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# The Cantor–Schröder–Bernstein Theorem for $\infty$-groupoids

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**Abstract**
We show that the Cantor–Schröder–Bernstein Theorem for homotopy types, or $\infty$-groupoids, holds in the following form: For any two types, if each one is embedded into the other, then they are equivalent. The argument is developed in the language of homotopy type theory, or Voevodsky's univalent foundations (HoTT/UF), and requires classical logic. It follows that the theorem holds in any boolean $\infty$-topos.

**Keywords** Cantor–Schröder–Bernstein Theorem · $\infty$-groupoid · Homotopy type theory · Univalent foundations · $\infty$-topos

## 1 Introduction

The classical Cantor–Schröder–Bernstein Theorem of set theory, formulated by Cantor and first proved by Bernstein, states that for any pair of sets, if there is an injection of each one into the other, then the two sets are in bijection. There are proofs that use the principle of excluded middle but not the axiom of choice. That the principle excluded middle is absolutely necessary has been recently established Pradic and Brown [5].

The appropriate *principle of excluded middle* for HoTT/UF [8] says that every subsingleton (or proposition, or truth value) is either empty or pointed. The statement that *every type* is either empty or pointed is much stronger, and amounts to the axiom of *global choice*, which is incompatible with univalence [8, Theorem 3.2.2]. In fact, in the presence of global choice, every type is a set by Hedberg's Theorem, but univalence gives types that are not sets. The principle of excluded middle, however, is known to be compatible with univalence, and is validated in Voevodsky's model of simplicial sets. And so is the axiom of (non-global) choice, but it is not needed for our purposes.

Even assuming the principle of excluded middle, it may seem unlikely at first sight that the Cantor–Schröder–Bernstein Theorem can be generalized from sets to arbitrary homotopy types, or $\infty$-groupoids:

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1. The Cantor–Schröder–Bernstein property fails for 1-categories. In fact, it already fails for posets. For example, the intervals (0, 1) and [0, 1] are order-embedded into each other, but they are not order isomorphic, or equivalent as categories.
2. The known proofs of the Cantor–Schröder–Bernstein Theorem for sets rely on deciding equality of elements of sets, but, in the presence of of the principle of excluded middle, the types that have decidable equality are precisely the sets, by Hedberg's Theorem.

In set theory, a map $f : X \to Y$ is an injection if and only if it is left-cancellable, in the sense that $f(x) = f(x')$ implies $x = x'$. But, for types $X$ and $Y$ that are not sets, this notion is too weak, and, moreover, is not a proposition as the identity type $x = x'$ has multiple elements in general. The appropriate notion of *embedding* for a function $f$ of *arbitrary* types $X$ and $Y$ is given by any of the following two equivalent conditions:

1. The canonical map $\mathrm{ap}(f, x, x') : x = x' \to f(x) = f(x')$ is an equivalence for any $x, x' : X$.
2. The fibers of $f$ are all subsingletons.

A map of sets is an embedding if and only if it is left-cancellable. However, for example, any map $1 \to Y$ that picks a point $y : Y$ is left-cancellable, but it is an embedding if and only if the point $y$ is homotopy isolated, which amounts to saying that the identity type $y = y$ is contractible. This fails, for instance, when the type $Y$ is the homotopical circle $S^1$, for any point $y$, or when $Y$ is a univalent universe and $y : Y$ is the two-point type, or any type with more than one automorphism.

***Example 1.1*** (Pradic [4]) There is a pair of left-cancellable maps between the types $\mathbb{N} \times S^1$ and $1 + \mathbb{N} \times S^1$ (taking inl going forward and, going backward, mapping $\mathrm{inl}(*)$ to (0, base) and shifting the indices of the circles by one), but no equivalence between these two types.

## 2 Cantor–Schröder–Bernstein for $\infty$-groupoids

As explained in the introduction, our argument is in the language of HoTT/UF and requires the principle of excluded middle. The following theorem holds in all *boolean* $\infty$-toposes, because HoTT/UF with the principle of excluded middle can be interpreted in them [6]. We assume the terminology and notation of the HoTT book [8].

**Theorem 2.1** *From given embeddings of two types into each other, we can construct an equivalence between them using the principle of excluded middle.*

We adapt the proof for sets presented in Halmos' book [3]. We need to reformulate the argument so that the principle of excluded middle is applied to truth-valued, rather than arbitrary type-valued, mathematical statements, and this is the contribution in this note (see Remark 2.3 below). We don't need to invoke univalence, the existence of propositional truncations or any other higher inductive type for our construction. But we do rely on function extensionality. An version of the following argument formalized in Agda [7] is available at [1,2].

***Proof*** Let $f : X \to Y$ and $g : Y \to X$ be embeddings of arbitrary types $X$ and $Y$. We say that $x : X$ is a $g$-point if for any $x_0 : X$ and $n : \mathbb{N}$ with $(g \circ f)^n(x_0) = x$, the $g$-fiber of $x_0$ is pointed. Using the assumption that $g$ is an embedding, we see that being a $g$-point is property rather than data, because subsingletons are closed under products by function extensionality.

Considering $x_0 = x$ and $n = 0$, we see that if $x$ is a $g$-point then the $g$-fiber of $x$ is pointed, and hence we get a function $g^{-1}$ of $g$-points of $X$ into $Y$. By construction, we have that $g(g^{-1}(x)) = x$. In particular, if $g(y)$ is a $g$-point for a given $y : Y$, we conclude that $g(g^{-1}(g(y))) = g(y)$, and because $g$, being an embedding, is left-cancellable, we get $g^{-1}(g(y)) = y$.

Now define $h : X \to Y$ by

$$h(x) = \begin{cases} g^{-1}(x) & \text{if } x \text{ is a } g\text{-point,} \\ f(x) & \text{otherwise.} \end{cases}$$

To conclude the proof, it is enough to show that $h$ is left-cancellable and split-surjective, as any such map is an equivalence.

To see that $h$ is left-cancellable, it is enough to show that the images of $f$ and $g^{-1}$ in the definition of $h$ are disjoint, because $f$ and $g^{-1}$ are left-cancellable. For that purpose, let $x$ be a non-$g$-point and $x'$ be a $g$-point, and, for the sake of contradiction, assume $f(x) = g^{-1}(x')$. Then $g(f(x)) = g(g^{-1}(x')) = x'$. Now, because if $g(f(x))$ were a $g$-point then so would be $x$, we conclude that it isn't, and hence neither is $x'$, which contradicts the assumption.

To see that $h$ is a split surjection, say that $x : X$ is an $f$-point if there are designated $x_0 : X$ and $n : \mathbb{N}$ with $(g \circ f)^n(x_0) = x$ and the $g$-fiber of $x_0$ empty. This is data rather than property, and so this notion could not have been used for the construction of $h$. But every non-$f$-point is a $g$-point, as we see by applying the principle of excluded middle to the $g$-fiber of $x_0$ in the definition of $g$-point.

**Claim 2.2** *If $g(y)$ is not a $g$-point, then there is a designated point $(x, p)$ of the $f$-fiber of $y$, with $x : X$ and $p : f(x) = y$, such that $x$ is not a $g$-point either.*

To prove the claim, first notice that it is impossible that $g(y)$ is not an $f$-point, by the above observation. But this is not enough to conclude that it is an $f$-point, because the principle of excluded middle applies to subsingletons only, which the notion of $f$-point isn't. However, it is readily seen that if $g(y)$ is an $f$-point, then there is a designated point $(x, p)$ in the $f$-fiber of $y$. From this it follows that it is impossible that the subtype of the fiber consisting of the elements $(x, p)$ with $x$ not a $g$-point is empty. But the $f$-fiber of $y$ is a subsingleton because $f$ is an embedding, and hence so is the subtype, and therefore the claim follows by double-negation elimination, which is a standard consequence of the principle of excluded middle.

We can now resume the proof that $h$ is a split surjection. For any $y : Y$, we check whether $g(y)$ is a $g$-point. If it is, we map $y$ to $g(y)$, and if it isn't we map $y$ to the point $x : X$ given by the claim, which concludes the proof of the theorem. □

***Remark 2.3*** So, in this argument we don't apply the principle of excluded middle to equality directly, which we wouldn't be able to as the types $X$ and $Y$ are not necessarily sets. We instead apply it to (1) the property of being a $g$-point, defined in terms of the fibers of $g$, to define $h$, (2) a fiber of $g$, and (3) a subtype of a fiber of $f$. These three types are propositions because the functions $f$ and $g$ are embeddings rather than merely left-cancellable maps.

***Remark 2.4*** If the type $X$ in the proof is connected, then every map of $X$ into a set is constant. In particular, the property of being a $g$-point is constant, because the type of truth values is a set (assuming univalence for subsingletons). Hence, by the principle of excluded middle, it is constantly true or constantly false, and so $h = g^{-1}$ or $h = f$, which means that one of the embeddings $f$ and $g$ is already an equivalence. Mike Shulman (personal communication) observed that this is true even without the principle of excluded middle: If $X$ is connected and we have an embedding $g : Y \to X$ and any function at all $f : X \to Y$, then $g$ is an equivalence. For any $x : X$, we have $\|g(f(x)) = x\|$ since $X$ is connected; thus $g$ is (non-split) surjective. But a surjective embedding is an equivalence.



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