

$$
\begin{equation*}
\mathfrak{m}\left(\Sigma_{\mathfrak{m}}^{*}\right)>\frac{14}{16} . \tag{4.6}
\end{equation*}
$$

Again we define the level sets by

$$
\Sigma_{\mathfrak{m}, n}^{*}:=\left\{\mathbf{a} \in \mathcal{A}^{n}:\left[a_{1} \ldots a_{n}\right] \cap \Sigma_{\mathfrak{m}}^{*} \neq \emptyset\right\}
$$

and we note that $\Sigma_{\mathfrak{m}, n}^{*} \subseteq \Sigma_{\mathfrak{m}, n}$ for all $n$. Therefore (4.4) also applies to elements of $\Sigma_{\mathfrak{m}, n}^{*}$.
Using the measure bounds coming from the definition of $\Sigma_{m}^{*}$ we have the following upper bound for the cardinality of $\Sigma_{\mathfrak{m}, n}^{*}$ :

$$
\begin{equation*}
\# \Sigma_{\mathfrak{m}, n}^{*} \leq C_{5} e^{n\left(h(\mathfrak{m})+\varepsilon_{3}\right)} \tag{4.7}
\end{equation*}
$$

We can bound the expected number of words

$$
B_{\mathfrak{m}, n}(\omega):=\left\{\mathbf{a} \in \Sigma_{\mathfrak{m}, n}^{*}:\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right| \notin\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{1}}, C_{1} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)\right\}
$$

that do not have good Lyapunov exponent using (4.4):

$$
\begin{aligned}
\int_{\Omega} \# B_{\mathfrak{m}, n}(\omega) d \eta & =\int_{\Omega} \#\left\{\mathbf{a} \in \Sigma_{\mathfrak{m}, n}^{*}:\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right| \notin\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{2}}, C_{2} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)\right\} d \eta \\
& =\sum_{\mathbf{a} \in \Sigma_{\mathfrak{m}, n}^{*}} \int_{\Omega} \chi\left(\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right| \notin\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{2}}, C_{2} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)\right) d \eta \\
& \leq C_{4} \# \sum_{\mathbf{a} \in \Sigma_{\mathfrak{m}, n}^{*}} \rho^{n} \\
& \leq C_{4} \# \mathcal{A} \cdot \rho^{n} \# \Sigma_{\mathfrak{m}, n}^{*} .
\end{aligned}
$$

By Markov's inequality, we have

$$
\begin{aligned}
\eta\left(\omega: \# B_{\mathfrak{m}, n}(\omega) \geq \# \Sigma_{\mathfrak{m}, n}^{*} \cdot \theta^{n}\right) & \leq C_{4} \# \mathcal{A} \cdot \rho^{n} \# \Sigma_{\mathfrak{m}, n}^{*} \theta^{-n}\left(\# \Sigma_{\mathfrak{m}, n}^{*}\right)^{-1} \\
& \leq C_{4} \# \mathcal{A} \rho^{n} \theta^{-n} .
\end{aligned}
$$

Therefore by (4.5)

$$
\sum_{n \in \mathbb{N}} \eta\left(\omega: \# B_{\mathfrak{m}, n}(\omega) \geq \# \Sigma_{\mathfrak{m}, n}^{*} \cdot \theta^{n}\right)<\infty
$$

It follows from the Borel-Cantelli Lemma that for $\eta$-almost every $\omega \in \Omega$ there exists $N=N(\omega) \in$ $\mathbb{N}$ such that

$$
\begin{equation*}
\# B_{\mathfrak{m}, n}(\omega) \leq \# \Sigma_{\mathfrak{m}, n}^{*} \cdot \theta^{n} \tag{4.8}
\end{equation*}
$$

for all $n \geq N$. It follows now from the definition of $\Sigma_{\mathfrak{m}, n}^{*}$ that for $\eta$ almost every $\omega$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$
\begin{align*}
\mathfrak{m}\left(\bigcup_{a_{1} \ldots a_{n} \in B_{\mathfrak{m}, n}(\omega)}\left[a_{1} \ldots a_{n}\right]\right) & \leq C_{5} e^{\left.-n\left(h(\mathfrak{m})-\varepsilon_{3}\right)\right)} \cdot \# B_{\mathfrak{m}, n}(\omega) \\
& \leq C_{5} e^{\left.-n\left(h(\mathfrak{m})-\varepsilon_{3}\right)\right)} \cdot \# \Sigma_{\mathfrak{m}, n}^{*} \cdot \theta^{n}  \tag{4.8}\\
& \leq\left(C_{5}\right)^{2} e^{\left.-n\left(h(\mathfrak{m})-\varepsilon_{3}\right)\right)} e^{n\left(h(\mathfrak{m})+\varepsilon_{3}\right)} \theta^{n} \tag{4.7}
\end{align*}
$$

$$
\begin{align*}
& \leq\left(C_{5}\right)^{2} e^{2 n \varepsilon_{3}} \theta^{n} \\
& \leq\left(C_{5}\right)^{2}\left(\theta^{\prime}\right)^{n} \tag{4.5}
\end{align*}
$$

Replacing $N$ with some larger value if necessary, we may assume that

$$
\sum_{n=N}^{\infty} \mathfrak{m}\left(\bigcup_{a_{1} \ldots a_{n} \in B_{\mathfrak{m}, n}(\omega)}\left[a_{1} \ldots a_{n}\right]\right) \leq \sum_{n=N}^{\infty}\left(C_{5}\right)^{2}\left(\theta^{\prime}\right)^{n}<1 / 16
$$

holds for $\eta$-almost every $\omega \in \Omega$. Therefore, for $\eta$-almost every $\omega$ we have

$$
\mathfrak{m}\left(\bigcup_{n=N}^{\infty} \bigcup_{a_{1} \ldots a_{n} \in B_{\mathfrak{m}, n}(\omega)}\left[a_{1} \ldots a_{n}\right]\right)<\frac{1}{16}
$$

Combining this inequality with (4.6) we see that for $\eta$-almost every $\omega$, for $N$ sufficiently large we have

$$
\mathfrak{m}\left(\Sigma_{\mathfrak{m}}^{*} \backslash \bigcup_{n=N}^{\infty} \bigcup_{a_{1} \ldots a_{n} \in B_{\mathfrak{m}, n}(\omega)}\left[a_{1} \ldots a_{n}\right]\right)>\frac{13}{16}
$$

Finally, we observe that if

$$
\mathbf{a} \in \Sigma_{\mathfrak{m}}^{*} \backslash \bigcup_{n=N}^{\infty} \bigcup_{a_{1} \ldots a_{n} \in B_{\mathfrak{m}, n}(\omega)}\left[a_{1}, \ldots, a_{n}\right]
$$

then a satisfies

$$
\left|\operatorname{Det}\left(\widehat{A}_{\omega, a_{1}, \ldots, a_{n}}\right)\right| \in\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{1}}, C_{1} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)
$$

for all $n \geq N$. This completes our proof.
We now adapt Proposition 4.1 into a meaningful statement regarding the level sets $L_{\mathfrak{m}, n}$. Instead of dealing with $L_{\mathfrak{m}, n}$ directly it is useful to restrict to the following large subset upon which we have strong measure decay estimates. For any slowly decaying $\sigma$-invariant ergodic probability measure $\mathfrak{m}$ and $\varepsilon_{1}>0$, we can use the Shannon-McMillan Breiman theorem and Egorov's theorem to choose $C_{2}\left(\mathfrak{m}, \varepsilon_{1}\right)>0$ such that the set $L_{\mathfrak{m}, n, \varepsilon_{1}} \subseteq L_{\mathfrak{m}, n}$ defined as follows

$$
\begin{equation*}
L_{\mathfrak{m}, n, \varepsilon_{1}}:=\left\{\mathbf{a} \in L_{\mathfrak{m}, n}: \frac{e^{-k\left(h(\mathfrak{m})+\varepsilon_{1}\right)}}{C_{2}} \leq \mathfrak{m}\left(\left[a_{1} \ldots a_{k}\right]\right) \leq C_{2} e^{-k\left(h(\mathfrak{m})-\varepsilon_{1}\right)} \text { for all } 1 \leq k \leq|\mathbf{a}|\right\} \tag{4.9}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
\mathfrak{m}\left(\bigcup_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}}[\mathbf{a}]\right)>15 / 16 \tag{4.10}
\end{equation*}
$$

Proposition 4.2. Fix a RIFS and a slowly decaying $\sigma$-invariant ergodic probability measure $\mathfrak{m}$. Then for any $\varepsilon_{1}>0$, there exists $C=C\left(\mathfrak{m}, \eta, \varepsilon_{1}\right)>0$ such that for almost every $\omega \in \Omega$, there exists $N_{1}=N_{1}(\omega) \in \mathbb{N}$ and $N_{2}=N_{2}(\omega) \in \mathbb{N}$ such that for all $n \geq N_{2}$ there exists $\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \subseteq L_{\mathfrak{m}, n, \varepsilon_{1}}$ satisfying:

1. For each $\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$ we have

$$
\left|\operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a}}\right)\right| \in\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C}, C e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)
$$

for all $N_{1} \leq n \leq|\mathbf{a}|$.
2. $\# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}} \asymp c_{\mathrm{m}}^{-n}$ for all $n \geq N_{2}$.

Proof. Let $\omega$ belong to the full measure set whose existence is asserted by Proposition 4.1. Let $N_{1}=N_{1}(\omega)$ denote the large $N$ whose existence is also guaranteed by this proposition. Since $\mathfrak{m}$ is non atomic, we may choose $N_{2}=N_{2}(\omega)$ sufficiently large such that for all $n \geq N_{2}$, each $\mathbf{a} \in L_{\mathfrak{m}, n}$ satisfies $|\mathbf{a}| \geq N_{1}$.

By Proposition 4.1 we have $\mathfrak{m}(H(\omega))>13 / 16$, where

$$
H(\omega):=\left\{\mathbf{a} \in \mathcal{A}^{\mathbb{N}}:\left|\operatorname{Det}\left(\widehat{A}_{\omega, a_{1} \ldots a_{n}}\right)\right| \in\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{1}}, C_{1} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right) \text { for all } n \geq N_{1}\right\}
$$

and $C_{1}>0$ is the constant guaranteed by Proposition 4.1.
For $n \geq N_{2}$ define

$$
\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):=\left\{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}:[\mathbf{a}] \cap H(\omega) \neq \emptyset\right\} .
$$

Notice that Property 1. is immediately satisfied by $\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$. To see that Property 2 . holds notice that $\mathfrak{m}\left(H(\omega) \cap \cup_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}}[a]\right)>12 / 16$ follows from from the bounds $\mathfrak{m}\left(\cup_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}}[a]\right)>$ $15 / 16$ and $\mathfrak{m}(H(\omega))>13 / 16$. Our cardinality bound now follows because $\mathfrak{m}([\mathbf{a}]) \asymp c_{\mathfrak{m}}^{n}$ for each $\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$.

### 4.2 Transversality estimates

To prove Theorem 3.4 we need the following transversality lemma that bounds the probability that two points in the attractor are close. This is the only part in the proof where we use our non-singularity assumptions.

Lemma 4.3. Let $\left(\left\{\Omega_{i}\right\}_{i \in \mathcal{A}},\left\{\eta_{i}\right\}_{i \in \mathcal{A}},\left\{t_{i}\right\}_{i \in \mathcal{A}}\right)$ be a RIFS and assume one of the following:
A. Assume that $\Omega_{i} \in S_{d}$ for all $i \in \mathcal{A}$ and that the RIFS is distantly non-singular;
B. Assume that $\Omega_{i} \in M_{d}$ for all $i \in \mathcal{A}$ and the RIFS is non-singular.

Let $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$ and $\mathbf{a}, \mathbf{a}^{\prime} \in \mathcal{A}^{*}$ be two distinct words such that neither one is the prefix of the other. Then for any $C>0$ and $s>0$ and all $0<\varepsilon<s$,

$$
\begin{align*}
& \int_{\Omega} \chi_{[0, r]}\left(\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|\right) \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1} \ldots a_{n}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right) \text { for all } 1 \leq n \leq|\mathbf{a}|\right) \\
& \quad \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1}^{\prime} \ldots a_{n}^{\prime}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right) \text { for all } 1 \leq n \leq\left|\mathbf{a}^{\prime}\right|\right) d \eta \\
& =\mathcal{O}\left(r^{d} \cdot C \cdot e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)}\right) . \tag{4.11}
\end{align*}
$$

Proof. We split our proof into two parts.
Proof under assumption A. First, assume that $\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right| \geq 1$, i.e. that $\mathbf{a}$ and $\mathbf{a}^{\prime}$ share a common prefix. Note that by assumption we also have $\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|<\min \left\{|\mathbf{a}|,\left|\mathbf{a}^{\prime}\right|\right\}$. Let $\mathbf{c}$ and $\mathbf{c}^{\prime}$ be the unique words such that $\mathbf{a b}=\left(\mathbf{a} \wedge \mathbf{a}^{\prime}\right) \mathbf{c b}$ and $\mathbf{a}^{\prime} \mathbf{b}=\left(\mathbf{a} \wedge \mathbf{a}^{\prime}\right) \mathbf{c}^{\prime} \mathbf{b}$. We emphasise that $\mathbf{c b}$ and $\mathbf{c}^{\prime} \mathbf{b}$ must have distinct first letter. We highlight the following inequality

$$
\chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1} \ldots a_{n}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right), \forall 1 \leq n \leq|\mathbf{a}|\right)
$$

$$
\begin{aligned}
& \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1}^{\prime} \ldots a_{n}^{\prime}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right), \forall 1 \leq n \leq\left|\mathbf{a}^{\prime}\right|\right) \\
& \leq \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\left.\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}\right)}\right) \in\left(\frac{e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)}}{C}, C e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s-\varepsilon)}\right)\right)
\end{aligned}
$$

This implies

$$
\begin{align*}
\chi_{[0, r]}\left(\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|\right) \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1} \ldots a_{n}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right), \forall 1 \leq n \leq|\mathbf{a}|\right) \\
\cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1}^{\prime} \ldots a_{n}^{\prime}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right), \forall 1 \leq n \leq\left|\mathbf{a}^{\prime}\right|\right) \\
\leq \chi\left(\omega: \Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right) \in B(0, r)\right) \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\right) \in\left(\frac{e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)}}{C}, C e^{-\left|\mathbf{a} \wedge \wedge^{\prime}\right|(s-\varepsilon)}\right)\right) \\
=\chi\left(\omega: \Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}(\mathbf{c b})-\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right) \in\left(\widehat{A}_{\left.\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}\right)^{-1}}(B(0, r))\right)\right. \\
\cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\right) \in\left(\frac{e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)}}{C}, C e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s-\varepsilon)}\right)\right) . \tag{4.12}
\end{align*}
$$

 (4.12) can be bounded above by

$$
\begin{equation*}
\chi_{1}:=\chi\left(\omega: \Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}(\mathbf{c b})-\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right) \in B\left(0, r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}\right)\right) . \tag{4.13}
\end{equation*}
$$

We remark that the iterative definition of the random maps give the identity

$$
\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}(\mathbf{c b})=A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)+t_{c_{1}} .
$$

Write $r^{*}=\min _{i \neq j}\left|t_{i}-t_{j}\right|$. Since $\mathbf{c b}$ and $\mathbf{c}^{\prime} \mathbf{b}$ differ in their first letter we have $\left|t_{c_{1}}-t_{c_{1}^{\prime}}\right| \geq r^{*}$. Note that

$$
\begin{aligned}
1 & =\chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right|<r^{*} / 4\right) \cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right)\right|<r^{*} / 4\right) \\
& +\chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right| \geq r^{*} / 4\right) \cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right)\right|<r^{*} / 4\right) \\
& +\chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right|<r^{*} / 4\right) \cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right)\right| \geq r^{*} / 4\right) \\
& +\chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right| \geq r^{*} / 4\right) \cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right)\right| \geq r^{*} / 4\right) \\
& =: \chi<\cdot \chi_{<}^{\prime}+\chi \geq \cdot \chi_{<}^{\prime}+\chi<\chi_{\geq}^{\prime}+\chi \geq \cdot \chi_{\geq}^{\prime} .
\end{aligned}
$$

We use this identity to split write (4.13) as four summands

$$
\chi_{1}=\chi_{1} \cdot \chi_{<} \cdot \chi_{<}^{\prime}+\chi_{1} \cdot \chi_{\geq} \cdot \chi_{<}^{\prime}+\chi_{1} \cdot \chi_{<} \cdot \chi_{\geq}^{\prime}+\chi_{1} \cdot \chi_{\geq} \cdot \chi_{\geq}^{\prime} .
$$

The first of these summands is

$$
\begin{aligned}
\chi_{1} \cdot \chi_{<} \cdot \chi_{<}^{\prime}=\chi\left(\omega: \Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}(\mathbf{c b})-\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right) \in B\left(0, r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}\right)\right) \\
\cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right|<r^{*} / 4\right) \\
\cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right)\right|<r^{*} / 4\right)
\end{aligned}
$$

Since we are interested in asymptotic behaviour with respect to $C r^{d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)} \rightarrow 0$, we can without loss of generality, assume that $r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}<r^{*} / 8$. In which case we have $\chi_{1} \cdot \chi_{<} \cdot \chi_{<}^{\prime}=0$. This is because, if $\chi_{1} \cdot \chi_{<} \cdot \chi_{<}^{\prime}=1$ then we would have

$$
\begin{aligned}
r^{*} / 2 & <\left|t_{c_{1}}-t_{c_{1}^{\prime}}\right|-\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)-A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right)\right| \\
& \leq\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)+t_{c_{1}}-A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}^{\prime}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right)-t_{c_{1}^{\prime}}\right| \\
& =\left|\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}(\mathbf{c b})-\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right|<r^{*} / 8,
\end{aligned}
$$

which is not possible.
Summarising the above, we have shown that the left hand side of inequality (4.11) satisfies

$$
\begin{aligned}
& \int_{\Omega} \chi_{[0, r]}\left(\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|\right) \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1}, \ldots, a_{n}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right), \forall 1 \leq n \leq|\mathbf{a}|\right) \\
& \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, a_{1}^{\prime}, \ldots, a_{n}^{\prime}}\right) \in\left(\frac{e^{-n(s+\varepsilon)}}{C}, C e^{-n(s-\varepsilon)}\right), \forall 1 \leq n \leq\left|\mathbf{a}^{\prime}\right|\right) d \eta \\
& \leq \int_{\Omega}\left(\chi_{1} \cdot \chi_{\geq} \cdot \chi_{<}^{\prime}+\chi_{1} \cdot \chi_{<} \cdot \chi_{\geq}^{\prime}+\chi_{1} \cdot \chi \geq \cdot \chi_{\geq}^{\prime}\right) d \eta+\mathcal{O}\left(r^{d} \cdot C \cdot e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)}\right)
\end{aligned}
$$

It remains to appropriately bound the above integral. Manipulating this integral we have

$$
\begin{aligned}
& \int_{\Omega}\left(\chi_{1} \cdot \chi_{\geq} \cdot \chi_{<}^{\prime}+\chi_{1} \cdot \chi_{<} \cdot \chi_{\geq}^{\prime}+\chi_{1} \cdot \chi_{\geq} \cdot \chi_{\geq}^{\prime}\right) d \eta . \\
& =\int_{\Omega}\left(2 \chi_{1} \cdot \chi_{\geq} \cdot \chi_{<}^{\prime}+\chi_{1} \cdot \chi_{\geq} \cdot \chi_{\geq}^{\prime}\right) d \eta \quad \quad \text { (By symmetry) } \\
& \leq 2 \int_{\Omega}\left(\chi_{1} \cdot \chi_{\geq}\right) d \eta \\
& =2 \int_{\prod_{\mathbf{d} \in \mathcal{A}^{*}} \Omega_{l(\mathbf{d})}}\left(\chi_{1} \cdot \chi_{\geq}\right) d \prod_{\mathbf{d} \in \mathcal{A}^{*}} \eta_{\mathbf{d}} . \\
& =2 \int_{\prod_{\mathbf{d} \in \mathcal{A}^{*}} \Omega_{l(\mathbf{d})}} \chi\left(\omega: \Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}(\mathbf{c b})-\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right) \in B\left(0, r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}\right)\right) \\
& \cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right| \geq r^{*} / 4\right) d \prod_{\mathbf{d} \in \mathcal{A}^{*}} \eta_{\mathbf{d}} . \\
& =2 \int_{\prod_{\mathbf{d} \in \mathcal{A}^{*}} \Omega_{l(\mathbf{d})}} \chi\left(\omega: A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right) \in B\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-t_{c_{1}}, r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}\right)\right) \\
& \cdot \chi\left(\omega:\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right| \geq r^{*} / 4\right) d \prod_{\mathbf{d} \in \mathcal{A}^{*}} \eta_{\mathbf{d}} .
\end{aligned}
$$

As stated above, there is no loss of generality in assuming that $r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}<r^{*} / 8$. Therefore, if $\left|A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right)\right| \geq r^{*} / 4$ and $A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right) \in B\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-\right.$ $\left.\left.t_{c_{1}}, r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}\right)\right)$ then we must have $\left|\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-t_{c_{1}}\right| \geq r^{*} / 8$. Using this fact and then Fubini's theorem, we may bound the integral above by

$$
\begin{aligned}
2 \int_{\prod_{\mathbf{d} \in \mathcal{A}^{*} \backslash\left\{\mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}\right\}} \Omega_{l(\mathbf{d})}} \int_{\Omega_{c_{1}}} \chi\left(\omega: A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right) \in B\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-t_{c_{1}}, r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}\right)\right) \\
\cdot \chi\left(\omega:\left|\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-t_{c_{1}}\right| \geq r^{*} / 8\right) \quad d \eta_{\mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}} d \prod_{\mathbf{d} \in \mathcal{A}^{*}} \eta_{\mathbf{d}}
\end{aligned}
$$

Notice that the inner integrand is the probability that the image of a certain point under the linear part of the map associated with $\mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}$ lies in a certain ball away from the origin. By
our distantly non-singular assumption, this probability is bounded above by a constant times the volume of the ball. Therefore we obtain the upper bound

$$
\begin{aligned}
& 2 \int_{\prod_{\mathbf{d} \in \mathcal{A}^{*} \backslash\left\{\mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}\right\}} \Omega_{l(\mathbf{d})}} \int_{\Omega_{c_{1}}} \chi\left(\omega: A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\sigma\left(\mathbf{c}^{\prime} \mathbf{b}\right)\right)\right) \in B\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-t_{c_{1}}, r \cdot C^{1 / d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon) / d}\right)\right) \\
& \cdot \chi\left(\omega:\left|\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-t_{c_{1}}\right| \geq r^{*} / 8\right) \quad d \eta_{\mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}} d \prod_{\substack{\mathbf{d} \in \mathcal{A}^{*} \\
\mathbf{d} \neq \mathbf{a}^{\prime} \mathbf{a}_{1}}} \eta_{\mathbf{d}} \\
& \leq 2 \int_{\prod_{\mathbf{d} \in \mathcal{A}^{*} \backslash\left\{\mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}\right\}} \Omega_{l(\mathbf{d})}} C^{\prime} \cdot r^{d} \cdot C e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)} d \prod_{\mathbf{d} \in \mathcal{A}^{*}} \eta_{\mathbf{d}} \\
& \leq 2 C^{\prime} C r^{d} e^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)},
\end{aligned}
$$

where $C^{\prime}>0$ is the constant given by the distantly non-singular condition. This shows the correct upper bound when $\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right| \geq 1$. The proof where $\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|=0$ follows along similar lines, noting that this implies the first letters of $\mathbf{a}$ and $\mathbf{a}^{\prime}$ differ and we can directly apply the distantly non-singular condition.

## Proof under assumption B.

The proof under assumption B is similar to the proof under assumption A. However, since we are no longer dealing with similarities, the set $\left(\widehat{A}_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\right)^{-1}(B(0, r))$ in (4.12) is not a ball but rather an ellipse. Writing $E_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}=\left(\widehat{A}_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\right)^{-1}(B(0, r))$ we obtain

$$
\begin{aligned}
& \int_{\Omega} \chi\left(\omega: \Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}(\mathbf{c b})-\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right) \in E_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\right) \\
& \cdot \\
& \quad \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\right) \in\left(\frac{e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)}}{C}, C e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s-\varepsilon)}\right)\right) d \eta \\
& =\int_{\Omega} \chi\left(\omega: A_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}\left(\Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime} c_{1}}(\sigma(\mathbf{c b}))\right) \in \Pi_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\left(\mathbf{c}^{\prime} \mathbf{b}\right)-t_{c_{1}}+E_{\left.\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}\right)}\right. \\
& \\
& \cdot \chi\left(\omega: \operatorname{Det}\left(\widehat{A}_{\omega, \mathbf{a} \wedge \mathbf{a}^{\prime}}\right) \in\left(\frac{e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s+\varepsilon)}}{C}, C e^{-\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|(s-\varepsilon)}\right)\right) d \eta
\end{aligned}
$$

as an upper bound for the left hand side of (4.11). The proof then follows by an analogous argument where we appeal to Fubini's theorem, the conditions imposed by the second characteristic function, and Lemma 3.3 which allows us to handle the more general case of ellipses instead of balls. It is a consequence of our stronger assumption that the RIFS in non-singular that we do not need to include the initial conditioning argument that was necessary under assumption $A$.

### 4.3 General results

To prove Theorem 3.4 we will use the following results from [5] and [9].
Given $r>0$, we say that a set $Y \subset \mathbb{R}^{d}$ is an $r$-separated set if $\left|z-z^{\prime}\right|>r$ for all distinct $z, z^{\prime} \in Y$. Given a finite $Y \subset \mathbb{R}^{d}$ and $r>0$ we let

$$
T(Y, r):=\sup \left\{\# Y^{\prime}: Y^{\prime} \subset Y \text { and } Y^{\prime} \text { is an } r \text {-separated set }\right\} .
$$

Now suppose that we have a metric space $\Omega$ and $\widetilde{X}$ is some compact subset of $\mathbb{R}^{d}$. Suppose that for each $n \in \mathbb{N}$ there exists a finite set of functions $\left\{f_{l, n}: \Omega \rightarrow \widetilde{X}\right\}_{l=1}^{R_{n}}$. For each $\omega \in \Omega$ we let

$$
Y_{n}(\omega):=\left\{f_{l, n}(\omega)\right\}_{l=1}^{R_{n}}
$$

Moreover, given $c>0, s>0$, and $n \in \mathbb{N}$, we let

$$
B(c, s, n):=\left\{\omega \in \Omega: \frac{T\left(Y_{n}(\omega), \frac{s}{R_{n}^{1 / d}}\right)}{R_{n}}>c\right\} .
$$

The following proposition was proved in [5].
Proposition 4.4. Let $\omega \in \Omega$ and $g: \mathbb{N} \rightarrow[0, \infty)$. Assume that the following properties are satisfied:

- There exists $\gamma>1$ such that

$$
R_{n} \asymp \gamma^{n} .
$$

- There exists $c>0$ and $s>0$ such that

$$
\sum_{\substack{n \in \mathbb{N} \\ \omega \in B(c, s, n)}} g(n)=\infty
$$

Then

$$
\left\{x \in \mathbb{R}^{d}:\left|x-f_{l, n}(\omega)\right| \leq\left(\frac{g(n)}{R_{n}}\right)^{1 / d} \text { for i.m. }(l, n) \in\left\{1, \ldots, R_{n}\right\} \times \mathbb{N}\right\}
$$

has positive Lebesgue measure.
We will also use the following lemma which follows from Lemma 1 of [9].
Lemma 4.5. Let $\left(x_{j}\right)$ be a sequence of points in $\mathbb{R}^{d}$ and $\left(r_{j}\right),\left(r_{j}^{\prime}\right)$ be two sequences of positive real numbers both converging to zero. If $r_{j} \asymp r_{j}^{\prime}$ then

$$
\mathcal{L}\left(x: x \in B\left(x_{j}, r_{j}\right) \text { for i.m. } j\right)=\mathcal{L}\left(x: x \in B\left(x_{j}, r_{j}^{\prime}\right) \text { for i.m. } j\right) .
$$

## 5 Proof of Theorem 3.4

In this section we will prove Theorem 3.4. We begin by remarking that it is a consequence of Lemma 4.5 that Statement 2. follows from Statement 1. To prove Theorem 3.4 it therefore suffices to prove Statement 1.

For the rest of this section we fix a RIFS satisfying either assumption A or assumption B, we fix $\mathfrak{m}$ a slowly decaying $\sigma$-invariant ergodic probability measure such that $\frac{h(\mathfrak{m})}{\lambda(\eta, \mathfrak{m})}>1$, and $\mathbf{b} \in \mathcal{A}^{\mathbb{N}}$ is fixed. We also let $\varepsilon_{1}>0$ be sufficiently small such that

$$
\begin{equation*}
h(\mathfrak{m})-\lambda(\mathfrak{m}, \eta)-2 \varepsilon_{1}>0 \tag{5.1}
\end{equation*}
$$

Such an $\varepsilon_{1}>0$ must exist because of our assumption that $\frac{h(\mathfrak{m})}{\lambda(\mathfrak{m}, \eta)}>1$.
Let us fix a parameter $0<\varepsilon_{0}<1 / 2$. By Proposition 4.2 we may fix a large $N_{1}^{\prime}=N_{1}^{\prime}(\eta) \in \mathbb{N}$ and $N_{2}^{\prime}=N_{2}^{\prime}(\eta) \in \mathbb{N}$ such that for a set of $\omega$ with $\eta$-measure at least $1-\varepsilon_{0}$, there exists $\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \subseteq L_{\mathfrak{m}, n, \varepsilon_{1}}$ for each $n \geq N_{2}^{\prime}$ that satisfies the following:

1. For each $\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$ we have

$$
\left|\operatorname{Det}\left(\widehat{A}_{\omega, a_{1}, \ldots, a_{n}}\right)\right| \in\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{1}}, C_{1} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)
$$

for all $N_{1}^{\prime} \leq n \leq|\mathbf{a}|$.
2. $\# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \asymp c_{\mathfrak{m}}^{-n}$.

We denote the set of $\omega$ for which these properties hold by $\Omega^{\prime}$. By construction $\eta\left(\Omega^{\prime}\right) \geq 1-\varepsilon_{0}$. We will show that the conclusion of Statement 1 . from Theorem 3.4 is satisfied by almost every element of $\Omega^{\prime}$. Since $\varepsilon_{0}$ is arbitrary this will complete our proof.

Note that since $N_{1}^{\prime}$ only depends upon $\eta$ we can in fact strengthen Property 1 . on the set $\Omega^{\prime}$. By letting $C_{1}$ depend upon $\Omega^{\prime}$, we can replace Property 1 with the following stronger statement:
3. For each $\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$ we have

$$
\left|\operatorname{Det}\left(\widehat{A}_{\omega, a_{1}, \ldots, a_{n}}\right)\right| \in\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{1}}, C_{1} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right)
$$

for all $1 \leq n \leq|\mathbf{a}|$.
The following proposition tells us that for a typical $\omega \in \Omega^{\prime}$ there are not too many ( $\mathbf{a}, \mathbf{a}^{\prime}$ ) $\in$ $\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \times \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$ for which $\Pi_{\omega}(\mathbf{a b})$ and $\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)$ are close. The proof is based upon arguments given in [5], which in turn are an appropriate adaptation of arguments due to [7] and [28].

Proposition 5.1. Let $\Omega^{\prime}$ be as above. For any $s>0$ and $n \geq N_{2}^{\prime}$ we have

$$
\int_{\Omega^{\prime}} \frac{\#\left\{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \times \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right| \leq \frac{s}{\# L_{\mathbf{m}, n}^{1 / d}}, \mathbf{a} \neq \mathbf{a}^{\prime}\right\}}{\# L_{\mathfrak{m}, n}} d \eta=\mathcal{O}\left(s^{d}\right)
$$

Proof. We begin by observing that

$$
\begin{aligned}
& \#\left\{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \times \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right| \leq \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}, \mathbf{a} \neq \mathbf{a}^{\prime}\right\} \\
= & \sum_{\substack{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in \tilde{L}_{\mathbf{m}}, n, \varepsilon_{1}(\omega) \times \tilde{L}_{\mathbf{m}}, n, \varepsilon_{1}(\omega) \\
\mathbf{a} \neq \mathbf{a}^{\prime}}} \chi_{\left[0, \frac{s}{\# L_{\mathbf{m}, n}^{1 / d}}\right]}\left(\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|\right) .
\end{aligned}
$$

By Property 3. above we know that for any $\omega \in \Omega^{\prime}$ each $\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$ satisfies

$$
\begin{equation*}
\left|\operatorname{Det}\left(\widehat{A}_{\omega, a_{1}, \ldots, a_{n}}\right)\right| \in\left(\frac{e^{-n\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{C_{1}}, C_{1} e^{-n\left(\lambda(\eta, \mathfrak{m})-\varepsilon_{1}\right)}\right) \tag{5.2}
\end{equation*}
$$

for all $1 \leq n \leq|\mathbf{a}|$. Therefore we have the following upper bound for our counting function

$$
\begin{aligned}
& \#\left\{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \times \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right| \leq \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}, \mathbf{a} \neq \mathbf{a}^{\prime}\right\} \\
& \leq \sum_{\substack{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in L_{\mathbf{m}, n, \varepsilon_{1}, L_{m}}^{\mathbf{a} \neq \mathbf{a}^{\prime}}}} \chi_{\left[0, n, \frac{s}{} \frac{s}{\left.\# L_{\mathbf{m}, n}^{1 /]}\right]}\right.}\left(\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|\right) \cdot \chi(\omega: \mathbf{a} \text { satisfies }(5.2)) \\
& \cdot \chi\left(\omega: \mathbf{a}^{\prime} \text { satisfies }(5.2)\right)
\end{aligned}
$$

Recall that $L_{\mathfrak{m}, n, \varepsilon_{1}}$ was defined in (4.9). Notice that we are now summing over all pairs in $L_{\mathfrak{m}, n, \varepsilon_{1}} \times L_{\mathfrak{m}, n, \varepsilon_{1}}$ such that $\mathbf{a} \neq \mathbf{a}^{\prime}$. In particular the terms in this sum no longer depend upon $\omega$. Since $\mathfrak{m}([\mathbf{a}]) \asymp \# L_{\mathfrak{m}, n}^{-1}$ for each $\mathbf{a} \in L_{\mathfrak{m}, n}$ we have

$$
\int_{\Omega^{\prime}} \frac{\#\left\{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \times \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right| \leq \frac{s}{\# L_{\mathfrak{m}, n}^{1, d}}, \mathbf{a} \neq \mathbf{a}^{\prime}\right\}}{\# L_{\mathfrak{m}, n}} d \eta
$$

$$
\begin{aligned}
& \ll \# L_{\mathfrak{m}, n} \sum_{\substack{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in L_{\mathfrak{m}, n, n, \varepsilon_{1}} \times L_{\mathfrak{m}, n, n, \varepsilon_{1}}^{\mathbf{a} \neq \mathbf{a}^{\prime}}}} \mathfrak{m}([\mathbf{a}]) \mathfrak{m}\left(\left[\mathbf{a}^{\prime}\right]\right) \int_{\Omega^{\prime}} \chi_{\left[0, \frac{s}{\left.\# L_{\mathbf{m}, n}^{1 /]}\right]}\right.}\left(\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|\right) \\
& \cdot \chi\left(\omega \text { : a satisfies (5.2)) } \cdot \chi\left(\omega: \mathbf{a}^{\prime} \text { satisfies (5.2) }\right) d \eta\right. \\
& \ll \# L_{\mathfrak{m}, n} \sum_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}} \sum_{\substack{ \\
k=0}}^{|\mathbf{a}|} \sum_{\substack{\mathbf{a}^{\prime} \in L_{\mathfrak{m}}, n, \varepsilon_{1} \\
\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|=k}} \mathfrak{m}([\mathbf{a}]) \mathfrak{m}\left(\left[\mathbf{a}^{\prime}\right]\right) \int_{\Omega} \chi_{\left[0, \frac{s}{\left.\# L_{\mathbf{m}, n}^{1 / d}\right]}\right.}\left(\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|\right) \\
& \text { - } \chi\left(\omega \text { : a satisfies (5.2)) } \chi \chi\left(\omega: \mathbf{a}^{\prime} \text { satisfies (5.2) }\right) d \eta\right. \text {. }
\end{aligned}
$$

The integrals appearing in the sum above are in a form where we can apply Lemma 4.3. Applying Lemma 4.3 and the definition of $L_{\mathfrak{m}, n, \varepsilon_{1}}$, we see that we can bound the above by

$$
\begin{aligned}
& C \# L_{\mathfrak{m}, n} \sum_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}} \sum_{\substack{k=0}}^{|\mathbf{a}|} \sum_{\substack{\mathbf{a}^{\prime} \in L_{\mathfrak{m}, n, \varepsilon_{1}}^{\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right|=k}}} \mathfrak{m}([\mathbf{a}]) \mathfrak{m}\left(\left[\mathbf{a}^{\prime}\right]\right) \frac{s^{d} C_{1} e^{k\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)}}{\# L_{\mathfrak{m}, n}} \\
& \ll s^{d} \sum_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}} \mathfrak{m}([\mathbf{a}]) \sum_{\substack{|\mathbf{a}|}} \sum_{\substack{\mathbf{a}^{\prime} \in L_{\mathfrak{m}}, n, \varepsilon_{1} \\
\left|\mathbf{a} \wedge \mathbf{a}^{\prime}\right| \mid=k}} \mathfrak{m}\left(\left[\mathbf{a}^{\prime}\right]\right) e^{k\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)} \\
& \ll s^{d} \sum_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}} \mathfrak{m}([\mathbf{a}]) \sum_{k=0}^{|\mathbf{a}|} \mathfrak{m}\left(\left[a_{1} \ldots a_{k}\right]\right) e^{k\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)} \\
& \ll s^{d} \sum_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}} \mathfrak{m}([\mathbf{a}]) \sum_{k=0}^{|\mathbf{a}|} e^{-k\left(h(\mathfrak{m})-\varepsilon_{1}\right)} e^{k\left(\lambda(\eta, \mathfrak{m})+\varepsilon_{1}\right)} \\
& \ll s^{d} \sum_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}} \mathfrak{m}([\mathbf{a}]) \sum_{k=0}^{|\mathbf{a}|} e^{-k\left(h(\mathfrak{m})-\lambda(\eta, \mathfrak{m})-2 \varepsilon_{1}\right)} \\
& \ll s^{d} \sum_{\mathbf{a} \in L_{\mathfrak{m}, n, \varepsilon_{1}}} \mathfrak{m}([\mathbf{a}]) \\
& \ll s^{d .}
\end{aligned}
$$

In the penultimate line we used that $\sum_{k=0}^{\infty} e^{-k\left(h(\mathfrak{m})-\lambda(\eta, \mathfrak{m})-2 \varepsilon_{1}\right)}<\infty$. This is a consequence of the definition of $\varepsilon_{1}$. Since all constants are universal, the proof follows.

We now show how Proposition 5.1 can be used to construct a large separated subset of projections for a large set of $n$ for almost every $\omega \in \Omega^{\prime}$.

For each $n \in \mathbb{N}$ and $\omega \in \Omega^{\prime}$ we let

$$
Y_{n}(\omega):=\left\{\Pi_{\omega}(\mathbf{a b})\right\}_{\mathbf{a} \in \widetilde{L}_{\mathbf{m}, n, \varepsilon_{1}}(\omega)} .
$$

Moreover, given $s>0, \omega \in \Omega^{\prime}$, and $n \geq N_{2}^{\prime}$ we let

$$
\mathrm{CP}(s, \omega, n):=\left\{\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \times \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right| \leq \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}, \mathbf{a} \neq \mathbf{a}^{\prime}\right\} .
$$

Recall that $T(Y, r)$ is the maximal cardinality of $r$-separated subsets of $Y$. We will need the following technical result.

Lemma 5.2. For any $\omega \in \Omega^{\prime}$ and $n \in N_{2}^{\prime}$ we have

$$
\# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) \leq T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right)+\# \mathrm{CP}(s, \omega, n)
$$

Proof. We start by observing that

$$
\begin{aligned}
\widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega) & =\left\{\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|>\frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}} \forall \mathbf{a}^{\prime} \neq \mathbf{a}\right\} \\
& \cup\left\{\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega): \exists \mathbf{a}^{\prime} \neq \mathbf{a} \text { s.t. }\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right| \leq \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right\}
\end{aligned}
$$

is a disjoint union. Notice also that the set of images corresponding to those a belonging to the first set in this union is $\frac{s}{\# L_{\mathrm{m}, n}^{11 /}}$-separated. Therefore

$$
\#\left\{\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega):\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right|>\frac{s}{\# L_{\mathbf{m}, n}^{1 / d}} \forall \mathbf{a}^{\prime} \neq \mathbf{a}\right\} \leq T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathbf{m}, n}^{1 / d}}\right) .
$$

Similarly, for the second set in this union we have

$$
\left\{\mathbf{a} \in \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega): \exists \mathbf{a}^{\prime} \neq \mathbf{a} \text { s.t. }\left|\Pi_{\omega}(\mathbf{a b})-\Pi_{\omega}\left(\mathbf{a}^{\prime} \mathbf{b}\right)\right| \leq \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right\} \leq \# \mathrm{CP}(s, \omega, n)
$$

This follows because the map $\left(\mathbf{a}, \mathbf{a}^{\prime}\right) \rightarrow \mathbf{a}$ from $\mathrm{CP}(s, \omega, n)$ to this set is surjective. The desired inequality now follows.

Given $n \in \mathbb{N}$ and $\omega \in \Omega^{\prime}$ let

$$
Y_{n}^{\prime}(\omega):=\left\{\Pi_{\omega}(\mathbf{a b})\right\}_{\mathbf{a} \in L_{\mathbf{m}, n}}
$$

Notice that $Y_{n}(\omega) \subset Y_{n}^{\prime}(\omega)$ therefore $T\left(Y_{n}(\omega), r\right) \leq T\left(Y_{n}^{\prime}(\omega), r\right)$ for any $r>0$. Given $c>0, s>0$, and $n \in \mathbb{N}$ we also let

$$
B(c, s, n):=\left\{\omega \in \Omega^{\prime}: \frac{T\left(Y_{n}^{\prime}(\omega), \frac{s}{\# L_{\mathrm{m}, n}^{1 / d}}\right)}{\# L_{\mathfrak{m}, n}}>c\right\}
$$

Recall that we define the upper density of a set $B \subset \mathbb{N}$ to be

$$
\bar{d}(B):=\limsup _{n \rightarrow \infty} \frac{\#\{1 \leq j \leq n: j \in B\}}{n} .
$$

Proposition 5.3. The following equality holds

$$
\eta\left(\bigcap_{\varepsilon>0} \bigcup_{c, s>0}\left\{\omega \in \Omega^{\prime}: \bar{d}(n: \omega \in B(c, s, n)) \geq 1-\varepsilon\right\}\right)=\eta\left(\Omega^{\prime}\right)
$$

Proof. Let $\varepsilon>0$ be arbitrary. Notice that by Proposition 5.1 and Markov's inequality, for any $c>0$ we have

$$
c \cdot \eta\left(\omega \in \Omega^{\prime}: \# \mathrm{CP}(s, \omega, n) \geq c \# L_{\mathfrak{m}, n}\right)=\mathcal{O}\left(s^{d}\right)
$$

for any $n \geq N_{2}^{\prime}$. Therefore, since $\# L_{\mathfrak{m}, n} \asymp \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)$ we can choose $c, s>0$ such that

$$
\eta\left(\omega \in \Omega^{\prime}: \# \operatorname{CP}(s, \omega, n) \geq c \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)\right)<\varepsilon .
$$

Therefore by Lemma 5.2, for this choice of $c, s$ we have

$$
\eta\left(\omega \in \Omega^{\prime}: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right) \geq \eta\left(\Omega^{\prime}\right)-\varepsilon
$$

for any $n \geq N_{2}^{\prime}$. Using this inequality and apply Fatou's lemma we have

$$
\begin{align*}
& \int_{\Omega^{\prime}} \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right) d \eta \\
= & \int_{\Omega^{\prime}} \limsup _{N \rightarrow \infty} \frac{\#\left\{1 \leq n \leq N: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{11, d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right\}}{N} d \eta \\
\geq & \limsup _{N \rightarrow \infty} \int_{\Omega^{\prime}} \frac{\sum_{n=1}^{N} \chi\left(\omega: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathbf{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right)}{N} d \eta \\
\geq & \eta\left(\Omega^{\prime}\right)-\varepsilon . \tag{5.3}
\end{align*}
$$

We now show that this implies that the occurrence of a large separated set for a set of $n$ with high upper density has large probability. That is, we will prove the inequality

$$
\begin{equation*}
\eta\left(\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right) \geq 1-\sqrt{\varepsilon}\right) \geq \eta\left(\Omega^{\prime}\right)-\sqrt{\varepsilon} \tag{5.4}
\end{equation*}
$$

Assume for a contradiction that

$$
\eta\left(\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right)<1-\sqrt{\varepsilon}\right)>\sqrt{\varepsilon}
$$

As the density is always bounded above by 1 we have

$$
\begin{align*}
& \int_{\Omega^{\prime}} \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right) d \eta \\
& \leq \eta\left(\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{11 d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right)<1-\sqrt{\varepsilon}\right)(1-\sqrt{\varepsilon}) \\
& +\eta\left(\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right) \geq 1-\sqrt{\varepsilon}\right) . \tag{5.5}
\end{align*}
$$

The second term on the right hand side of (5.5) is equal to

$$
\eta\left(\Omega^{\prime}\right)-\eta\left(\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right)<1-\sqrt{\varepsilon}\right) .
$$

Therefore the right hand side of (5.5) can be bounded above by

$$
\eta\left(\Omega^{\prime}\right)-\sqrt{\varepsilon} \eta\left(\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right)<1-\sqrt{\varepsilon}\right)
$$

$$
<\eta\left(\Omega^{\prime}\right)-\varepsilon
$$

Where in the final line we used our underlying assumption. However, this contradicts (5.3). Therefore (5.4) holds.

We have proved that for any $\varepsilon>0$ we can chose $c, s>0$ such that (5.4) holds. In particular, letting $\varepsilon_{k} \rightarrow 0$ with $\varepsilon_{k}<\varepsilon$ and picking appropriate sequences $s_{k}, c_{k}$ we may conclude that

$$
\eta\left(\bigcup_{c, s>0}\left\{\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right) \geq 1-\varepsilon\right\}\right)=\eta\left(\Omega^{\prime}\right) .
$$

Recall that $\varepsilon>0$ was arbitrary. Therefore taking the intersection over all $\varepsilon>0$ we have

$$
\begin{equation*}
\eta\left(\bigcap_{\varepsilon>0} \bigcup_{c, s>0}\left\{\omega \in \Omega^{\prime}: \bar{d}\left(n: T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n, \varepsilon_{1}}(\omega)(1-c)\right) \geq 1-\varepsilon\right\}\right)=\eta\left(\Omega^{\prime}\right) \tag{5.6}
\end{equation*}
$$

Because $T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathbf{m}, n}^{1 / d}}\right) \leq T\left(Y_{n}^{\prime}(\omega), \frac{s}{\# L_{\mathbf{m}, n}^{11 / d}}\right)$ and $\# L_{\mathfrak{m}, n} \asymp \# \widetilde{L}_{\mathfrak{m}, n}$, there exists $K>0$ such that

$$
T\left(Y_{n}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \# \widetilde{L}_{\mathfrak{m}, n}(\omega)(1-c) \Rightarrow T\left(Y_{n}^{\prime}(\omega), \frac{s}{\# L_{\mathfrak{m}, n}^{1 / d}}\right) \geq \frac{\# L_{\mathfrak{m}, n}(1-c)}{K}
$$

Therefore by (5.6) we have

$$
\eta\left(\bigcap_{\varepsilon>0} \bigcup_{c, s>0}\left\{\omega \in \Omega^{\prime}: \bar{d}\left(n: \omega \in B\left(\frac{1-c}{K}, s, n\right)\right) \geq 1-\varepsilon\right\}\right)=\eta\left(\Omega^{\prime}\right)
$$

This completes our proof.
With Proposition 5.3 we are now in a position to prove Statement 1 from Theorem 3.4.
Theorem 3.4, Statement 1. Let us fix

$$
\omega \in \bigcap_{\varepsilon>0} \bigcup_{c, s>0}\left\{\omega \in \Omega^{\prime}: \bar{d}(n: \omega \in B(c, s, n)) \geq 1-\varepsilon\right\}
$$

Let $g \in G$ be arbitrary. By definition there exists $\varepsilon^{*}>0$ such that $g \in G_{\varepsilon^{*}}$. By the definition of $G_{\varepsilon^{*}}$ we can choose $c>0, s>0$ such that

$$
\sum_{n: w \in B(c, s, n)} g(n)=\infty
$$

Moreover, notice that $\# L_{\mathfrak{m}, n} \asymp c_{\mathfrak{m}}^{-n}$. Therefore the two assumptions of Proposition 4.4 are satisfied and so the set

$$
\left\{x \in \mathbb{R}^{d}:\left|x-\Pi_{\omega}(\mathbf{a b})\right| \leq\left(\frac{g(n)}{\# L_{\mathfrak{m}, n}}\right)^{1 / d} \text { for some } \mathbf{a} \in L_{\mathfrak{m}, n} \text { for i.m. } n\right\}
$$

has positive Lebesgue measure. Recall that $\# L_{\mathfrak{m}, n}^{-1} \asymp \mathfrak{m}([\mathbf{a}])$ for any $\mathbf{a} \in L_{\mathfrak{m}, n}$. Therefore Lemma 4.5 implies that $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ has positive Lebesgue measure for any $g \in G$. By Proposition 5.3 it follows that for almost every $\omega \in \Omega^{\prime}$ the set $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ has positive Lebesgue measure for any $g \in G$. Since $\eta\left(\Omega^{\prime}\right)>1-\epsilon_{0}$, and $\epsilon_{0}$ was arbitrary, it follows that for almost every $\omega \in \Omega$, for any $g \in G$ the set $U_{\omega}(\mathbf{b}, \mathfrak{m}, g)$ has positive Lebesgue measure. This completes our proof.

## 6 Proof of Corollary 3.5

We now show how Corollary 3.5 follows from Theorem 3.4. The proof is essentially the same as the proof of Corollary 2.3 from [5]. We include the details for completion.

Proof of Corollary 3.5. Let us fix a RIFS such either assumption A or assumption B is satisfied. Let us also fix a probability vector $\left(p_{i}\right)_{i \in \mathcal{A}}$ such that $\frac{-\sum_{i \in A} p_{i} \log p_{i}}{\sum p_{i} \lambda^{\lambda^{\prime}}\left(\eta_{i}\right)}>1$.

By Theorem 3.4, to prove our result it suffices to show that if we let $\Psi: \mathcal{A}^{*} \rightarrow[0, \infty)$ be given by $\Psi(\mathbf{a})=\frac{\prod_{k=1}^{|\mathfrak{a}|} p_{a_{k}}}{|\mathbf{a}|}$, then $\Psi$ is equivalent to $(\mathfrak{m}, g)$ for some $g \in G$. Here we let $\mathfrak{m}$ denote the Bernoulli measure corresponding to $\left(p_{i}\right)_{i \in \mathcal{A}}$.

Let $g(n)=\frac{1}{n}$, then using the well known identity

$$
\sum_{n=1}^{N} \frac{1}{n}=\log N+\mathcal{O}(1)
$$

it can be shown that $g \in G$. For any $\mathbf{a} \in \mathcal{A}^{*}$ we have

$$
\begin{equation*}
\left(\min _{i \in \mathcal{A}} p_{i}\right)^{|\mathbf{a}|} \leq \mathfrak{m}([\mathbf{a}]) \leq\left(\max _{i \in \mathcal{A}} p_{i}\right)^{|\mathbf{a}|} \tag{6.1}
\end{equation*}
$$

Using (6.1) and the fact each $\mathbf{a} \in L_{\mathfrak{m}, n}$ satisfies $\mathfrak{m}([\mathbf{a}]) \asymp c_{\mathfrak{m}}^{n}$, we may deduce that

$$
|\mathbf{a}| \asymp n
$$

for any $\mathbf{a} \in L_{\mathfrak{m}, n}$. This implies

$$
\frac{\prod_{k=1}^{|\mathbf{a}|} p_{a_{k}}}{|\mathbf{a}|} \asymp \frac{\mathfrak{m}([\mathbf{a}])}{n}
$$

for any $\mathbf{a} \in L_{\mathfrak{m}, n}$. Therefore $\Psi$ is equivalent to $(\mathfrak{m}, g)$ for our choice of $g$. This completes our proof.

## $7 \quad$ Examples

In this section we detail some examples of RIFSs to which our results can be applied. The first example is stochastically self-similar and is distantly non-singular, whereas the second is stochastically self-affine and non-singular.

Example 7.1. For each $i \in \mathcal{A}$ assume that there exists $0 \leq r_{i}^{-}<r_{i}^{+}<1$ such that

$$
\Omega_{i}:=\left\{\lambda \cdot O: \lambda \in\left[r_{i}^{-}, r_{i}^{+}\right] \text {and } O \in \mathcal{O}(d)\right\} .
$$

Here $\mathcal{O}(d)$ is the set of $d \times d$ orthogonal matrices. Note that $\Omega_{i} \subset S_{d}$ for all $i \in \mathcal{A}$. For each $i \in \mathcal{A}$ we define a measure $\eta_{i}$ on $\Omega_{i}$ according to the law where $\lambda$ and $O$ are chosen independently with respect to the normalised Lebesgue measure on $\left[r_{i}^{-}, r_{i}^{+}\right]$and the Haar measure on $\mathcal{O}(d)$ respectively. For any $x \in \mathbb{R}^{d}$ we define the map $P_{x}: \Omega_{i} \rightarrow \mathbb{R}^{d}$ given by $P_{x}(A)=A x$. It can be shown that the pushforward measure $\left(P_{x}\right)_{*} \eta_{i}$ is the normalised Lebesgue measure on the annulus

$$
\left\{y \in \mathbb{R}^{d}: r_{i}^{-}\|x\| \leq\|y\| \leq r_{i}^{+}\|x\|\right\}
$$

Note that $\left(P_{x}\right)_{*} \eta_{i}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{d}$. Moreover, for any $\varepsilon>0$, there exists $C=C(\varepsilon)>0$ such that for any $x$ satisfying $\|x\| \geq \varepsilon$, the Radon-Nikodym derivative of $\left(P_{x}\right)_{*} \eta_{i}$ is uniformly bounded above by $C$.

Now let us fix a collection $\left\{t_{i}\right\}_{i \in \mathcal{A}}$ of distinct translation vectors and let $r_{0}:=\frac{\min _{i \neq j}\left|t_{i}-t_{j}\right|}{16}$. If $\|x\|<r_{0}$ and $r<r_{0}$ then for any $y \in \mathbb{R}^{d} \backslash B\left(0, \frac{\min _{i \neq j}\left|t_{i}-t_{j}\right|}{8}\right)$ we have $\left\{A \in \Omega_{i}: A(x) \in\right.$ $B(y, r)\}=\emptyset$. Therefore

$$
\begin{equation*}
\eta_{i}\left(\mathcal{A} \in \Omega_{i}: A(x) \in B(y, r)\right)=0 . \tag{7.1}
\end{equation*}
$$

If $\|x\|<r_{0}$ and $r \geq r_{0}$, we can choose $C_{1}>0$ sufficiently large in a way that only depends upon $r_{0}$ such that

$$
\begin{equation*}
\eta_{i}\left(\mathcal{A} \in \Omega_{i}: A(x) \in B(y, r)\right) \leq C_{1} r^{d} \tag{7.2}
\end{equation*}
$$

for all $y \in \mathbb{R}^{d}$. If $\|x\| \geq r_{0}$ then it follows by our above remarks regarding the Radon-Nikodym derivative that there exists $C_{2}>0$ independent of $x$ such that

$$
\begin{equation*}
\eta_{i}\left(\mathcal{A} \in \Omega_{i}: A(x) \in B(y, r)\right)=\left(\left(P_{x}\right)_{*} \eta_{i}\right)(B(y, r)) \leq C_{2} r^{d} \tag{7.3}
\end{equation*}
$$

for all $y \in \mathbb{R}^{d}$ and $r>0$.
Combining (7.1), (7.2), and (7.3) we see that the RIFS $\left(\left\{\Omega_{i}\right\}_{i \in \mathcal{A}},\left\{\eta_{i}\right\}_{i \in \mathcal{A}},\left\{t_{i}\right\}_{i \in \mathcal{A}}\right)$ is distantly non-singular. We need to check that the logarithmic condition (3.2) holds for $|s|$ sufficiently small. We see that

$$
\log \int_{\Omega_{i}}|\operatorname{Det}(A)|^{s} d \eta_{i}(A)=\log \int_{r_{i}^{-}}^{r_{i}^{+}} \frac{r^{s}}{r_{i}^{+}-r_{i}^{-}} d r=\log \frac{\left(r_{i}^{+}\right)^{s+1}-\left(r_{i}^{-}\right)^{s+1}}{\left(r_{i}^{+}-r_{i}^{-}\right)(s+1)}
$$

which is finite for all $s>-1$. Therefore (3.2) holds for all $|s|<1$.
Now by an appropriate choice of parameters, it is straightforward to construct many slowly decaying $\sigma$-invariant ergodic probability measures $\mathfrak{m}$ such that $\frac{h(\mathfrak{m})}{\lambda(\eta, \mathfrak{m})}>1$. Therefore Theorem 3.4 , Corollary 3.5, and Corollary 3.6 can be applied to this random model.

Example 7.2. For each $i \in \mathcal{A}$ let $Z_{i}$ be a compact subset of $M_{d}$ such that each $A \in Z_{i}$ satisfies $\|A x\| \geq c_{i}\|x\|$ for all $x \in \mathbb{R}^{d}$ for some $c_{i}>0$. Also assume that for each $i \in \mathcal{A}$ there exists a Borel probability measure $\nu_{i}$ supported on $Z_{i}$. For each $i \in \mathcal{A}$ let $0 \leq r_{i}^{-}<r_{i}^{+}<1$ and

$$
\Omega_{i}:=\left\{A=\lambda \cdot O B: \lambda \in\left[r_{i}^{-}, r_{i}^{+}\right], O \in \mathcal{O}(d), \text { and } B \in Z_{i}\right\} .
$$

We define a measure $\eta_{i}$ on $\Omega_{i}$ by choosing $\lambda, O$, and $B$ independently with respect to the normalised Lebesgue measure $\mathcal{L}$ on $\left[r_{i}^{-}, r_{i}^{+}\right]$, the Haar measure $m$ on $\mathcal{O}(d)$, and $\nu_{i}$ respectively. Let $\left\{t_{i}\right\}_{i \in \mathcal{A}}$ be a finite set of distinct vectors. We assume $\left\{t_{i}\right\}_{i \in \mathcal{A}}$ and $\left\{\Omega_{i}\right\}$ are such that there exists $\delta>0$ for which

$$
B(0, \delta) \cap \bigcup_{\omega \in \Omega} \Pi_{\omega}\left(\mathcal{A}^{\mathbb{N}}\right)=\emptyset
$$

This property is satisfied for example if each $t_{i}$ satisfies $\left\|t_{i}\right\|=1$ and each $A \in \cup_{i \in \mathcal{A}} \Omega_{i}$ satisfies $\|A\|<1 / 2$.

We now show that if the above conditions are satisfied then the RIFS is non-singular. Let us fix an ellipse $E$ and $x \in \cup_{\omega} \Pi_{\omega}\left(\mathcal{A}^{\mathbb{N}}\right)$. By Fubini's theorem

$$
\begin{equation*}
\eta_{i}\left(A \in \Omega_{i}: A(x) \in E\right)=\int_{Z_{i}}(\mathcal{L} \times m)((r, O): r \cdot O B x \in E) d \nu_{i}(B) . \tag{7.4}
\end{equation*}
$$

By construction $B x$ is a point with norm $\|B x\| \geq c \delta$. Therefore by the same reasoning as given in Example 7.1, by considering appropriate pushforwards it can be shown that an analogue of (7.3) holds, i.e. there exists $C>0$ independent of $x$ and $i$ such that

$$
(\mathcal{L} \times m)((r, O): r \cdot O B x \in E) \leq C \cdot \operatorname{Vol}(E)
$$

for any $B \in Z_{i}$. Substituting this bound into (7.4) we obtain

$$
\eta_{i}\left(A \in \Omega_{i}: A(x) \in E\right) \leq C \cdot \operatorname{Vol}(E)
$$

Hence our RIFS is non-singular. The logarithmic condition (3.2) holds for this RIFS for all $|s|$ sufficiently small by analogous reasoning to that given in Example 1. Therefore Theorem 3.4, Corollary 3.5, and Corollary 3.6 can be applied to this random model.

We conclude by remarking that to show that a RIFS is non-singular it is sufficient to show that the pushforward measure $\left(P_{x}\right)_{*} \eta_{i}$ is absolutely continuous for all $x \in \cup_{\omega} \Pi_{\omega}\left(\mathcal{A}^{\mathbb{N}}\right)$ and $i \in \mathcal{A}$, and that the Radon-Nikodym derivative can be bounded above by some constant independent of $x$ and $i$. This is the technique we have used in Example 2.

## 8 Final discussion

Remark 8.1. In a random recursive model one usually expects the threshold quantities to be defined in terms of the arithmetic average of random variables, as opposed to the geometric average that is the expected behaviour in 1-variable models. Here, this means that one naïvely suspects the Lyapunov exponent to be

$$
\lambda^{\prime}\left(\eta_{i}\right)=\log \int_{\Omega_{i}}|\operatorname{Det}(A)| d \eta_{i}(A)
$$

instead of

$$
\lambda^{\prime}\left(\eta_{i}\right)=\int_{\Omega_{i}} \log |\operatorname{Det}(A)| d \eta_{i}(A) .
$$

While we cannot exclude the possibilities that our work could be improved, the near optimal usage of large deviations in our work suggests that the second Lyapunov exponent is the correct one to use. This is unexpected and could be explained by us requiring level specific information on worst cases, as opposed to "eventually averaging" of behaviour of the descendants of each node.

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[^0]:    ${ }^{1}$ We say that $\Phi$ contains an exact overlap if there exists two words $\left(a_{1}, \ldots, a_{n}\right)$ and $\left(b_{1}, \ldots, b_{m}\right)$ such that $\phi_{a_{1}} \circ \cdots \circ \phi_{a_{n}}=\phi_{b_{1}} \circ \cdots \circ \phi_{b_{m}}$

[^1]:    ${ }^{2}$ This result in fact holds whenever $\Phi$ consists of conformal maps.

[^2]:    ${ }^{3}$ The lower bound of $13 / 16$ is arbitrary and can be replaced by any value less than 1 .

