

# The Cantor–Schröder–Bernstein Theorem for $\infty$ -groupoids

Escardo, Martin

DOI:

[10.1007/s40062-021-00284-6](https://doi.org/10.1007/s40062-021-00284-6)

License:

Creative Commons: Attribution (CC BY)

*Document Version*

Publisher's PDF, also known as Version of record

*Citation for published version (Harvard):*

Escardo, M 2021, 'The Cantor–Schröder–Bernstein Theorem for  $\infty$ -groupoids', *Journal of Homotopy and Related Structures*, vol. 16, no. 3, pp. 363-366. <https://doi.org/10.1007/s40062-021-00284-6>

[Link to publication on Research at Birmingham portal](#)

## General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

## Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact [UBIRA@lists.bham.ac.uk](mailto:UBIRA@lists.bham.ac.uk) providing details and we will remove access to the work immediately and investigate.



# The Cantor–Schröder–Bernstein Theorem for $\infty$ -groupoids

Martín Hötzel Escardó<sup>1</sup>

Received: 26 August 2020 / Accepted: 15 June 2021  
© The Author(s) 2021

## Abstract

We show that the Cantor–Schröder–Bernstein Theorem for homotopy types, or  $\infty$ -groupoids, holds in the following form: For any two types, if each one is embedded into the other, then they are equivalent. The argument is developed in the language of homotopy type theory, or Voevodsky’s univalent foundations (HoTT/UF), and requires classical logic. It follows that the theorem holds in any boolean  $\infty$ -topos.

**Keywords** Cantor–Schröder–Bernstein Theorem ·  $\infty$ -groupoid · Homotopy type theory · Univalent foundations ·  $\infty$ -topos

## 1 Introduction

The classical Cantor–Schröder–Bernstein Theorem of set theory, formulated by Cantor and first proved by Bernstein, states that for any pair of sets, if there is an injection of each one into the other, then the two sets are in bijection. There are proofs that use the principle of excluded middle but not the axiom of choice. That the principle excluded middle is absolutely necessary has been recently established Pradic and Brown [5].

The appropriate *principle of excluded middle* for HoTT/UF [8] says that every subsingleton (or proposition, or truth value) is either empty or pointed. The statement that *every type* is either empty or pointed is much stronger, and amounts to the axiom of *global choice*, which is incompatible with univalence [8, Theorem 3.2.2]. In fact, in the presence of global choice, every type is a set by Hedberg’s Theorem, but univalence gives types that are not sets. The principle of excluded middle, however, is known to be compatible with univalence, and is validated in Voevodsky’s model of simplicial sets. And so is the axiom of (non-global) choice, but it is not needed for our purposes.

Even assuming the principle of excluded middle, it may seem unlikely at first sight that the Cantor–Schröder–Bernstein Theorem can be generalized from sets to arbitrary homotopy types, or  $\infty$ -groupoids:

---

Communicated by Bjorn Dundas.

---

✉ Martín Hötzel Escardó  
m.escardo@cs.bham.ac.uk

<sup>1</sup> School of Computer Science, University of Birmingham, Birmingham, UK

1. The Cantor–Schröder–Bernstein property fails for 1-categories. In fact, it already fails for posets. For example, the intervals  $(0, 1)$  and  $[0, 1]$  are order-embedded into each other, but they are not order isomorphic, or equivalent as categories.
2. The known proofs of the Cantor–Schröder–Bernstein Theorem for sets rely on deciding equality of elements of sets, but, in the presence of the principle of excluded middle, the types that have decidable equality are precisely the sets, by Hedberg’s Theorem.

In set theory, a map  $f : X \rightarrow Y$  is an injection if and only if it is left-cancellable, in the sense that  $f(x) = f(x')$  implies  $x = x'$ . But, for types  $X$  and  $Y$  that are not sets, this notion is too weak, and, moreover, is not a proposition as the identity type  $x = x'$  has multiple elements in general. The appropriate notion of *embedding* for a function  $f$  of arbitrary types  $X$  and  $Y$  is given by any of the following two equivalent conditions:

1. The canonical map  $\text{ap}(f, x, x') : x = x' \rightarrow f(x) = f(x')$  is an equivalence for any  $x, x' : X$ .
2. The fibers of  $f$  are all subsingletons.

A map of sets is an embedding if and only if it is left-cancellable. However, for example, any map  $1 \rightarrow Y$  that picks a point  $y : Y$  is left-cancellable, but it is an embedding if and only if the point  $y$  is homotopy isolated, which amounts to saying that the identity type  $y = y$  is contractible. This fails, for instance, when the type  $Y$  is the homotopical circle  $S^1$ , for any point  $y$ , or when  $Y$  is a univalent universe and  $y : Y$  is the two-point type, or any type with more than one automorphism.

**Example 1.1** (Pradic [4]) There is a pair of left-cancellable maps between the types  $\mathbb{N} \times S^1$  and  $1 + \mathbb{N} \times S^1$  (taking  $\text{inl}$  going forward and, going backward, mapping  $\text{inl}(\ast)$  to  $(0, \text{base})$  and shifting the indices of the circles by one), but no equivalence between these two types.

## 2 Cantor–Schröder–Bernstein for $\infty$ -groupoids

As explained in the introduction, our argument is in the language of HoTT/UF and requires the principle of excluded middle. The following theorem holds in all *boolean*  $\infty$ -toposes, because HoTT/UF with the principle of excluded middle can be interpreted in them [6]. We assume the terminology and notation of the HoTT book [8].

**Theorem 2.1** *From given embeddings of two types into each other, we can construct an equivalence between them using the principle of excluded middle.*

We adapt the proof for sets presented in Halmos’ book [3]. We need to reformulate the argument so that the principle of excluded middle is applied to truth-valued, rather than arbitrary type-valued, mathematical statements, and this is the contribution in this note (see Remark 2.3 below). We don’t need to invoke univalence, the existence of propositional truncations or any other higher inductive type for our construction. But we do rely on function extensionality. An version of the following argument formalized in Agda [7] is available at [1,2].

**Proof** Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be embeddings of arbitrary types  $X$  and  $Y$ . We say that  $x : X$  is a  $g$ -point if for any  $x_0 : X$  and  $n : \mathbb{N}$  with  $(g \circ f)^n(x_0) = x$ , the  $g$ -fiber of  $x_0$  is pointed. Using the assumption that  $g$  is an embedding, we see that being a  $g$ -point is property rather than data, because subsingletons are closed under products by function extensionality.

Considering  $x_0 = x$  and  $n = 0$ , we see that if  $x$  is a  $g$ -point then the  $g$ -fiber of  $x$  is pointed, and hence we get a function  $g^{-1}$  of  $g$ -points of  $X$  into  $Y$ . By construction, we have that  $g(g^{-1}(x)) = x$ . In particular, if  $g(y)$  is a  $g$ -point for a given  $y : Y$ , we conclude that  $g(g^{-1}(g(y))) = g(y)$ , and because  $g$ , being an embedding, is left-cancellable, we get  $g^{-1}(g(y)) = y$ .

Now define  $h : X \rightarrow Y$  by

$$h(x) = \begin{cases} g^{-1}(x) & \text{if } x \text{ is a } g\text{-point,} \\ f(x) & \text{otherwise.} \end{cases}$$

To conclude the proof, it is enough to show that  $h$  is left-cancellable and split-surjective, as any such map is an equivalence.

To see that  $h$  is left-cancellable, it is enough to show that the images of  $f$  and  $g^{-1}$  in the definition of  $h$  are disjoint, because  $f$  and  $g^{-1}$  are left-cancellable. For that purpose, let  $x$  be a non- $g$ -point and  $x'$  be a  $g$ -point, and, for the sake of contradiction, assume  $f(x) = g^{-1}(x')$ . Then  $g(f(x)) = g(g^{-1}(x')) = x'$ . Now, because if  $g(f(x))$  were a  $g$ -point then so would be  $x$ , we conclude that it isn't, and hence neither is  $x'$ , which contradicts the assumption.

To see that  $h$  is a split surjection, say that  $x : X$  is an  $f$ -point if there are designated  $x_0 : X$  and  $n : \mathbb{N}$  with  $(g \circ f)^n(x_0) = x$  and the  $g$ -fiber of  $x_0$  empty. This is data rather than property, and so this notion could not have been used for the construction of  $h$ . But every non- $f$ -point is a  $g$ -point, as we see by applying the principle of excluded middle to the  $g$ -fiber of  $x_0$  in the definition of  $g$ -point.

**Claim 2.2** *If  $g(y)$  is not a  $g$ -point, then there is a designated point  $(x, p)$  of the  $f$ -fiber of  $y$ , with  $x : X$  and  $p : f(x) = y$ , such that  $x$  is not a  $g$ -point either.*

To prove the claim, first notice that it is impossible that  $g(y)$  is not an  $f$ -point, by the above observation. But this is not enough to conclude that it is an  $f$ -point, because the principle of excluded middle applies to subsingletons only, which the notion of  $f$ -point isn't. However, it is readily seen that if  $g(y)$  is an  $f$ -point, then there is a designated point  $(x, p)$  in the  $f$ -fiber of  $y$ . From this it follows that it is impossible that the subtype of the fiber consisting of the elements  $(x, p)$  with  $x$  not a  $g$ -point is empty. But the  $f$ -fiber of  $y$  is a subsingleton because  $f$  is an embedding, and hence so is the subtype, and therefore the claim follows by double-negation elimination, which is a standard consequence of the principle of excluded middle.

We can now resume the proof that  $h$  is a split surjection. For any  $y : Y$ , we check whether  $g(y)$  is a  $g$ -point. If it is, we map  $y$  to  $g(y)$ , and if it isn't we map  $y$  to the point  $x : X$  given by the claim, which concludes the proof of the theorem.  $\square$

**Remark 2.3** So, in this argument we don't apply the principle of excluded middle to equality directly, which we wouldn't be able to as the types  $X$  and  $Y$  are not necessarily

sets. We instead apply it to (1) the property of being a  $g$ -point, defined in terms of the fibers of  $g$ , to define  $h$ , (2) a fiber of  $g$ , and (3) a subtype of a fiber of  $f$ . These three types are propositions because the functions  $f$  and  $g$  are embeddings rather than merely left-cancellable maps.

**Remark 2.4** If the type  $X$  in the proof is connected, then every map of  $X$  into a set is constant. In particular, the property of being a  $g$ -point is constant, because the type of truth values is a set (assuming univalence for subsingletons). Hence, by the principle of excluded middle, it is constantly true or constantly false, and so  $h = g^{-1}$  or  $h = f$ , which means that one of the embeddings  $f$  and  $g$  is already an equivalence. Mike Shulman (personal communication) observed that this is true even without the principle of excluded middle: If  $X$  is connected and we have an embedding  $g : Y \rightarrow X$  and any function at all  $f : X \rightarrow Y$ , then  $g$  is an equivalence. For any  $x : X$ , we have  $\|g(f(x)) = x\|$  since  $X$  is connected; thus  $g$  is (non-split) surjective. But a surjective embedding is an equivalence.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

## References

1. Escardó, M.H.: Introduction to univalent foundations of mathematics with Agda. arXiv e-prints [arXiv:1911.00580](https://arxiv.org/abs/1911.00580) (2019)
2. Escardó, M.H.: Agda proof of the Cantor-Schröder-Bernstein Theorem for  $\infty$ -groupoids. Available at <https://www.cs.bham.ac.uk/~mhe/agda-new/CantorSchröderBernstein.html> with source code at <https://github.com/martinescardo/TypeTopology/> (2020)
3. Halmos, P.R.: Naive set theory. Reprint of the 1960 edition, Undergraduate Texts in Mathematics. Springer-Verlag, New York (1974)
4. Pradic, P.: Comment <https://homotopytypetheory.org/2020/01/26/the-cantor-schroder-bernstein-theorem-for-%e2%88%9e-groupoids/#comment-153480>
5. Pradic, P., Brown, C.E.: Cantor-Bernstein implies Excluded Middle. arXiv e-prints [arXiv:1904.09193](https://arxiv.org/abs/1904.09193) (2019)
6. Shulman, M.: All  $(\infty, 1)$ -toposes have strict univalent universes. arXiv e-prints [arXiv:1904.07004](https://arxiv.org/abs/1904.07004) (2019)
7. The Agda Team.: Agda documentation. <https://agda.readthedocs.io/en/latest/>
8. The Univalent Foundations Program: Homotopy type theory: univalent foundations of mathematics. Institute for Advanced Study <https://homotopytypetheory.org/book> (2013)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.