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Abstract

Until recently, it was an important open problem in Fractal Geometry to determine whether there exists an iterated function system acting on $\mathbb{R}$ with no exact overlaps for which cylinders are super-exponentially close at all small scales. Iterated function systems satisfying these properties were shown to exist by the author and by Bárány and Käenmäki. In this paper we prove a general theorem on the existence of such iterated function systems within a parameterised family. This theorem shows that if a parameterised family contains two independent subfamilies, and the set of parameters that cause exact overlaps satisfies some weak topological assumptions, then the original family will contain an iterated function system satisfying the desired properties. We include several explicit examples of parameterised families to which this theorem can be applied.

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1 Introduction

Let $\Phi = \{\phi_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i \in \mathcal{I}}$ denote a finite set of contracting similarities acting on $\mathbb{R}^d$. We call Φ an iterated function system or IFS for short. Iterated function systems are useful tools for generating fractal sets. A well known result due to Hutchinson [9] states that for any IFS Φ there exists a unique non-empty compact set $X \subset \mathbb{R}^d$ satisfying

$$X = \bigcup_{i \in \mathcal{I}} \phi_i(X).$$

We call X the self-similar set of Φ. Self-similar sets often exhibit fractal properties. The middle-third Cantor set and the von-Koch curve are well known examples of self-similar sets.

A well studied and difficult problem is to determine the Hausdorff dimension of a general self-similar set. Given an IFS $\Phi = \{\phi_i\}_{i \in \mathcal{I}}$ we denote by r_i the similarity ratio of ϕ_i. We call the unique $s \geq 0$ satisfying $\sum_{i \in \mathcal{I}} r_i^s = 1$ the similarity dimension of Φ and denote it by $\dim_S \Phi$. The following upper bound for the Hausdorff dimension of a self-similar set is well known

$$\dim_H(X) \leq \min\{\dim_S \Phi, d\}. \tag{1.1}$$

It is often the case that equality holds in (1.1). To make progress with the problem of determining the Hausdorff dimension of a self-similar set one often uses self-similar measures. These are defined as follows: Given an IFS Φ and a probability vector $\mathbf{p} = (p_i)_{i \in \mathcal{I}}$, then there exists a unique Borel probability measure $\mu_{\mathbf{p}}$ satisfying

$$\mu_{\mathbf{p}} = \sum_{i \in \mathcal{I}} p_i \cdot \mu_{\mathbf{p}} \circ \phi_i^{-1}.$$

We call $\mu_{\mathbf{p}}$ the self-similar measure corresponding to Φ and $\mathbf{p}$. We define the dimension of a Borel probability measure μ to be

$$\dim \mu := \inf\{\dim_H(A) : \mu(A) = 1\}.$$

We remark that there are other well-studied notions of dimension for measures. Importantly for self-similar measures these alternative definitions typically give the same value as our definition. This is a consequence of the exact dimensionality of self-similar measures (see [5]). The following upper bound for the dimension of a self-similar measure is well known

$$\dim \mu_{\mathbf{p}} \leq \min \left\{ \frac{\sum_{i \in \mathcal{I}} p_i \log p_i}{\sum_{i \in \mathcal{I}} p_i \log r_i}, d \right\}. \tag{1.2}$$

Note that if we take $\mathbf{p} = (r_i^{\dim_S \Phi})_{i \in \mathcal{I}}$ then equality in (1.2) implies equality in (1.1). Self-similar measures are often easier to analyse than self-similar sets. As such, proving that equality holds in (1.2) often provides an easier route to proving equality in (1.1).

Self-similar measures are well known examples of multifractal measures, i.e. measures which exhibit different rates of scaling on small balls. A useful tool for describing the multifractal behaviour of a measure is provided by the L^q dimension. Let $q > 1$ and μ be a Borel probability measure on $\mathbb{R}^d$, we define the L^q dimension of μ to be

$$D(\mu, q) := \liminf_{n \to \infty} \frac{-\log_2 \sum_{(j_1, \ldots, j_d) \in \mathbb{Z}^d} \mu([j_1 \cdot 2^{-n}, (j_1 + 1) \cdot 2^{-n}) \times \cdots \times [j_d \cdot 2^{-n}, (j_d + 1) \cdot 2^{-n}))^q}{(q - 1)n}.$$

Let $T(\mu_{\mathbf{p}}, q)$ be the unique solution to the equation $\sum_{i \in \mathcal{I}} p_i^q r_i^{-s} = 1$. The following upper bound for the L^q dimension of a self-similar measure always holds

$$D(\mu_{\mathbf{p}}, q) \leq \min \left\{ \frac{T(\mu_{\mathbf{p}}, q)}{q - 1}, d \right\}. \tag{1.3}$$

Determining when we have equality in (1.1), (1.2), and (1.3) is an active and important area of research (see [6, 7, 8, 11, 12, 13, 14] and the references therein). When Φ is an IFS acting on $\mathbb{R}$ the only known mechanism preventing equality in (1.1), (1.2), or (1.3) is the existence of distinct $\mathbf{a}, \mathbf{b} \in \cup_{n=1}^{\infty} \mathcal{I}^n$ such that $\phi_{\mathbf{a}} = \phi_{\mathbf{b}}$. When such an $\mathbf{a}$ and $\mathbf{b}$ exists we say that Φ contains an exact overlap. Here and throughout we adopt the notational convention that $\phi_{\mathbf{a}} := \phi_{a_1} \circ \cdots \circ \phi_{a_n}$ for $\mathbf{a} = (a_1, \ldots, a_n)$. In higher dimensions there are other mechanisms preventing equality. In particular an IFS could force the self-similar set into a lower dimensional affine subspace of $\mathbb{R}^d$ without it containing an exact overlap. For more on the mechanisms preventing equality in higher dimensions see [8]. For iterated function systems acting on $\mathbb{R}$, it is conjectured that the only mechanism preventing equality in (1.1), (1.2), and (1.3) is the presence of an exact overlap. This conjecture is commonly referred to as the exact overlaps conjecture. Significant progress on this conjecture has been made in recent years. Rapaport in [11] proved that if Φ is an IFS acting on $\mathbb{R}$ with algebraic similarity ratios, then either Φ contains an exact overlap or we have equality in (1.1) and (1.2) for all self-similar measures. Building upon the work of Hochman in [7], Varju recently proved that for the unbiased Bernoulli convolution we have equality in (1.2)

when the underlying IFS does not contain an exact overlap [14]. The motivation behind this work comes from two important results on this topic due to Hochman [7] and Shmerkin [13]. Combining their results it follows that if Φ is an IFS acting on $\mathbb{R}$ and we have strict inequality in one of (1.1), (1.2), or (1.3), then Φ comes extremely close to containing an exact overlap. This notion of closeness is provided by the following distance function. Given two contracting similarities $\phi(x) = rx + t$ and $\phi'(x) = r'x + t'$, we let

$$d(\phi, \phi') := \begin{cases} \infty, & r \neq r' \\ |t - t'|, & r = r'. \end{cases}$$

Importantly $d(\phi, \phi') = 0$ if and only if $\phi = \phi'$. Given an IFS Φ, for any $n \in \mathbb{N}$ we let

$$\Delta_n(\Phi) := \min\{d(\phi_{\mathbf{a}}, \phi_{\mathbf{b}}) : \mathbf{a}, \mathbf{b} \in \mathcal{I}^n, \mathbf{a} \neq \mathbf{b}\}.$$

The results of Hochman and Shmerkin can now be more accurately summarised as follows.

Theorem 1.1. *Let Φ be an IFS acting on $\mathbb{R}$.*

- *[7, Theorem 1.1] If $\limsup_{n \to \infty} \frac{\log \Delta_n}{n} > -\infty$ then we have equality in (1.1) and (1.2) for all self-similar measures.*

- *[13, Theorem 6.6] If $\limsup_{n \to \infty} \frac{\log \Delta_n}{n} > -\infty$ then we have equality in (1.3) for all self-similar measures and $q > 1$.*

With Theorem 1.1 and the exact overlaps conjecture in mind, it is natural to ask whether there exists an IFS Φ, such that Φ does not contain an exact overlap yet $\Delta_n(\Phi) \to 0$ super-exponentially fast. This question was posed by Hochman in [6]. If no such Φ exists then Theorem 1.1 would imply the exact overlaps conjecture. Using ideas from [2] it was shown in [1] that such Φ do exist. Interestingly it was shown that $(\Delta_n(\Phi))_{n=1}^{\infty}$ can converge to zero arbitrarily fast without Φ containing an exact overlap. At the same time and using a different method Bárány and Käenmäki obtained the same result [3]. Recently Chen [4] altered the construction given in [1] to allow for algebraic contraction ratios.

In this paper we prove a general result for the existence of an IFS Φ within a parameterised family of IFSs such that $(\Delta_n(\Phi))_{n=1}^{\infty}$ converges to zero arbitrarily fast and Φ does not contain an exact overlap. This serves several purposes. First of all, the argument given is more general and intuitive than the one presented in [1]. The argument shows that if a parameterised family of IFSs contains two independent subfamilies, then it is reasonable to expect that the original family will contain a Φ satisfying our desired properties. Secondly, our more general result provides new examples of Φ satisfying these properties. In the final section of this paper we include several examples of families to which our result can be applied.

2 Preliminaries and our main result

In the statement of our main result we will be working in $\mathbb{R}^d$ and so require a higher dimensional analogue of $d(\cdot, \cdot)$ and $\Delta_n(\cdot)$. It is a well known fact that any contracting similarity ϕ acting on $\mathbb{R}^d$ can be uniquely written as $\phi(\mathbf{x}) = r \cdot O\mathbf{x} + t$ for some $r \in (0, 1)$, $O \in \mathcal{O}(d)$, and $\mathbf{t} \in \mathbb{R}^d$. Here $\mathcal{O}(d)$ denotes the group of $d \times d$ orthogonal matrices. As such, given two contracting similarities $\phi(\mathbf{x}) = r \cdot O\mathbf{x} + t$ and $\phi'(\mathbf{x}) = r'O'\mathbf{x} + t'$ the following quantity is well defined

$$d(\phi, \phi') = |t - t'| + |\log r - \log r'| + \|O - O'\|.$$

Here $\| \cdot \|$ denotes the operator norm. This is the distance function defined by Hochman in [8]. Importantly it has the property that $d(\phi, \phi') = 0$ if and only if $\phi = \phi'$. Given an IFS Φ and $n \in \mathbb{N}$ we define

$$\Delta_n(\Phi) := \min\{d(\phi_{\mathbf{a}}, \phi_{\mathbf{b}}) : \mathbf{a}, \mathbf{b} \in \mathcal{I}^n, \mathbf{a} \neq \mathbf{b}\}.$$

Note that the function $d(\cdot, \cdot)$ behaves differently to the distance function defined previously for similarities acting on $\mathbb{R}$. With this new distance function it is possible for $d(\phi, \phi')$ to take small values for ϕ and ϕ' with different similarity ratios. If we equip $\mathcal{O}(d)$ with the topology obtained under the usual identification with a subset of $\mathbb{R}^{d \times d}$, and identify the space of contracting similarities with $(0, 1) \times \mathcal{O}(d) \times \mathbb{R}^d$, then $d(\cdot, \cdot)$ is a continuous function from the space of pairs of similarities into $[0, \infty)$. This fact will be important in our proofs.

Our main result will be phrased in terms of parameterised families of IFSs. The general framework we use for such a parameterisation is as follows. Let $U \subset \mathbb{R}^{k_1}$ and $V \subset \mathbb{R}^{k_2}$ be open subsets of their respective Euclidean spaces. Let $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ be finite sets that are disjoint. We assume that for each $i \in \mathcal{I}_1$ there exists continuous functions $O_{i,1} : U \to \mathcal{O}(d)$, $r_{i,1} : U \to (0, 1)$, and $t_{i,1} : U \to \mathbb{R}^d$. Similarly, for each $i \in \mathcal{I}_2$ we assume that there exists continuous functions $O_{i,2} : V \to \mathcal{O}(d)$, $r_{i,2} : V \to (0, 1)$, and $t_{i,2} : V \to \mathbb{R}^d$. Also, for each $i \in I_3$ we assume that there exists continuous functions $O_{i,3} : U \times V \to \mathcal{O}(d)$, $r_{i,3} : U \times V \to (0, 1)$, and $t_{i,3} : U \times V \to \mathbb{R}^d$. Equipped with these functions we can define three parameterised families of iterated function systems. Given $u \in U$ we let

$$\Phi_u := \{\phi_{i,u}(x) = r_{i,1}(u) \cdot O_{i,1}(u)x + t_{i,1}(u)\}_{i \in \mathcal{I}_1},$$

and similarly given $v \in V$ we let

$$\Phi_v := \{\phi_{i,v}(x) = r_{i,2}(v) \cdot O_{i,2}(v)x + t_{i,2}(v)\}_{i \in \mathcal{I}_2}.$$

Given $(u, v) \in U \times V$ we define

$$\Phi_{u,v} := \Phi_u \cup \Phi_v \cup \{\phi_{i,u,v}(x) = r_{i,3}(u, v) \cdot O_{i,3}(u, v)x + t_{i,3}(u, v)\}_{i \in \mathcal{I}_3}.$$

Note that all three of these iterated function systems are acting on $\mathbb{R}^d$. In our applications we may simply take $\mathcal{I}_3 = \emptyset$ and $\Phi_{u,v} = \Phi_u \cup \Phi_v$.

We define

$$H_1(n) := \{u \in U : \phi_{\mathbf{a},u} = \phi_{\mathbf{b},u} \text{ for some } \mathbf{a}, \mathbf{b} \in \mathcal{I}_1^n, \mathbf{a} \neq \mathbf{b}\}$$

and

$$H_1^*(n) := H_1(n) \setminus \cup_{j=1}^{n-1} H_1(j).$$

We use $H_2(n)$ and $H_2^*(n)$ to denote the corresponding sets for the family $\{\Phi_v\}_{v \in V}$. We let

$$H_3(n) := \{(u, v) \in U \times V : \phi_{\mathbf{a},u,v} = \phi_{\mathbf{b},u,v} \text{ for some } \mathbf{a}, \mathbf{b} \in (\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3)^n, \mathbf{a} \neq \mathbf{b}\}$$

and

$$H_3^*(n) := H_3(n) \setminus \cup_{j=1}^{n-1} H_3(j).$$

The following statement is the main result of this paper.

Theorem 2.1. *Let* $\{\Phi_u\}_{u \in U}, \{\Phi_v\}_{v \in V},$ *and* $\{\Phi_{u,v}\}_{(u,v) \in U \times V}$ *be parameterised families of iterated function systems as defined above. Suppose that the following properties are satisfied:*

1. $\cup_{n=1}^{\infty} H_1(n) \neq \emptyset$ *and* $\cup_{n=1}^{\infty} H_2(n) \neq \emptyset$.

2. *For any* $u \in H_1(n_0)$, $v \in H_2(m_0)$, *and* $\epsilon > 0$, *there exists* $u_1 \in \cup_{n=\max\{n_0,m_0\}+1}^{\infty} H_1^*(n)$ *such that:*

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(a) $\|u - u_1\|_\infty < \epsilon$.

(b) For any $\epsilon' > 0$, there exists $v_1 \in \cup_{n=\max\{n_0,m_0\}+1}^\infty H_2^*(n)$ such that $\|v - v_1\|_\infty < \epsilon'$ and $(u_1, v_1) \notin H_3(\max\{n_0, m_0\})$.

Then for any sequence $(\omega_n)_{n=1}^\infty$ of strictly positive real numbers, there exists $(u^*, v^*) \in U \times V$ such that $\Delta_n(\Phi_{u^*, v^*}) \leq \omega_n$ for all n sufficiently large and Φ_{u^*, v^*} contains no exact overlaps.

We emphasise that the parameter ϵ' appearing in 2b. can be chosen to depend upon u_1. We will use this fact in our proof.

Theorem 2.1 can be used to recover and strengthen the results of [1] and [4], see Example 3.1. Importantly the argument given bypasses the need to rely on properties of continued fractions. The following corollary is also implied by this theorem.

Corollary 2.2. *Let $\{\Phi_u\}_{u \in U}, \{\Phi_v\}_{v \in V}$ and $\{\Phi_{u,v}\}_{(u,v) \in U \times V}$ be parameterised families of iterated function systems as defined above. Suppose that the following properties are satisfied:*

1. *$\cup_{n=1}^\infty H_1(n)$ and $\cup_{n=1}^\infty H_2(n)$ are dense in U and V respectively.*

2. *For any $n \in \mathbb{N}$ the sets $H_1(n)$ and $H_2(n)$ are both nowhere dense.*

3. *Let $n_0 < n_1$. For any $u \in H_1^*(n_1)$ the set*

$$\{v \in V : \phi_{\mathbf{a},u,v} = \phi_{\mathbf{b},u,v} \text{ for some } \mathbf{a}, \mathbf{b} \in (\mathcal{I}_1 \cup \mathcal{I}_2 \cup I_3)^{n_0}, \mathbf{a} \neq \mathbf{b}\}$$

   is nowhere dense.

Then for any sequence $(\omega_n)_{n=1}^\infty$ of strictly positive real numbers, there exists $(u^*, v^*) \in U \times V$ such that $\Delta_n(\Phi_{u^*, v^*}) \leq \omega_n$ for all n sufficiently large and Φ_{u^*, v^*} contains no exact overlaps.

The hypotheses appearing in Corollary 2.2 are natural. For many parameterised families of overlapping IFSs the set of parameters causing exact overlaps are dense. For any $n \in \mathbb{N}$ the set $H_1(n)$ is closed, and so if it failed to be nowhere dense then there would be an non-empty open subset contained in $H_1(n)$. Such a set would mean that there exists a sizeable part of U for which the corresponding IFSs have overlaps of length n effectively built in. The same is true for $H_2(n)$. Similarly, if the third condition was not satisfied it would mean that there exists $u' \in H_1^*(n_1)$ for which the parameterised family of IFSs given by $\{\Phi_{u',v}\}_{v \in V}$ would effectively have exact overlaps of length n_0 built in within some non-empty open subset of V.

2.1 Proof of Theorem 2.1

The following lemma records several elementary facts that we will need in our proof. These facts follow immediately from the definitions and their proofs are omitted.

Lemma 2.3.   *1. For any IFS Φ the sequence $(\Delta_n(\Phi))_{n=1}^\infty$ is decreasing.*

2. *Let Φ_1, Φ_2, and Φ_3 be three IFSs satisfying $\Phi_1 \subseteq \Phi_3$ and $\Phi_2 \subseteq \Phi_3$. Then for any $n \in \mathbb{N}$ we have*

$$\Delta_n(\Phi_3) \leq \min\{\Delta_n(\Phi_1), \Delta_n(\Phi_2)\}.$$

3. *For any $n \in \mathbb{N}$ the sets $H_1(n)$, $H_2(n)$, and $H_3(n)$ are closed.*

The proof of Theorem 2.1 relies on a simple strategy that we outline here. Given a sequence of positive real numbers (ω_n), it is a relatively easy task to construct an element $u^* \in U$ such that $\Delta_n(\Phi_{u^*}) \leq \omega_n$ for all n belonging to infinitely many long stretches of the natural numbers. This can be achieved by requiring u^* be extremely well approximated at infinitely many scales

by parameters that cause exact overlaps. The issue here is that there might exist gaps between those stretches of natural numbers for which $\Delta_n(\Phi_{u^*}) \leq \omega_n$ holds. The trick is to use the family $\{\Phi_v\}_{v \in V}$ to find an element $v^* \in V$ for which the set of n for which $\Delta_n(\Phi_{v^*}) \leq \omega_n$ fills in these gaps. Ensuring Φ_{u^*,v^*} contains no exact overlaps can be achieved by a topological argument.

In our proof we let $B(x,r)$ denote the open ball of radius r centred at x with respect to the infinity norm.

Proof of Theorem 2.1. Let us start by fixing $(\omega_n)_{n=1}^{\infty}$ a sequence of strictly positive real numbers. The element (u^*, v^*) we construct will belong to the countable intersection of a nested collection of closed balls. We define these balls below via an inductive argument.

Base case. Pick $u_0 \in H_1(n_0)$ and $v_0 \in H_2(m_0)$ for some $n_0, m_0 \in \mathbb{N}$. Both u_0 and v_0 exist by property 1. By definition there exists distinct $\mathbf{a}_0, \mathbf{b}_0 \in \mathcal{I}_1^{n_0}$ such that $\phi_{\mathbf{a}_0,u_0} = \phi_{\mathbf{b}_0,u_0}$. Since the maps $U_{i,1}$, $r_{i,1}$, and $t_{i,1}$ are continuous, and the distance function d is continuous, there exists $\epsilon_0 > 0$ such that

$$B(u_0, \epsilon_0) \subseteq \left\{ u : d(\phi_{\mathbf{a}_0,u}, \phi_{\mathbf{b}_0,u}) \leq \min_{n_0 \leq n \leq \max\{n_0,m_0\}} \omega_n \right\}.$$

It follows from the definition of $\Delta_n(\cdot)$ and Lemma 2.3.1 that

$$B(u_0, \epsilon_0) \subseteq \{u : \Delta_n(\Phi_u) \leq \omega_n \text{ for all } n_0 \leq n \leq \max\{n_0, m_0\}\}. \tag{2.1}$$

Let $u_1 \in U$ be the element whose existence is guaranteed by property 2. for the choice of parameters u_0, v_0 and ϵ_0. Therefore $u_1 \in H_1^*(n_1)$ for some $n_1 > \max\{n_0, m_0\}$ and $u_1 \in B(u_0, \epsilon_0)$.

Since $v_0 \in H_2(m_0)$ there exists distinct $\mathbf{a}_0', \mathbf{b}_0' \in \mathcal{I}_2^{m_0}$ such that $\phi_{\mathbf{a}_0',v_0} = \phi_{\mathbf{b}_0',v_0}$. By the same reasoning as above, we may pick $\epsilon_0' > 0$ such that

$$B(v_0, \epsilon_0') \subseteq \{v : \Delta_n(\Phi_v) \leq \omega_n \text{ for all } m_0 \leq n \leq n_1\}. \tag{2.2}$$

Since u_1 is the point whose existence is guaranteed by property 2, we know that there exists $v_1 \in H_2^*(m_1)$ for some $m_1 > \max\{n_0, m_0\}$ such that $v_1 \in B(v_0, \epsilon_0')$ and $(u_1, v_1) \notin H_3(\max\{n_0, m_0\})$. By Lemma 2.3.3 we know that $H_3(\max\{n_0, m_0\})$ is closed. Therefore there exists sufficiently small $\delta_1 > 0$ such that
$$\overline{B((u_1, v_1), \delta_1)} \cap H_3(\max\{n_0, m_0\}) = \emptyset. \tag{2.3}$$

Moreover, since $u_1 \in B(u_0, \epsilon_0)$ and $v_1 \in B(v_0, \epsilon_0)$, we may also assume that δ_1 is sufficiently small that
$$\overline{B((u_1, v_1), \delta_1)} \subseteq B(u_0, \epsilon_0) \times B(v_0, \epsilon_0').$$

Therefore by (2.1), (2.2), and Lemma 2.3.2, we may deduce that

$$\Delta_n(\Phi_{u,v}) \leq \omega_n \text{ for all } n_0 \leq n \leq n_1$$

for all $(u,v) \in \overline{B((u_1, v_1), \delta_1)}$.

Inductive step. Suppose that we have constructed sequences $(u_k)_{k=0}^{K}$, $(v_k)_{k=0}^{K}$, $(n_k)_{k=0}^{K}$, $(m_k)_{k=0}^{\infty}$, and $(\delta_k)_{k=1}^{K}$ such that the following properties are satisfied:

(a) $u_k \in H_1^*(n_k)$ and $v_k \in H_2^*(m_k)$ for all $1 \leq k \leq K$.

(b) Both $(n_k)_{k=0}^{K}$ and $(m_k)_{k=0}^{K}$ are strictly increasing sequences.

(c) $(\overline{B((u_k, v_k), \delta_k)})_{k=1}^{K}$ is a nested sequence.

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(d) $\overline{B((u_k, v_k), \delta_k)} \cap H_3(\max\{n_{k-1}, m_{k-1}\}) = \emptyset$ for all $1 \leq k \leq K$.

(e) For all $(u, v) \in \overline{B((u_K, v_K), \delta_K)}$ we have

$$\Delta_n(\Phi_{u,v}) \leq \omega_n \text{ for all } n_0 \leq n \leq n_K.$$

In the base case we constructed the relevant sequences for $K = 1$. We now show how these sequences are defined for $K + 1$.

By definition there exists distinct $\mathbf{a}_K, \mathbf{b}_K \in \mathcal{I}_1^{n_K}$ such that $\phi_{\mathbf{a}_K, u_K} = \phi_{\mathbf{b}_K, u_K}$. Therefore by analogous reasoning to that given in the base case, there exists $\epsilon_K > 0$ such that

$$B(u_K, \epsilon_K) \subseteq \{u : \Delta_n(\Phi_u) \leq \omega_n \text{ for all } n_K \leq n \leq \max\{n_K, m_K\}\} \cap B(u_K, \delta_K). \quad (2.4)$$

Let $u_{K+1} \in U$ be the element whose existence is guaranteed by property 2. for the choice of parameters u_K, v_K, and ϵ_K. Therefore $u_{K+1} \in H_1^*(n_{K+1})$ for some $n_{K+1} > \max\{n_K, m_K\}$ and $u_{K+1} \in B(u_K, \epsilon_K)$.

Since $v_K \in H_2^*(m_K)$ there exists distinct $\mathbf{a}_K', \mathbf{b}_K' \in \mathcal{I}_2^{m_k}$ such that $\phi_{\mathbf{a}_K', v_K} = \phi_{\mathbf{b}_K', v_K}$. By analogous reasoning to that given in the base case, it follows that there exists $\epsilon_K' > 0$ such that

$$B(v_K, \epsilon_K') \subseteq \{v : \Delta_n(\Phi_v) \leq \epsilon_n \text{ for all } m_k \leq n \leq n_{k+1}\} \cap B(v_K, \delta_K). \quad (2.5)$$

By the definition of u_{K+1}, we know that there exists $v_{K+1} \in H_2^*(m_{K+1})$ for some $m_{K+1} > \max\{n_K, m_K\}$ such that $v_{K+1} \in B(v_K, \epsilon_K')$ and $(u_{K+1}, v_{K+1}) \notin H_3(\max\{n_K, m_K\})$. By Lemma 2.3.3 we know that the set $H_3(\max\{n_K, m_K\})$ is closed. Therefore there exists $\delta_{K+1} > 0$ such that

$$\overline{B((u_{K+1}, v_{K+1}), \delta_{K+1})} \cap H_3(\max\{n_K, m_K\}) = \emptyset.$$

Since $u_{K+1} \in B(u_K, \epsilon_K)$ and $v_{K+1} \in B(v_K, \epsilon_K')$ we may also assume that δ_{K+1} is sufficiently small so that

$$\overline{B((u_{K+1}, v_{K+1}), \delta_{K+1})} \subseteq B(u_K, \epsilon_K) \times B(v_K, \epsilon_K'). \quad (2.6)$$

It follows from Lemma 2.3.2, (2.4), (2.5), and (2.6) that if $(u, v) \in \overline{B((u_{K+1}, v_{K+1}), \delta_{K+1})}$ then

$$\Delta_n(\Phi_{u,v}) \leq \epsilon_n \text{ for } n_K \leq n \leq n_{K+1}.$$

Moreover, it follows from (2.4), (2.5), and (2.6) that $\overline{B((u_{K+1}, v_{K+1}), \delta_{K+1})} \subseteq \overline{B((u_K, v_K), \delta_K)}$. Therefore by property (e) we may conclude that

$$\Delta_n(\Phi_{u,v}) \leq \epsilon_n \text{ for } n_0 \leq n \leq n_{K+1}$$

for all $(u, v) \in \overline{B((u_{K+1}, v_{K+1}), \delta_{K+1})}$. For these choices of $u_{K+1}, v_{K+1}, n_{K+1}, m_{K+1}$ and δ_{K+1} we see that properties $(a) - (e)$ are satisfied for level $K + 1$. This completes our inductive step.

Conclusion of proof. Repeating the inductive step indefinitely we can define sequences $(u_k)_{k=0}^\infty$, $(v_k)_{k=0}^\infty$, $(n_k)_{k=0}^\infty$, $(m_k)_{k=0}^\infty$, and $(\delta_k)_{k=1}^\infty$ such that properties $(a) - (e)$ hold for any choice of K.

Since $(\overline{B((u_k, v_k), \delta_k)})_{k=1}^\infty$ is a nested sequence of closed balls we have

$$\bigcap_{k=1}^\infty \overline{B((u_k, v_k), \delta_k)} \neq \emptyset.$$

Taking (u^*, v^*) in this intersection it follows from property (e) that $\Delta_n(\Phi_{u^*, v^*}) \leq \omega_n$ for all $n \geq n_0$. Moreover by property (d) we may also conclude that $(u^*, v^*) \notin H_3(\max\{n_k, m_k\})$ for

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any $k \in \mathbb{N}$. Since $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ are both strictly increasing sequences and $H_3(n) \subseteq H_3(n')$ for $n \leq n'$, it follows that $(u^*, v^*) \notin H_3(n)$ for any n. Therefore Φ_{u^*,v^*} also contains no exact overlaps as required.

$\square$

Remark 2.4. Adapting the proof of Theorem 2.1 it can be shown that the set

$$\{(u,v) : \Phi_{u,v} \text{ contains no exact overlaps and } \Delta_n(\Phi_{u,v}) \leq \omega_n \text{ for all } n \text{ sufficiently large}\}$$

is dense in $U \times V$ under the additional assumption that both $\cup_{n=1}^\infty H_1(n)$ and $\cup_{n=1}^\infty H_2(n)$ are dense in U and V respectively.

Remark 2.5. In the statement of Theorem 2.1 it is possible to strengthen the assertion $\Delta_n(\Phi_{u^*,v^*}) \leq \omega_n$ for all n sufficiently large. A careful inspection of the proof shows that we can assert that there exists (u^*, v^*) such that $\Delta_n(\Phi_{u^*,v^*}) \leq \omega_n$ for all $n \geq \min\{n : H_1(n) \neq \emptyset\}$.

3 Examples

In this section we detail two examples of parameterised families of IFSs to which we can apply Theorem 2.1. Our first family does not impose any algebraic conditions on the similarity ratio and this quantity can be taken to be transcendental. This is particularly relevant given the aforementioned result of Rapaport [11] which establishes the exact overlaps conjecture when the underlying contraction ratios are all algebraic. The second family is inspired by the Bernoulli convolution and establishes the existence of a non-equicontractive IFS with super-exponentially close cylinders and no exact overlaps.

Note that the function $\Delta_n(\cdot)$ appearing in Theorem 2.1 is defined using a distance function for which similarities with different contraction ratios can be close. For the examples given below we may insist $\Delta_n(\cdot)$ be defined using the following stricter distance function and still guarantee the existence of $\Phi_{\mathbf{u}^*,\mathbf{v}^*}$ without exact overlaps such that $(\Delta_n(\Phi_{\mathbf{u}^*,\mathbf{v}^*}))_{n=1}^\infty$ converges to zero at any desired speed. Given two contracting similarities $\phi(\mathbf{x}) = r \cdot O\mathbf{x} + t$ and $\phi'(\mathbf{x}) = r' \cdot O'\mathbf{x} + t'$ acting on $\mathbb{R}^d$ we define our stricter distance function via the equation

$$d(\phi, \phi') := \begin{cases} \infty, & r \neq r' \text{ or } O \neq O' \\ |t - t'|, & r = r' \text{ and } O = O'. \end{cases}$$

Example 3.1. In this example $\lambda \in (0, 1/2]$ is some fixed parameter. We could take $U = V = \mathbb{R}^d$, however for ease of exposition it is more convenient to set

$$U = V := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_j \notin \{0, 1\} \text{ for all } 1 \leq j \leq d\}.$$

Given a vector $\mathbf{u} = (u_1, \ldots, u_d) \in U$ we associate the IFS

$$\Phi_{\mathbf{u}} := \left\{ \lambda(\mathbf{x} + \mathbf{a}) : \mathbf{a} \in \prod_{j=1}^d \{0, 1, u_j, 1 + u_j\} \right\}.$$

Similarly, given $\mathbf{v} = (v_1, \ldots, v_d) \in V$ we let

$$\Phi_{\mathbf{v}} := \left\{ \lambda(\mathbf{x} + \mathbf{a}) : \mathbf{a} \in \prod_{j=1}^d \{0, 1, v_j, 1 + v_j\} \right\}.$$

Moreover, given $\mathbf{u}$ and $\mathbf{v}$ let $\mathbf{T_{u,v}} \subset \mathbb{R}^d$ be any finite set of vectors satisfying

$$\prod_{j=1}^{d}\{0,1,u_j,1+u_j\} \cup \prod_{j=1}^{d}\{0,1,v_j,1+v_j\} \subseteq \mathbf{T_{u,v}} \subseteq \prod_{j=1}^{d}\{0,1,u_j,v_j,1+u_j,1+v_j,u_j+v_j,1+u_j+v_j\}.$$

Let

$$\Phi_{\mathbf{u,v}} = \{\lambda(\mathbf{x}+\mathbf{a}) : \mathbf{a} \in \mathbf{T_{u,v}}\}.$$

Notice that the translation vectors for $\Phi_{\mathbf{u}}$ depend continuously on $\mathbf{u}$, the translation vectors for $\Phi_{\mathbf{v}}$ depend continuously on of $\mathbf{v}$, and the remaining translation vectors for $\Phi_{\mathbf{u,v}}$ depend continuously on $(\mathbf{u},\mathbf{v})$. Thus $\{\Phi_{\mathbf{u}}\}, \{\Phi_{\mathbf{v}}\}$ and $\{\Phi_{\mathbf{u,v}}\}$ are three parameterised families of IFSs as considered in Theorem 2.1.

After a little calculation it can be shown that $\mathbf{u} \in H_1(n)$ if and only if[1]

$$\mathbf{u} \in \bigcup_{j=1}^{d}\left\{\mathbf{u} \in U : u_j = \frac{\sum_{i=1}^{n}\kappa_i\lambda^i}{\sum_{i=1}^{n}\delta_i\lambda^i}, \ (\kappa_i)_{i=1}^{n}, (\delta_i)_{i=1}^{n} \in \{-1,0,1\}^n \setminus \{(0)^n\}\right\}.$$

Obviously the same holds for $H_2(n)$ with the parameter $\mathbf{v}$ replacing $\mathbf{u}$. This shows that $H_1(n)$ and $H_2(n)$ are both non-empty for any $n \in \mathbb{N}$. Therefore property 1. from Theorem 2.1 holds for these families.

To see that property 2 from Theorem 2.1 holds for this family we fix $\mathbf{u} \in H_1(n_0)$, $\mathbf{v} \in H_2(m_0)$ and $\epsilon > 0$. Let $L_0 := \max\{n_0, m_0\}$ and

$$\mathcal{D}(\mathbf{u}) = \left\{j : u_j = \frac{\sum_{i=1}^{L_0}\kappa_i\lambda^i}{\sum_{i=1}^{L_0}\delta_i\lambda^i}, \ \text{for some } (\kappa_i)_{i=1}^{L_0}, (\delta_i)_{i=1}^{L_0} \in \{-1,0,1\}^{L_0} \setminus \{(0)^{L_0}\}\right\}.$$

$D(\mathbf{u})$ is the set of coordinate positions of $\mathbf{u}$ that give rise to an exact overlap caused by two distinct words of length at most L_0. Since $\mathbf{u} \in H_1(n_0)$ it follows from the characterisation of $H_1(n_0)$ above that the set $D(\mathbf{u})$ is non-empty. For each $N \in \mathbb{N}$, if $j \in \mathcal{D}(\mathbf{u})$ and

$$u_j = \frac{\sum_{i=1}^{L_0}\kappa_i\lambda^i}{\sum_{i=1}^{L_0}\delta_i\lambda^i}$$

then we define

$$u_{j,N} := \frac{\sum_{i=1}^{L_0}\kappa_i\lambda^i + \lambda^N}{\sum_{i=1}^{L_0}\delta_i\lambda^i},$$

and for all $j \notin \mathcal{D}(\mathbf{u})$ we set $u_{j,N} := u_j$. For each $N \in \mathbb{N}$ we then let $\mathbf{u}_N := (u_{1,N},\ldots,u_{d,N})$. The set

$$\left\{\frac{\sum_{i=1}^{L_0}\kappa_i\lambda^i}{\sum_{i=1}^{L_0}\delta_i\lambda^i}, \ \text{for some } (\kappa_i)_{i=1}^{L_0}, (\delta_i)_{i=1}^{L_0} \in \{-1,0,1\}^{L_0} \setminus \{(0)^{L_0}\}\right\}$$

is finite. Therefore for any N sufficiently large the parameter $u_{j,N}$ does not belong to this set for any j. It follows from the characterisation of $H_1(n)$ stated above that $\mathbf{u}_N \notin H_1(L_0)$ for N sufficiently large. Using the characterisation of $H_1(n)$ stated above again, and the fact $H_1(L_0) = \cup_{n=1}^{L_0}H_1(n)$, it follows that $\mathbf{u}_N \in \cup_{n=L_0+1}^{\infty}H_1^*(n)$ for N sufficiently large. Moreover, we may also assume that N is sufficiently large that $\mathbf{u}_N$ satisfies $\|\mathbf{u}-\mathbf{u}_N\|_{\infty} < \epsilon$. We define $\mathbf{u}_1 = \mathbf{u}_N$ for any N sufficiently large such that both of these properties hold.

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[1]Here we use our assumption $\lambda \in (0,1/2]$.

Let $\mathbf{u}_1 = (u_{1,1}, \ldots, u_{d,1})$. We've shown that $\mathbf{u}_1$ satisfies $2a$ from Theorem 2.1. To show that property $2b$ holds for this choice of $\mathbf{u}_1$ let us fix $\epsilon' > 0$. By a simple calculation, it can be shown that for $\mathbf{v}^* \in V$, if $(\mathbf{u}_1, \mathbf{v}^*) \in H_3(L_0)$ then

$$\mathbf{v}^* \in \bigcup_{j=1}^{d} \left\{ \mathbf{v} \in V : v_j = \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i + u_{j,1} \sum_{i=1}^{L_0} \delta_i \lambda^i}{\sum_{i=1}^{L_0} \gamma_i \lambda^i}, \text{ for some } (\gamma_i) \in \{-1, 0.1\}^{L_0} \setminus \{(0)^{L_0}\} \right.$$

$$\left. \text{and } (\kappa_i), (\delta_i) \in \{-1, 0, 1\}^{L_0} \right\}.$$

For our previously fixed value of $\mathbf{v}$ let

$$\mathcal{D}'(\mathbf{v}) = \left\{ j : v_j = \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i}{\sum_{i=1}^{L_0} \delta_i \lambda^i} \text{ for some } (\kappa_i), (\delta_i) \in \{-1, 0, 1\}^{L_0} \setminus \{(0)^{L_0}\} \right\}$$

and

$$\mathcal{D}''(\mathbf{v}) = \left\{ j : v_j = \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i + u_{j,1} \sum_{i=1}^{L_0} \delta_i \lambda^i}{\sum_{i=1}^{L_0} \gamma_i \lambda^i}, (\gamma_i) \in \{-1, 0, 1\}^{L_0} \setminus \{(0)^{L_0}\}, (\kappa_i), (\delta_i) \in \{-1, 0, 1\}^{L_0} \right\}.$$

Since $\mathbf{v} \in H_2(m_0)$ it follows that $\mathbf{v} \in H_2(L_0)$ and therefore $\mathcal{D}'(\mathbf{v})$ is non-empty.

For each $N \in \mathbb{N}$, if $j \in \mathcal{D}'(\mathbf{v})$ and

$$v_j = \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i}{\sum_{i=1}^{L_0} \delta_i \lambda^i},$$

then we define

$$v_{j,N} := \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i + \lambda^N}{\sum_{i=1}^{L_0} \delta_i \lambda^i},$$

if $j \in \mathcal{D}''(\mathbf{v}) \setminus \mathcal{D}'(\mathbf{v})$ and

$$v_j = \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i + u_{j,1} \sum_{i=1}^{L_0} \delta_i \lambda^i}{\sum_{i=1}^{L_0} \gamma_i \lambda^i},$$

then define

$$v_{j,N} := \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i + \lambda^N + u_{j,1} \sum_{i=1}^{L_0} \delta_i \lambda^i}{\sum_{i=1}^{L_0} \gamma_i \lambda^i},$$

and if $j \notin \mathcal{D}'(\mathbf{v}) \cup \mathcal{D}''(\mathbf{v})$ then

$$v_{j,N} := v_j.$$

For each $N \in \mathbb{N}$ set $\mathbf{v}_N := (v_{1,N}, \ldots, v_{j,N})$.

The sets

$$\left\{ \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i}{\sum_{i=1}^{L_0} \delta_i \lambda^i}, (\kappa_i), (\delta_i) \in \{-1, 0, 1\}^{L_0} \setminus \{(0)^{L_0}\} \right\}$$

and

$$\left\{ \frac{\sum_{i=1}^{L_0} \kappa_i \lambda^i + u_{j,1} \sum_{i=1}^{L_0} \delta_i \lambda^i}{\sum_{i=1}^{L_0} \gamma_i \lambda^i}, (\gamma_i) \in \{-1, 0.1\}^{L_0} \setminus \{(0)^{L_0}\}, (\kappa_i), (\delta_i) \in \{-1, 0, 1\}^{L_0} \right\}$$

are both finite. Therefore for any N sufficiently large, $v_{j,N}$ does not belong to either of these sets for any j. Therefore $\mathbf{v}_N \in \cup_{n=L_0+1}^{\infty} H_2^*(n)$ and $(\mathbf{u}_1, \mathbf{v}_N) \notin H_3(L_0)$ for N sufficiently large. Moreover, for N sufficiently large we also have $\|\mathbf{v} - \mathbf{v}_N\|_\infty < \epsilon'$. We take $\mathbf{v}_1 = \mathbf{v}_N$ for any N sufficiently large such that each of these properties hold. For this choice of $\mathbf{v}_1$ we see that property $2b$ from Theorem 2.1 is satisfied. As such this theorem can be applied to conclude that given any sequence (ω_n) of strictly positive real numbers, there exists an IFS $\Phi_{\mathbf{u}^*, \mathbf{v}^*}$ within this family without exact overlaps such that $\Delta_n(\Phi_{\mathbf{u}^*, \mathbf{v}^*}) \leq \omega_n$ for all n sufficiently large.

Example 3.2. In this example $U = V = (1/2, 1)$. For $u \in (1/2, 1)$ we let

$$\Phi_u := \{u(x+1), u(x+2)\},$$

and for $v \in (1/2, 1)$ we let

$$\Phi_v := \{v(x+1), v(x+2)\}.$$

Moreover, for $(u, v) \in (1/2, 1) \times (1/2, 1)$ let

$$\Phi_{u,v} := \{u(x+1), u(x+2), v(x+1), v(x+2)\}.$$

It is straightforward to show that $u \in H_1(n)$ if and only if

$$u \in \left\{ u : \sum_{i=1}^{n} \kappa_i u^i = 0 \text{ for some } (\kappa_i) \in \{-1, 0, 1\}^n \setminus \{(0)^n\} \right\}.$$

An equivalent characterisation exists for $H_2(n)$ with u replaced by v. The set of zeros of non-zero polynomials with coefficients belonging to the set $\{-1, 0, 1\}$ is well known to be dense in $(1/2, 1)$. For a proof of this statement see the proof of Corollary 3 from [10]. It follows that $\cup_{n=1}^{\infty} H_1(n)$ and $\cup_{n=1}^{\infty} H_2(n)$ are both non-empty and property 1 from Theorem 2.1 is satisfied. To show that property 2 holds for this family fix $u \in H_1(n_0), v \in H_2(m_0)$ and $\epsilon > 0$. Let $L_0 = \max\{n_0, m_0\}$. Since the set of zeros of non-zero polynomials with coefficients belonging to the set $\{-1, 0, 1\}$ is dense, we can choose $u_1 \in \cup_{n=L_0+1}^{\infty} H_1^*(n)$ such that $\|u - u_1\|_\infty < \epsilon$ and u_1 is not a zero for any non-zero polynomial of degree at most L_0 with coefficients belonging to the set $\{-2, -1, 0, 1, 2\}$. Here we used the fact that the set of zeros of non-zero polynomials of degree at most L_0 with coefficients belonging to the set $\{-2, -1, 0, 1, 2\}$ is a finite set. We've shown that property 2a from Theorem 2.1 is satisfied by this choice of u_1. It remains to show that property 2b. is also satisfied. As such let us fix $\epsilon' > 0$. For an arbitrary choice of $v^* \in V$ one can check that if $(u_1, v^*) \in H_3(L_0)$, then there exists $(\kappa_{i,0}), \ldots, (\kappa_{i,L_0}) \in \{-2, -1, 0, 1, 2\}^{L_0+1}$ such that one of these sequences is non-zero and

$$\sum_{i=0}^{L_0} \kappa_{i,0}(u_1)^i + \left(\sum_{i=0}^{L_0} \kappa_{i,1}(u_1)^i \right) v^* + \cdots \left(\sum_{i=0}^{L_0} \kappa_{i,L_0}(u_1)^i \right) (v^*)^{L_0} = 0. \tag{3.1}$$

Equation (3.1) shows that if $(u_1, v^*) \in H_3(L_0)$ then v^* is the zero of a polynomial of degree L_0 with coefficients belonging to the set $\{\sum_{i=0}^{L_0} \kappa_i(u_1)^i : (\kappa_i) \in \{-2, -1, 0, 1, 2\}^{L_0+1}\}$. Moreover, since one of $(\kappa_{i,0}), \ldots, (\kappa_{i,L_0})$ is non-zero and u_1 is not the zero of any polynomial of degree at most L_0 with coefficients belonging to the set $\{-2, -1, 0, 1, 2\}$, we may conclude that if $(u_1, v^*) \in H_3(L_0)$ then v^* is the root of a non-zero polynomial with coefficients belonging to the set $\{\sum_{i=0}^{L_0} \kappa_i(u_1)^i : (\kappa_i) \in \{-2, -1, 0, 1, 2\}^{L_0+1}\}$ of degree at most L_0. There are only finitely many such v^*. Therefore appealing to the density of $\cup_{n=1}^{\infty} H_2(n)$ we may pick $v_1 \in \cup_{n=L_0+1}^{\infty} H_2^*(n)$ such that $\|v - v_1\|_\infty < \epsilon'$ and $(u_1, v_1) \notin H_3(L_0)$. Therefore property 2b is satisfied for this choice of v_1. By Theorem 2.1 there exists $\Phi_{\mathbf{u}^*, \mathbf{v}^*}$ within this family with no exact overlaps such that $(\Delta_n(\Phi_{\mathbf{u}^*, \mathbf{v}^*}))$ converges to zero arbitrarily fast.

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