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Abstract

Let p be a prime. In this paper we investigate finite $\mathcal{K}_{\{2,p\}}$-groups G which have a subgroup $H \le G$ such that $K \le H = N_G(K) \le \mathrm{Aut}(K)$ for K a simple group of Lie type in characteristic p, and $|G : H|$ is coprime to p. If G is of local characteristic p, then G is called almost of Lie type in characteristic p. Here G is of local characteristic p means that p divides $|G|$ and for all non-trivial p-subgroups P of G, and Q the largest normal p-subgroup in $N_G(P)$ we have the containment $C_G(Q) \le Q$. We determine details of the structure of groups which are almost of Lie type in characteristic p. In particular, in the case that the rank of K is at least 3 we prove that $G = H$. If H has rank 2 and K is not $\mathrm{PSL}_3(p)$ we determine all the examples where $G \neq H$. We further investigate the situation above in which G is of parabolic characteristic p. This is a weaker assumption than local characteristic p. In this case, especially when $p \in \{2, 3\}$, many more examples appear.

In the appendices we compile a catalogue of results about the simple groups with proofs. These result may be of independent interest.

1. Introduction

The classification theorem of the finite simple groups asserts that a non-abelian finite simple group is one of an alternating group, a group of Lie type defined over a finite field of characteristic p or one of the 26 sporadic simple groups. A description and many properties of these simple groups and, in particular, a definition of groups of Lie type is provided in the appendices. The statement suggests that a "generic" finite simple group is a group of Lie type defined over a finite field of characteristic p where p is a prime. It is therefore useful and interesting to prove theorems which characterize just these groups. This is an objective of this memoir.

A property of a finite simple group that suggests G could be a group of Lie type in characteristic p, originates from a signature property of such groups: if X is a group of Lie type in characteristic p, then, for any non-trivial p-subgroup P of X,

$$C_X(O_p(N_X(P))) \le O_p(N_X(P))$$

where, for a group L, $O_p(L)$ denotes the maximal normal p-subgroup of L. This follows from the Borel–Tits Theorem [**27**, Theorem 3.1.3] and is the property that we shall impose on an arbitrary group G. Thus we say that G has *local characteristic* p provided p divides $|G|$ and for all non-trivial p-subgroups P of G,

$$C_G(O_p(N_G(P))) \le O_p(N_G(P)).$$

Definition. *A finite group G is almost a group of Lie type in characteristic p if and only if G is of local characteristic p, G has a subgroup H containing a Sylow p-subgroup of G such that $H = N_G(F^*(H))$ and $F^*(H)$ is a simple group of Lie type in characteristic p and of rank at least two.*

Recall that for a group L, $F^*(L)$ is the *generalized Fitting subgroup* of L and $K = F^*(L)$, is a non-abelian simple group if and only if $K \leq L \leq \mathrm{Aut}(K)$ (see [**2**, Chapter 11]).

Thus an almost group of Lie type G in characteristic p shares one of the significant properties of groups of Lie type and approximates a group of Lie type as it contains a group of Lie type defined in characteristic p which is close to being G in that it contains a Sylow p-subgroup of G. Main Theorem 2 determines all groups that are almost groups of Lie type under an additional assumption. It is remarkable that most groups that are almost of Lie type are indeed groups of Lie type. In fact if $F^*(H) \not\cong \mathrm{PSL}_3(p)$ and $p > 5$, then every group which is almost of Lie type is a group of Lie type defined in characteristic p. Furthermore, if G is almost a group of Lie type which is not a group of Lie type, then H turns out to be a maximal subgroup of G a fact which is not assumed in the definition.

In research that aims to identify the groups of Lie type, at a certain stage a subgroup which is a group of Lie type will be constructed. The aim is then to show that the subgroup is the whole group. This is precisely the point at which our theorems should be applied. The potential for our theorems to be applied to problems which aim to classify simple groups is the reason why we prove our theorems in an environment where they can be applied to a group which is not a a known simple group. This means that our theorems are applicable to all the programmes which aim to improve the classification of the finite simple groups. One such project [**47**] aims to understand the groups of local characteristic p via "unipotent" methods and our work is directly applicable in this case. For more on this see [**47**, Introduction]. In addition, especially, when $p = 2$ our theorems have the potential for application in the on-going programme of Gorenstein, Lyons and Solomon volumes to reclassify the simple groups [**25**]. With an eye to future developments and perhaps applications in instances where a large part of the classification has been proved by invoking an approach that comes from fusion systems [**4**], we are also interested in a less restrictive property requires that this containment holds for all non-trivial p-subgroups normal in some Sylow p-subgroup of G and in this case we say that G has *parabolic characteristic* p. For this we prove our Main Theorem 1 which just assumes G has parabolic characteristic p. There are many more groups in this case.

To state our main theorems we need further notation. A *p-local* subgroup of X, is by definition $N_X(P)$ for some non-trivial p-subgroup P of X. The proof of the classification of simple groups assumes inductively that G is a simple group of minimal order subject to not being included in the list of *known* finite simple groups as listed above. This means that if K is a proper subgroup of G, then all its composition factors are known simple groups. The subgroup K is called a $\mathcal{K}$-*group* and G is called $\mathcal{K}$-*proper*. For a set of prime numbers π, we say that G is a $\mathcal{K}_\pi$-*group* if for all $r \in \pi$, all subgroups of G which normalize a non-trivial r-subgroup are $\mathcal{K}$-groups. Note that our notion of a $\mathcal{K}_p$-group is stronger than that in [**47**] where only p-local subgroups are assumed to be $\mathcal{K}$-groups.

Main Theorem 1. *Suppose that p is a prime, G is a finite $\mathcal{K}_{\{2,p\}}$-group of parabolic characteristic p, and H is a subgroup of G of index coprime to p. Assume that $H = N_G(F^*(H))$ and $F^*(H)$ is a simple group of Lie type in characteristic p and of rank at least two.*

If there exists a p-local subgroup of G which contains a Sylow p-subgroup of H and is not contained in H, then either

(i) $p = 2$ *and* $(F^*(G), F^*(H)) = (\mathrm{Mat}(11), \mathrm{Sp}_4(2)')$, $(\mathrm{PSL}_4(3), \mathrm{SU}_4(2))$, $(\mathrm{G}_2(3), \mathrm{G}_2(2)')$, $(\mathrm{Mat}(23), \mathrm{PSL}_3(4))$, $(\mathrm{Alt}(10), \mathrm{SL}_4(2))$ *or* $(\mathrm{P}\Omega_8^+(3), \Omega_8^+(2))$;
(ii) $p = 3$ *and* $(F^*(G), F^*(H)) = (\mathrm{F}_4(2), \mathrm{PSL}_4(3))$, $(\mathrm{PSU}_6(2), \mathrm{PSU}_4(3))$, $(\mathrm{McL}, \mathrm{PSU}_4(3))$, $(\mathrm{Co}_2, \mathrm{PSU}_4(3))$, $({}^2\mathrm{E}_6(2), \mathrm{P}\Omega_7(3))$, $(\mathrm{M}(22), \mathrm{P}\Omega_7(3))$, $(\mathrm{M}(23), \mathrm{P}\Omega_8^+(3))$ *or* $(\mathrm{F}_2, \mathrm{P}\Omega_8^+(3))$;
(iii) $p = 5$ *and* $(G, F^*(H)) = (\mathrm{LyS}, \mathrm{G}_2(5))$; *or*
(iv) $p \in \{3, 5, 7, 13\}$ *and* $F^*(H) \cong \mathrm{PSL}_3(p)$.

If we impose the stronger restriction that G has local characteristic p, then we obtain the following almost complete description of the groups which are almost groups of Lie type.

Main Theorem 2. *Suppose that p is a prime, G is a finite $\mathcal{K}_{\{2,p\}}$-group which is almost a group of Lie type in characteristic p.*
If $G \neq H$, then one of the following holds:

(i) $p = 2$ *and* $(G, F^*(H)) = (\mathrm{Mat}(11), \mathrm{Sp}_4(2)')$, $(\mathrm{Mat}(23), \mathrm{PSL}_3(4))$, $(\mathrm{G}_2(3), \mathrm{G}_2(2)')$;
(ii) $p = 3$ *and* $F^*(H) = \mathrm{PSU}_4(3)$ *and* $G = \mathrm{McL}$ *or* $\mathrm{Aut}(\mathrm{McL})$;
(iii) $p = 5$ *and* $F^*(H) = \mathrm{G}_2(5)$ *and* $G = \mathrm{LyS}$; *or*
(iv) p *is odd and* $F^*(H) \cong \mathrm{PSL}_3(p)$.

We draw the following immediate corollary to Main Theorem 2.

Corollary. *Suppose that p is a prime, G is a finite $\mathcal{K}_{\{2,p\}}$-group which is almost a group of Lie type in characteristic p and that $F^*(H)$ has rank at least* 3. *Then $G = H$.*

The proof of Main Theorem 2 relies on the following theorem which also requires that G has local characteristic p.

THEOREM 1. *Suppose that p is a prime, G is a finite group which is almost a group of Lie type. If all p-local subgroups of G which contain a Sylow p-subgroup of H are contained in H, then*

(i) *either $G = H$ or one of the following holds:*
  (ia) *H is strongly p-embedded in G;*
  (ib) *$p = 5$, $F^*(H) \cong \mathrm{PSp}_4(5)$;*
  (ic) *$p = 7$, $F^*(H) \cong \mathrm{G}_2(7)$; or*
  (id) *$p = 3$ and $F^*(H) \cong \mathrm{PSL}_3(3)$ or $p = 7$ and $F^*(H) \cong \mathrm{PSL}_3(7)$.*

(ii) *If G is a $\mathcal{K}_2$-group, then either (ia) or (id) holds.*

(iii) *If G is a $\mathcal{K}_{\{2,p\}}$-group, then $G = H$ or p is odd and $F^*(H) \cong \mathrm{PSL}_3(p)$.*

Recall that for a prime r, a subgroup Y of a group X is *strongly r-embedded* in X if and only if Y has order divisible by r and $Y \cap Y^x$ has order coprime to r for all $x \in X \setminus Y$. Strongly 2-embedded subgroups are often referred to a *strongly embedded* subgroups.

The statement of Theorem 1 reveals our strategy for its proof and is formulated to show exactly which type of $\mathcal{K}$-group hypothesis is required at each step. Assuming that $G \neq H$, the main theorem from [**66**] can be applied so long as H has rank at least 3 or $p = 2$. Their theorem, which relies on the work of Bundy, Hebbinghaus and Stellmacher [**15**], does not require any $\mathcal{K}$-group assumption. Our proof of Theorem 2 extends the work of [**66**] to the case when the Lie rank is 2 and p is odd and is inspired by the work in the aforementioned paper. The anomalies listed as (ib) and (ic) are caused by the existence of certain exotic fusion systems [**62**]. This means that these cases cannot be eliminated using p-local methods alone. Thus we consider centralizers of involutions in these cases, and so we require a $\mathcal{K}_2$-group assumption to recognize composition facts in the centralizers of involutions. This leads to the elimination of (ib) and (ic) and thus proves (ii). Finally (iii) is obtained as an application of [**9**] and [**56, 57**]. The configuration when $F^*(H) \cong \mathrm{PSL}_3(p)$ and H is strongly p-embedded in G in Theorem 1 cannot be handled as we have no proof that this group cannot be strongly p-embedded. New ideas are needed to make progress with this problem.

In Main Theorem 1 (iv) where we have $p \in \{3, 5, 7, 13\}$ and $F^*(H) \cong \mathrm{PSL}_3(p)$, the embedding of $\mathrm{PSL}_3(3)$ into ${}^2\mathrm{F}_4(2)'$ and $\mathrm{PSL}_3(7)$ into O'N shows that the latter two groups are almost groups of Lie type. The configurations with $p = 5$ and 13 appear not to exist, however, in F_3

and F_1 there are fusion systems which carry all the p-local structure we can glean from H, but there is no subgroup $\mathrm{PSL}_3(5)$ in F_3 or $\mathrm{PSL}_3(13)$ in F_1. Using the classification theorem, we know there are no groups which are almost groups of Lie type with these latter structures. We also remark that the exceptions in Main Theorem 1 (iv) also appear in [**53**] and in the work on fusion systems by Ruiz and Viruel [**65**].

We would ideally like to weaken the requirement that G has local characteristic p to G has parabolic characteristic p in the definition of a group which is almost a group of Lie type however, in this case, to achieve a classification we would need a variant of Theorem 1 for groups of parabolic characteristic p. At the moment we do not see how to do this and so this is an open avenue for future research.

To discuss our approach to the proof of Main Theorem 1 we introduce our main hypothesis.

Main Hypothesis 1. *We have p is a prime, G is a finite group, $S_0 \in \mathrm{Syl}_p(G)$, $H \geq S_0$ is a subgroup of G such that $F^*(H)$ is a simple group of Lie type in characteristic p and of rank at least two and $H = N_G(F^*(H))$. We set $S = S_0 \cap F^*(H)$.*

To prove Main Theorem 1 we may assume that some p-local subgroup which contains S is not contained in H. We recall that a non-trivial element of a group X is *p-central* in X if and only if its centralizer in X contains a Sylow p-subgroup of X. We first consider the possibility that the p-local subgroup $N_G(O_p(C_H(t)))$ is not contained in H for some p-central element t of order p in H. We divide this case into two different projects. We consider first the case when $C_H(z)$ is not soluble for all non-trivial $z \in Z(S_0)$ as this is the situation we are most likely to encounter. We prove

Theorem 2. *Suppose that Main Hypothesis 1 holds with G a $\mathcal{K}_p$-group of parabolic characteristic p. Assume $N_G(O_p(C_G(z))) \not\leq H$ for some non-trivial $z \in Z(S_0)$ and that all p-central elements of H have non-soluble centralizers in H. Then $p = 5$ and $H \cong \mathrm{G}_2(5)$. Moreover, if G is a $\mathcal{K}_{\{2,p\}}$-group, then $G \cong \mathrm{LyS}$.*

When $p = 2$ and $C_H(z)$ is soluble for some non-trivial element z in $Z(S_0)$, then $|S_0|$ is rather small. Since $|S_0|$ is small, so are the 2-local subgroups of G and so these can be analysed without the help of a $\mathcal{K}_2$-hypothesis and as usual these small cases spawn a shoal of exotic examples.

Theorem 3. *Suppose that Main Hypothesis 1 holds with G a group of parabolic characteristic p. Assume $N_G(O_p(C_G(z))) \not\leq H$ for some non-trivial $z \in Z(S_0)$ and that some p-central element of H has soluble* *centralizer in H. Moreover, if p is odd, assume that G is a $\mathcal{K}_p$-group. Then one of the following holds:*

(i) $p = 2$ *and* $(F^*(G), F^*(H)) = (\mathrm{Mat}(11), \mathrm{Sp}_4(2)')$, $(\mathrm{Mat}(23), \mathrm{PSL}_3(4))$, $(\mathrm{G}_2(3), \mathrm{G}_2(2)')$ *or* $(\mathrm{P}\Omega_8^+(3), \Omega_8^+(2))$*;*

(ii) $p = 3$ *and* $(F^*(G), F^*(H)) = (\mathrm{F}_4(2), \mathrm{PSL}_4(3))$, $(\mathrm{PSU}_6(2), \mathrm{PSU}_4(3))$, $(\mathrm{McL}, \mathrm{PSU}_4(3))$, $(\mathrm{Co}_2, \mathrm{PSU}_4(3))$, $({}^2\mathrm{E}_6(2), \mathrm{P}\Omega_7(3))$, $(\mathrm{M}(22), \mathrm{P}\Omega_7(3))$, $(\mathrm{M}(23), \mathrm{P}\Omega_8^+(3))$ *or* $(\mathrm{F}_2, \mathrm{P}\Omega_8^+(3))$*; or*

(iii) $p \in \{3, 5, 7, 13\}$ *and* $F^*(H) \cong \mathrm{PSL}_3(p)$*.*

Having proved Theorem 2 and 3 we consider the possibility that some other p-local subgroup of G containing S_0 is not contained in H.

THEOREM 4. *Suppose that Main Hypothesis 1 holds with G a $\mathcal{K}_p$-group and that for all p-central elements z in H,*

$$C_G(z) \leq N_G(O_p(C_{F^*(H)}(z))) \leq H.$$

If there exists a p-local of G containing S_0 and not contained in H, then either

(i) $p = 2$ *and* $(F^*(G), F^*(H)) = (\mathrm{PSL}_4(3), \mathrm{SU}_4(2))$ *or* $(\mathrm{Alt}(10), \mathrm{SL}_4(2))$*; or*

(ii) $p = 3$ *and* $F^*(H) \cong \mathrm{PSL}_3(3)$ *or* $p = 7$ *and* $F^*(H) \cong \mathrm{PSL}_3(7)$*.*

We note that the condition $C_G(z) \leq H$ for all $z \in Z(S_0)^\#$ implies that G is of parabolic characteristic p (see Lemma 2.1 (iii)). Combining Theorems 2, 3 and 4 yields Main Theorem 1.

We now discuss the proofs of Theorems 2, 3 and 4 in some detail. For the convenience of the reader, the recognition results required to identify the groups appearing in these theorems are collected together in Section 3 and results about strongly p-embedded subgroups are collated in Section 4. The real proof commences in Section 5

Suppose that G has parabolic characteristic p and $N_G(O_p(C_G(z))) \not\leq H$. A common feature of the groups of Lie type X other than $\mathrm{Sp}_{2n}(2^e)$, $\mathrm{F}_4(2^e)$ and $\mathrm{G}_2(3^e)$ is that for $T \in \mathrm{Syl}_p(X)$ the subgroup embedding of $Q = O_p(N_X(Z(T)))$ into X satisfies

(L1) $Q = O_p(N_X(Q)) \geq C_X(Q)$; and

(L2) $N_X(U) \leq N_X(Q)$ for all $1 \neq U \leq C_X(Q)$.

In [**47**] groups which fulfill this property are called *large* in X. Furthermore, when $X \not\cong {}^2\mathrm{F}_4(2^{2e+1})$, Q is a special group and has some additional nice properties (see Lemma D.16). We prove the main theorems for the case when $F^*(H)$ is one of $\mathrm{PSL}_3(p^e)$, $\mathrm{PSp}_{2n}(2^e)'$, $\mathrm{F}_4(2^e)$, ${}^2\mathrm{F}_4(2^{2e+1})'$ and $\mathrm{G}_2(3^e)$ using various different arguments depending upon the group encountered. In Sections 10, 11, 12, and 13 we consider the candidates $\mathrm{PSL}_3(2^e)$, $\mathrm{PSp}_{2n}(2^e)'$, $\mathrm{F}_4(2^e)$, ${}^2\mathrm{F}_4(2^{2e+1})'$ for $F^*(H)$. In particular in Proposition 13.8 we prove:

Let G be a $\mathcal{K}_2$-group of parabolic characteristic 2. *If $H \leq G$, $F^*(H) \cong {}^2\mathrm{F}_4(2^{2e+1})'$, $\mathrm{F}_4(2^e)$, $\mathrm{Sp}_{2n}(2^e)$, $n \geq 3$, $\mathrm{Sp}_4(2^e)$, $e > 1$ or $\mathrm{PSL}_3(2^e)$, $e \neq 2$, $H = N_G(F^*(H))$, $|G : H|$ odd, then $G = H$.*

The groups Mat(11) and Mat(23) are almost groups of Lie type with $F^*(H) \cong \mathrm{Sp}_4(2)'$ and $\mathrm{PSL}_3(4)$ respectively. The first one is identified using Lemma 3.11. To identify Mat(23) we provide rank 3 amalgam characterisations first of Mat(22) in Lemma 3.1 and then of Mat(23) in Lemma 3.2. The identification of G with H in Proposition 13.8 is achieved by employing a result due to D. Holt, Lemma 4.4.

In Section 9, very detailed calculations for $F^*(H) \cong \mathrm{PSL}_3(p^e)$, p odd, yield the exceptional fusion systems mentioned earlier and otherwise show that H is strongly p-embedded in G. Section 15 considers the cases with $F^*(H) \cong \mathrm{G}_2(3^e)$ and shows that H is strongly 3-embedded in G. This then leads to a contradiction via Proposition 4.6.

Once the above candidates for $F^*(H)$ are handled, we may assume that $O_p(C_H(z))$ is large in H. In Section 7 and specifically in Lemma 7.2 we demonstrate that $Q = O_p(C_H(z))$ is large in G. We then prove Theorem 2 and Theorem 3 separately.

Suppose first that $N_G(Q)$ is not soluble. This case has been investigated by A. Seidel [**67**] when p is odd and by G. Pientka [**64**] when $p = 2$ in their Ph.D. theses under the assumption that the Lie rank of H is at least 3. We first intend to determine the structure of $N_G(Q)/Q$. The main tool for this is provided in Section 5 which might be of interest for other avenues of research.

In Section 5, we consider a vector space V over $\mathrm{GF}(p)$ and subgroups $L \leq M \leq \mathrm{GL}(V)$ with the property that $C_M(L)$ is a p'-group and $\mathrm{Syl}_p(L) \subseteq \mathrm{Syl}_p(M)$. We say that L is *Sylow embedded* in M, see Definition 5.1. Our objective is to find all the cases where L is not normal in M. We call L *Sylow maximal* in $\mathrm{GL}(V)$ if L is normal in every candidate for M. In Section 5 we consider Sylow embeddings with $V = Q/Z(Q)$ and $L = O^{p'}(N_H(Q)/Q)$. The structure of $O^{p'}(N_H(Q)/Q)$ and V is described in Lemma D.1. In this section we assume that all groups are $\mathcal{K}$-groups and this is one of the reasons why in our theorems we need the stronger version of the $\mathcal{K}_p$-property as the results in Section 5 sometimes require results about possible over-groups of L in $\mathrm{GL}(V)$ and this often needs the classification of all maximal subgroups of $\mathrm{GL}(V)$ (see for example [**14, 37**], which require the classification of the finite simple groups). The work in this section also needs almost all the results about representations that we provided in Appendix C.

The motivation for this Section 5 comes from the hypothesis in Theorem 2 that $N_G(Q) > N_H(Q) \geq S_0$. This means $L = O^{p'}(N_H(Q)/Q)$ is Sylow embedded in $N_G(Q)/Q$ acting on $Q/Z(Q)$ and we would like to show that L is Sylow maximal for then $N_G(Q)$ can usually be shown to normalize $F^*(H)$ and this yields the contradiction $N_G(Q) \leq H$.

In Section 8 we apply the Proposition 5.3, Lemmas 5.12, 5.14 and Proposition 5.15 from Section 5 to find that the hypothesis $N_G(Q) > N_H(Q)$ is fulfilled only when $F^*(H) \cong \mathrm{G}_2(5)$. The final identification of G with LyS can only be made with an additional $\mathcal{K}_2$-hypothesis. With this, we consider the subgroup $G_0 = \langle N_G(Q), H \rangle$ and Lemma 3.10 implies that $G_0 \cong$ LyS. After this a short argument shows that either $G = G_0$ or G_0 is strongly 5-embedded in G and [**56**] provides the result.

When $C_H(z)$ is soluble, it turns out that $F^*(H)$ is defined over GF(2) or GF(3) and Q is extraspecial of order at most 3^9 see Lemma D.15. The proof of the Theorem 3 starts in Section 14, where we treat $p = 2$, while Section 16 and Section 17 handle the case $p = 3$, here $\mathrm{P}\Omega_8^+(3) \cong F^*(H)$ needs special treatment. When $p = 2$ as the outer automorphism group of Q is an orthogonal group of the appropriate type and, when p is odd, as Q has exponent p, then it is a general symplectic group [**79**, Theorem 1]. The fact that Q has small order means that when $p = 2$ we can complete calculations without knowing the possibilities simple sections. Hence in this case we do not impose a $\mathcal{K}_2$ hypothesis. When $p = 3$, it is useful to use the maximal subgroups of $\mathrm{Sp}_6(3)$ and $\mathrm{Sp}_8(3)$ and so we have a $\mathcal{K}_3$ assumption exactly as in Section 5.

Once $N_G(Q)$ is determined we use characterization theorems to identify the groups from either 2-local or 3-local information. We have included the theorems in Section 3. As an illustrative example, consider the possibility that $H \cong \mathrm{PSL}_4(3)$ or $\mathrm{PSU}_4(3)$. In this case we show that Q is an extraspecial group of order 3^5 and then, using the subgroup structure of $\mathrm{Out}(Q) \cong \mathrm{GSp}_4(3)$, we show that $N_G(Q)/Q$ has restricted structure. We then further investigate the 3-local structure of G until we have sufficient information to apply the appropriate recognition results Lemmas 3.5 and 3.4, 3.3. This completes our discussion of the proofs of Theorem 2 and Theorem 3.

We now discuss the proof of Theorem 4. If $p = 2$ and $N_H(Q)$ is soluble, then in Section 14 we show that $F^*(G)$ is either $\mathrm{PSL}_4(3)$ or $\mathrm{Alt}(10)$ with $F^*(H)$ either $\mathrm{PSU}_4(2)$ or $\mathrm{SL}_4(2)$ respectively. The identification of G uses Lemmas 3.13 and 3.14 which recognizes G from its Sylow 2-subgroup. After this, our aim is to show that the hypothesis in Theorem 4 leads to a contradiction. In Section 19, we let M be a

p-local subgroup of G containing S_0 with $M \not\leq H$. We still have that $Q = O_p(C_G(z))$ is large in G and we assume that $N_G(Q) \leq H$. Our plan is to select a subgroup P of M containing S such that

- $O_p(P) \neq 1$;
- $H \cap P$ contains a Sylow p-subgroup S of $F^*(H)$ with $Q \leq S$;
- $P \not\leq H$;
- P is minimal with respect to the first three conditions.

Using the action of P on a subgroup Y of $\Omega_1(Z(O_p(P)))$, we show that Y is a dual F-module or a dual $2F$-module (see Definition C.18) with offender $Q/C_Q(Y)$. Applying results from Appendix C restricts the structure of $P/O_p(P)$ and also of Y. Here again, because P need not to be a p-local subgroup of G, we have to use the stronger $\mathcal{K}_p$-group assumption. A detailed analysis of the pair (P, Y) eventually shows that $P/O_p(P)$ and Y can be identified with the same factors of a minimal parabolic subgroup of H and then using the fact that $C_G(z) \leq N_G(Q) \leq H$, for z a p-central element in H, we obtain $P \leq H$ which is a contradiction.

Naturally, the proof of our theorems need explicit details about the finite simple groups that we come across. We have collected these results in a series of appendices. Some of these results are well-known and are included for the convenience of the readers and others, though possibly familiar, are presented with proofs as we could find no reference.

In Appendix A we present some properties of groups of Lie type and establish our main notation for these groups. In particular, root subgroups are introduced and the automorphism groups of groups of Lie type are presented.

In Appendix B we establish various facts about alternating groups that we require. For example, we determine which alternating groups have the Sylow 2- or 3-subgroup contained in a unique maximal subgroup.

In Appendix C we focus on small GF(p)-representations of simple groups. For example, theorems about quadratic modules, F-modules and $2F$-modules are presented. This appendix also contains cross characteristic information such as the Landazuri–Seitz–Zalesskii Theorem giving lower bounds for the dimensions of cross characteristic projective representations of groups of Lie type. These results are applied throughout the proof of our main theorems and are particularly heavily used in Section 5.

In Appendix D we study the parabolic subgroups of the groups of Lie type giving explicit descriptions of the normalizers of root subgroups. In Lemmas D.22 and D.23 we investigate a minimal parabolic subgroup which does not normalize a root group. This is the parabolic subgroup that we mentioned above which resembles the subgroup P constructed in Section 19.

Appendix E contains an assortment of different results which do not have a natural home anywhere else in the paper.

We almost always reference [**27**] for our facts about simple groups. We typically use classical notation for the groups of Lie type which have an alternative classical name. The dihedral group of order n is denoted by $\mathrm{Dih}(n)$. We denote the Mathieu groups of degree m by $\mathrm{Mat}(m)$, and the alternating and symmetric groups of degree n are written as $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ respectively. The remainder of our notation for the sporadic simple groups is compatible with [**27**, Table 5.3].

If X is a classical group, then we will call the associated module when considered as a module over the prime field, a *natural module* for X. We also extend this terminology to the 6-dimensional module for $\mathrm{G}_2(2^e)$ and the 7-dimensional module for $\mathrm{G}_2(p^e)$ when p is odd. The *natural modules* for the symmetric and alternating groups, $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$, are defined to be the non-trivial section of the standard permutation module of dimension n defined over $\mathrm{GF}(p)$. More information about naming modules can be found in [**47**, Appendix A2].

For an odd prime p and natural number n, the extraspecial group of exponent p and order p^{2n+1} is denoted by p_+^{1+2n}. The extraspecial 2-groups of order 2^{2n+1} are denoted by 2_+^{1+2n} if the maximal elementary abelian subgroups have order 2^{1+n} and otherwise we write 2_-^{1+2n}. If X and Y are groups then $X{:}Y$ denotes the split extension of X by Y with normal subgroup X and unspecified non-trivial action of Y on X. If Z is a group with normal subgroup X and $Z/X \cong Y$, then we write $Z \sim X.Y$, in the cases where this extension is known not to split we write $Z \sim X{\cdot}Y$. This notation allows us to give suggestive descriptions of groups which indicate the isomorphism type of certain composition factors. We refer to such descriptions as the *shape* of a group.

Our group theoretic notation is mostly standard and follows that in [**2**] or [**22**] for example. We assume the reader is familiar with group actions, including coprime action, the Fitting group, components and the generalized Fitting subgroup as far as can be found in the texts just mentioned. However, we list some regularly used terms and notation which may be less widely used. Suppose that X is a finite group and p is a prime. The set of non-identity elements of X is designated by $X^\#$.

For a subset Y of X, Y^X denotes that set of X-conjugates of Y. One element sets are often denoted by elements. Thus x often denotes $\{x\}$ and so, for example, $\mathcal{K}_{\{p\}}$-groups are $\mathcal{K}_p$-groups. The subgroup $O(X)$ is the largest normal subgroup of X of odd order. The number $m_p(X)$ is the maximal k such that X has an elementary abelian subgroup of order p^k. We call $m_p(X)$ the *p-rank* of X. On the other hand, for a natural number n, n_p denotes the *p-part* of n, so for example $45_3 = 9$. If $Y \le X$ and $Z \subseteq Y$. Then Y *controls X-fusion* of Z in Y if and only if whenever $Z^x \subseteq Y$ for some $x \in X$, there exists $y \in Y$ such that $Z^y = Z^x$. If X and Y are groups and W is a subgroup of $Z(X \times Y)$ which is not contained in either direct factor, then $X \circ Y = (X \times Y)/W$ is a *central product* of X and Y.

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2. Preliminary group theoretical results

Suppose that p is a prime and G is a finite group of order divisible by p. If $Y \le G$ and $F^*(Y) = O_p(Y)$, then we say that Y has *characteristic p*.

The group G is of *local characteristic p* if and only if $N_G(X)$ has characteristic p for all non-trivial p-subgroups X of G. Further, G has *parabolic characteristic p* if and only if, for all non-trivial p-subgroups X which are normal in some Sylow p-subgroup of G, $N_G(X)$ has characteristic p. For non-trivial p-subgroups X, we often use the equivalence $F^*(N_G(X)) = O_p(N_G(X))$ if and only if $C_G(O_p(N_G(X))) \le O_p(N_G(X))$.

We start this section with some results about groups of local and parabolic characteristic p.

Lemma 2.1. *Let G be a group, S a Sylow p-subgroup of G and $X \le S$.*

(i) *Suppose that Y is a normal subgroup of X. If $F^*(N_G(Y)) = O_p(N_G(Y))$, then $F^*(N_G(X)) = O_p(N_G(X))$.*
(ii) *If there is some $z \in Z(X)^\#$ with $F^*(C_G(z)) = O_p(C_G(z))$, then also $F^*(N_G(X)) = O_p(N_G(X))$.*
(iii) *G is of parabolic characteristic if and only if for all $1 \ne z \in \Omega_1(Z(S))$ we have that $F^*(C_G(z)) = O_p(C_G(z))$.*
(iv) *If $F^*(C_G(z)) = O_p(C_G(z))$ for all elements z of order p in S, then G is of local characteristic p.*

Proof. (i) Define $E = E(N_G(X))O_{p'}(N_G(X))$. Then, as $Y \leq X \leq O_p(N_G(X))$, $[E, X] = 1$, EX normalizes Y and so EX acts on $R = O_p(N_G(Y))$. We have $C_R(X)$ normalizes and is normalized by E. Hence $[C_R(X), E] = [C_R(X), E, E] = 1$ as R is a p-group. But then $E \leq C_G(R)$ by the $A \times B$-Lemma [**2**, 24.2]. By assumption we have $C_G(R) \leq R$. It follows that $E = 1$ and so $F^*(N_G(X)) = O_p(N_G(X))$.

(ii) follows from (i) by setting $Y = \langle z \rangle$ and noting that $F^*(N_G(X)) = O_p(N_G(X))$ if and only if $F^*(C_G(X)) = O_p(C_G(X))$.

(iii) It is obvious that if G has parabolic characteristic p, then $C_G(O_p(C_G(z))) \leq O_p(C_G(z))$ for all $z \in \Omega_1(Z(S))^{\#}$. For the converse direction, we remark that if X is normal in S, then $Z(X) \cap Z(S) \neq 1$ and so the assertion comes from (ii).

(iv) follows directly from (ii). □

We present the notion of a large subgroup from the introduction.

Definition 2.2. Suppose that Q is a p-subgroup of G. Then Q is a *large p-subgroup* of G if and only if

(L1) $F^*(N_G(Q)) = Q$; and
(L2) if $1 \neq U \leq G$ and $[U, Q] = 1$, then $N_G(U) \leq N_G(Q)$.

The basic lemma about large subgroups that we shall use (mostly without further specific reference) is just below. The statements are also included in [**47**, Lemmas 1.52 and 1.55]. Recall that for a group X and subgroups $Z \leq Y \leq X$, Z is *weakly closed* in Y with respect to X if and only if Z is the only X-conjugate of Z contained in Y.

Lemma 2.3. *Suppose that Q is a large p-subgroup of G and T is a non-trivial p-subgroup of G such that $N_G(T) \geq Q$.*

(i) *If $Q \leq T$, then $N_G(T) \leq N_G(Q)$.*
(ii) *$N_G(Q)$ contains the normalizer of every Sylow p-subgroup of G which contains Q.*
(iii) *Assume that $Q \leq S \in \mathrm{Syl}_p(G)$. Then Q is weakly closed in S with respect to G.*
(iv) *$F^*(N_G(T)) = O_p(N_G(T))$.*
(v) *G has parabolic characteristic p.*

Proof. Let $S \in \mathrm{Syl}_p(G)$ with $Q \leq S$.

To see (i), we observe that $1 \neq Z(T)$ and $[Z(T), Q] = 1$. Therefore, property (L2) yields $N_G(T) \leq N_G(Z(T)) \leq N_G(Q)$.

Taking $S = T$, (ii) follows from (i).

For (iii) suppose that $x \in G$ and $Q^x \leq S$. Then, by (ii), $N_G(S) \leq N_G(Q) \cap N_G(Q^x)$. Thus Q^x is normal in S and S^x, so there exists $y \in N_G(Q^x)$ with $S^{xy} = S$. Now by (i) $Q = Q^{xy} = Q^x$ and (iii) holds.

(iv) We have $Q \leq N_G(T)$ and so $1 \neq C_{Z(T)}(Q) \leq Z(Q)$ by (L1). Let $z \in C_{Z(T)}(Q)^{\#}$. Then, by (L2), $Q \leq C_{N_G(T)}(z) \leq N_{N_G(T)}(Q)$ and therefore

$$F^*(C_{N_G(T)}(z)) = O_p(C_{N_G(T)}(z)).$$

Hence Lemma 2.1 (ii) applies to yield (iv).

Assume that $N_G(T) \geq S$. Then certainly $N_G(T) \geq Q$ and so, by part (iv), $F^*(N_G(T)) = O_p(N_G(T))$. Hence G has parabolic characteristic p and (v) holds. □

LEMMA 2.4. *Suppose that G has parabolic characteristic p and $S \in \mathrm{Syl}_p(G)$. If G_1 is a normal subgroup of G and $1 \neq \Omega_1(Z(S \cap G_1)) \leq \Omega_1(Z(S))$, then G_1 has parabolic characteristic p.*

PROOF. Let $S_1 = S \cap G_1$ and let $z \in \Omega_1(Z(S_1))^{\#}$. Then $z \in Z(S)$ and so $C_G(z)$ has characteristic p. By [**46**, Lemma 1.2(a)] $C_{G_1}(z)$ has characteristic p and thus G_1 has parabolic characteristic p by Lemma 2.1(iii). □

LEMMA 2.5. *If G has parabolic characteristic 2 and $Z(G/O(G)) = 1$, then $O(G) = 1$. In particular, if $O_2(G/O(G)) = 1$, then $O(G) = 1$.*

PROOF. Assume $O(G) \neq 1$. Choose $1 \neq z \in Z(S)$, S a Sylow 2-subgroup of G. Then $F^*(C_G(z)) = O_2(C_G(z))$ as G has parabolic characteristic 2. In particular, $C_G(O(G))$ has odd order and z inverts $O(G)$. Hence $O(G) = C_G(O(G))$ and $z \in Z(G/O(G)) = 1$, a contradiction. □

The so-called p-minimal groups play an important role in this paper.

DEFINITION 2.6. A group H is *p-minimal*, if for some Sylow p-subgroup S of H we have $H = \langle S^H \rangle$ and S is contained in a unique maximal subgroup of H.

An easy application of the Frattini Argument shows that H is p-minimal if and only if S not normal in H and S is contained in a unique maximal subgroup of H.

For a group X, $\Phi_p(X)$ is the full preimage of $\Phi(X/O_p(X))$. The structure of p-minimal groups is described in the next lemma.

LEMMA 2.7. *Suppose that P is p-minimal and $S \in \mathrm{Syl}_p(P)$. Let M be the unique maximal subgroup of P containing S and set $F = \bigcap_{g \in P} M^g$. Then the following hold.*

(i) *$O_p(P) \in \mathrm{Syl}_p(F)$.*
(ii) *$F = \Phi_p(P)$ and, in particular, if $O_p(O^p(P)) = 1$, then F is nilpotent.*

(iii) *If N is a subnormal subgroup of P contained in M, then $N \cap S \le O_p(P)$.*
(iv) *If $O^p(P)$ is p-closed, then P is a $\{t, p\}$-group for some prime $t \ne p$.*
(v) *For $N \trianglelefteq P$, either $O^p(P) \le N$ or $N \le F$.*
(vi) *$O^p(P)/(F \cap O^p(P))$ is a minimal normal subgroup of $P/(F \cap O^p(P))$.*
(vii) *If P is soluble, then $O^p(P)$ is p-closed and P is a $\{t, p\}$-group for some prime $t \ne p$.*
(viii) *$O^p(P) = [O^p(P), P]$.*

Proof. See [**44**, Lemma 3.2]. □

Lemma 2.8. *Suppose that P is p-minimal, $O_p(P) = 1$ and K is a component of P. Let $S \in \mathrm{Syl}_p(P)$. Then $O^p(P) = F^*(P) = E(P) = \langle K^S \rangle$ and $KN_S(K)$ is p-minimal.*

Proof. Set $E = O^p(P)$ and let M be the unique maximal subgroup of P containing S. Then, by Lemma 2.7 (vi), $E/(E \cap F)$ is a minimal normal subgroup of P/F. We also have $K \le E$ and $P = ES$. Furthermore $K \not\le M$. It follows that $E = \langle K^S \rangle$. In particular,

$$E = \langle K^S \rangle = E(P) = F^*(P).$$

Now set $S_0 = N_S(K)$ and suppose that K_1 and K_2 are subgroups of K such that K_1S_0 and K_2S_0 are maximal subgroups of KS_0. Then, for $i = 1, 2$, the distinct elements of K_i^S pairwise commute and so $\langle K_i^S \rangle S$ is a proper subgroup of P containing S. Thus $K = \langle K_1, K_2 \rangle \le M$ but then $P = ES = \langle K^S \rangle S \le M < P$, which is absurd. Hence KS_0 is p-minimal. □

One of the major parts of this paper requires that we study representations of certain simple groups on p-groups which have a rather restricted structure. For this we now define semi-extraspecial groups, which are generalizations of extraspecial groups.

Definition 2.9. Suppose that p is a prime and X is a p-group. Then

(i) X is *special* if $X' = \Phi(X) = Z(X)$;
(ii) X is *extraspecial* if X is special and $|Z(X)| = p$; and
(iii) X *semi-extraspecial* if X is special and, for all maximal subgroup Y of $Z(X)$, X/Y is extraspecial.

Lemma 2.10. *Suppose that X is a semi-extraspecial p-group and $x \in X \setminus Z(X)$. Then $|X : C_X(x)| = |Z(X)|$ and $[x, X] = Z(X)$.*

Proof. Set $W = \langle x, Z(X)\rangle$. Then W is abelian. If $[W, X] < Z(X)$, then there exists a maximal subgroup Y of $Z(X)$, such that

$$Z(X)/Y < W/Y \leq Z(X/Y),$$

which is a contradiction to X being semi-extraspecial. Hence $[W, X] = Z(X)$. Define $\phi : X \to Z(X)$ by $y \mapsto [x, y]$. Then, as X has class two, ϕ is a homomorphism and, as $Z(X) = [W, X]$, ϕ is surjective. Therefore $X/\ker\phi \cong Z(X)$. Since $\ker\phi = C_G(x)$, this proves the result. □

Definition 2.11. Suppose that p and r are primes with $p \neq r$. Then $l(p, r)$ is the minimal dimension of a faithful action of a group of order p on an elementary abelian r-group.

Lemma 2.12. *Suppose that p and r are primes with $p \neq r$. Assume that E is an extraspecial group of order p^{2w+1} which acts faithfully on an elementary abelian r-group A. Then $|A| \geq r^{l(p,r)p^w}$.*

Proof. See [**22**, Chap. 5, Theorem 5.5]. □

The next lemma is just the Thompson $A \times B$ Lemma applied to the dual of the module $P/\Phi(P)$.

Lemma 2.13. *Suppose that P is a p-group and $A \times B$ acts on P with A a p-group and B a p'-group. If $[P, B] \leq [P, A]$, then B centralizes P.*

Proof. Suppose that $[P, B] \leq [P, A]$. It suffices to prove that B centralizes $\overline{P} = P/\Phi(P)$ by Burnside's Lemma [**22**, Chap. 5, Theorem 1.4]. Since B is a p'-group we have $\overline{P} = [\overline{P}, B] \times C_{\overline{P}}(B)$ and this is an A-invariant decomposition. Therefore $[\overline{P}, B] \leq [\overline{P}, A] = [[\overline{P}, B], A] \times [C_{\overline{P}}(B), A]]$. Since $[P, B]A$ is nilpotent, $[[\overline{P}, B], A] < [\overline{P}, B]$ and so we have a contradiction. □

Lemma 2.14. *Suppose that p is a prime, G is a group and E is a normal p-subgroup of G. Assume that U is a non-cyclic elementary abelian p-subgroup of G and that either*

(a) $C_E(U) = C_E(u)$ *for all* $u \in U^{\#}$; *or*
(b) $[E, U] = [E, u]$ *for all* $u \in U^{\#}$.

If R is a p'-subgroup of G which is normalized by U, then $[R, U]$ centralizes E.

Proof. Let R be an p'-subgroup of G which is normalized by U. Assume that U does not centralize R. Since U is not cyclic, [**26**, Lemma 11.25] yields

$$[R, U] = \langle [C_R(W), U] \mid |U : W| = p\rangle.$$

Let W be a maximal subgroup of U and set $R_0 = [C_R(W), U]$. If option (a) holds, then R_0U acts on $C_E(W) = C_E(U)$. The Three Subgroup Lemma implies that R_0 centralizes $C_E(W)$. Hence the Thompson $A \times B$-Lemma ([**2**, 24.2]) applied to the action of WR_0 on E implies $[E, R_0] = 1$. On the other hand, if option (b) holds, then $C_R(W)U$ normalizes $[E, W] = [E, U]$ and so $[E, U, C_R(W)] \le [E, U]$, $[E, C_R(W), U] \le [E, U]$ and so the Three Subgroups Lemma implies that $[E, R_0] \le [E, U] = [E, W]$. Now Lemma 2.13 applied to R_0W implies $[E, R_0] = 1$. Hence in both cases, $[E, R_0] = 1$. Since this is true for every maximal subgroup of U, it follows that $[R, U]$ centralizes E, as claimed. $\square$

LEMMA 2.15. *Suppose that G is a finite group, p is an odd prime which divides $|G|$ and $P \in \mathrm{Syl}_p(G)$. If $\Omega_1(P) \le Z(G)$, then G has a normal p-complement. In particular, if G is quasisimple, then $\Omega_1(P) \not\le Z(G)$.*

PROOF. Suppose that $1 \ne T \le P$. If $x \in N_G(T)$ has order coprime to p, then x centralizes $\Omega_1(T) \le \Omega_1(P) \le Z(G)$. Therefore, as p is odd, $x \in C_G(T)$ by [**2**, 24.8] and so $N_G(T)/C_G(T)$ is a p-group. Now the Frobenius Normal p-Complement Theorem [**2**, 39.4] yields that G has a normal p-complement.

If G is quasisimple, it does not have a normal p-complement and so $\Omega_1(P) \not\le Z(G)$. $\square$

We mention in passing that in the case $p = 2$, the statement in Lemma 2.15 does not hold as can bee seen in $\mathrm{SL}_2(q)$ when q is odd.

LEMMA 2.16. *Suppose that G is a finite group and $S \in \mathrm{Syl}_2(G)$. If $S \cong \mathbb{Z}_2 \times \mathrm{Dih}(8)$, then G has a subgroup of index two.*

PROOF. Suppose that $O^2(G) = G$ is perfect. Write $S = \langle z \rangle \times S_0$ where $S_0 \cong \mathrm{Dih}(8)$. As $G = O^2(G)$, the Thompson Transfer Lemma [**26**, Lemma 15.16] implies that z is conjugate to an element y of S_0 with $C_S(y) \in \mathrm{Syl}_2(C_G(y))$. Since z is 2-central, we have $y \in Z(S) \cap S_0$ and y is a square while z is not, a contradiction. $\square$

The following definition is repeated in Appendix C.

DEFINITION 2.17. Suppose that p is a prime, A is a group and V is a non-trivial $\mathrm{GF}(p)A$-module. Then

(i) A acts *quadratically* on V provided $[V, A, A] = 0$; and
(ii) A acts *cubically* on V provided $[V, A, A, A] = 0$.

If A acts cubically on V but not quadratically on V, then we say that A acts *strictly cubically* on V.

Suppose that T is a p-group. Then

$$J(T) = \langle A \mid A \le T, \Phi(A) = 1 \text{ and } m_p(A) = m_p(T)\rangle$$

is the Thompson subgroup of T and the *Baumann subgroup* of G is defined to be

$$B(T) = C_T(\Omega_1(Z(J(T)))).$$

DEFINITION 2.18. *Let p be an odd prime, $T \in \mathrm{Syl}_p(G)$ and assume $X \le G$. Set $Q = O_p(X)$ and $W = \Omega_1(Z(Q))$. Then X is a $B(T)$-*block *of G if*

(i) *$X = O^p(X) = [X, B(T)]$, $[O_p(X), X] = O_p(X)$, and $[X, \Omega_1(Z(T))] \ne 1$.*
(ii) *$X/O_p(X) \cong \mathrm{SL}_2(p^d)'$, and $W/C_W(X)$ is a natural $\mathrm{SL}_2(p^d)$-module for X/Q.*
(iii) *If $Q \ne W$, then*
  (a) *$p = 3$, and Q/W is a natural $\mathrm{SL}_2(3^n)$-module for X/Q,*
  (b) *$Q' = \Phi(Q) = Z(X) = C_W(X)$ and $|Z(X)| = 3^b$, and*
  (c) *no element of $B(T) \setminus C_{B(T)}(W)$ acts quadratically on $Q/Z(X)$.*

Moreover, if (iii) holds, then X is called an exceptional block.

We close this preliminary section with some results about modules.

Suppose that k is a field and V is a finite dimensional kG-module. Then the *dual* of V is

$$V^* = \mathrm{Hom}(V, k)$$

the set of k-linear transformations from V to k. For $\phi \in V^*$ and $g \in G$, the map

$$\phi g : V \to k$$

is defined by

$$v\phi g = (vg^{-1})\phi$$

for all $v \in V$ and this makes V^* into a kG-module. Note that $V^{**} \cong V$ as kG-modules. We say that V is *self-dual* as a kG-module if $V \cong V^*$ as kG-modules.

For a subspace U of V, we define

$$U^\dagger = \{\phi \in V^* \mid U \le \ker \phi\} \le V^*$$

and similarly, for $W \le V^*$, set

$$W^\dagger = \bigcap_{\phi \in W} \ker \phi \le V.$$

Notice that applying $\dagger$ twice returns the original subspace. We have the following well known-result

LEMMA 2.19. *If $U \leq V$ is a kG-submodule, then $V^*/U^\dagger \cong U^*$ as kG-modules. Furthermore, $[V^*, G]^\dagger = C_V(G)$, $C_V(G)^\dagger = [V^*, G]$, $[V, G]^\dagger = C_{V^*}(G)$ and $C_{V^*}(G)^\dagger = [V, G]$.*

PROOF. For $\phi \in V^*$. let ϕ_U denote the restriction of ϕ to U. Then $\phi_U \in U^*$ and the map $\phi \mapsto \phi_U$ is a kG-epimorphism with kernel $U^\dagger$. This proves $V^*/U^\dagger \cong U^*$ as kG-modules.

We calculate

$$\begin{aligned}
[V^*, G]^\dagger &= \bigcap_{\theta \in [V^*, G]} \ker \theta = \bigcap_{\phi \in V^*, g \in G} \ker(\phi g - \phi) \\
&= \bigcap_{\phi \in V^*, g \in G} \{w \in V \mid w\phi g = w\phi\} \\
&= \bigcap_{\phi \in V^*, g \in G} \{w \in V \mid w g^{-1}\phi = w\phi\} \\
&= \bigcap_{g \in G} \{w \in V \mid w g^{-1} = w\} = C_V(G).
\end{aligned}$$

The remaining claims follow similarly. □

LEMMA 2.20. *Suppose that V is a vector space defined over a field k and f is a non-degenerate sesquilinear form on V. If G is a subgroup of $\mathrm{GL}(V)$ which preserves f, then $[V, G]^\perp = C_V(G)$. In particular, if $C_V(G) \leq [V, G]$, then $C_V(G)$ is totally isotropic. Furthermore, if G is a finite group of order coprime to the characteristic of k, then $V = [V, G] \perp C_V(G)$.*

PROOF. For the first part see [**51**, Lemma 2.5.3]. The rest is easy to prove. □

LEMMA 2.21. *Suppose that V is a $2n$-dimensional orthogonal $\mathrm{GF}(p)$-vector space and U an n-dimensional isotropic subspace. Let $w \in O(V)$ be a p-element with $[w, U] = 0$. Then $[w, V] \leq U$.*

PROOF. As U is isotropic we have $U \leq U^\perp$. Now the dimension of U yields $U = U^\perp$. Let $0 \neq v \in U$. Then $[w, v] = 0$ and so also $[V/v^\perp, w] = 0$. This shows $[V, w] \leq \bigcap_{0 \neq v \in U} v^\perp = U^\perp = U$. □

LEMMA 2.22. *Suppose that p and r are primes with $p \neq r$, V is a finite dimensional vector space over a field of characteristic r and E is a finite elementary abelian p-subgroup of $\mathrm{GL}(V)$. Assume that E is not cyclic and let Γ denote the set of all maximal subgroups of E. If $V = [V, E]$, then*

$$V = \bigoplus_{F \in \Gamma} C_V(F).$$

Furthermore, if V supports an E-invariant non-degenerate bilinear form, then the direct sum above is an orthogonal sum.

PROOF. For the main statement, see [**2**, page 50]. The second statement follows from Lemma 2.20. $\square$

LEMMA 2.23. *Suppose that p and r are primes with $p \neq r$, V is a $\mathrm{GF}(r)X$-module and E is a non-trivial elementary abelian normal p-subgroup of X. Assume that m is the length of a minimal orbit of X on the maximal subgroups of E. Then $\dim V \geq ml(p,r)$.*

PROOF. This comes from Lemma 2.22 as, for F a maximal subgroup of E with $C_V(F) \neq 0$, E/F is cyclic and acts faithfully on when $C_V(F)$. $\square$

LEMMA 2.24. *Let X be a group, which acts faithfully and irreducibly on a vector space V over $\mathrm{GF}(p)$. Let U be a group of order r coprime to p which also acts on V but centralizes X. Then U is cyclic and, if n is the order of p modulo r, then V can be considered as an $\mathrm{GF}(p^n)X$-module and U induces field multiplication.*

PROOF. By assumption we have that $U \subseteq \mathrm{End}_X(V)$. By Schur's Lemma [**2**, 12.4] we have that $\mathrm{End}_X(V)$ is a division ring and so, as $|\mathrm{End}_X(V)|$ is finite, $\mathrm{End}_X(V)$ is a field by Wedderburn's little theorem. In particular, U is contained in the subfield F which is generated by U over the prime field $\mathrm{GF}(p)$ and U is cyclic. We have that $F \cong \mathrm{GF}(p^n)$. Thus V is an FX-module. $\square$

For the next lemma we recall that $\Gamma_n(q)$ represents the group of all semilinear transformations of an n-dimensional vector space over $\mathrm{GF}(q)$.

LEMMA 2.25. *Let V be a finite dimensional, faithful $\mathrm{GF}(p)G$-module. Assume that there is an abelian normal subgroup A in G, such that A acts irreducibly on V. If $|V| = q$, then G is isomorphic to a subgroup of $\Gamma_1(q)$.*

PROOF. See [**33**, Satz II 3.11]. $\square$

LEMMA 2.26. *Suppose that p is an odd prime and V, V_1 are p-groups of equal order on which some group G acts irreducibly. Suppose that $Z(G)$ contains some cyclic group Z such that for all $z \in Z^{\#}$, $C_V(z) = 0 = C_{V_1}(z)$. Let n be the order of p modulo $|Z|$. Then as Z-modules V and V_1 are equivalent if and only if they are conjugate under a Galois automorphism of $\mathrm{GF}(p^n)$.*

Proof. By Lemma 2.24 V and V_1 may be considered as vector spaces over $\mathrm{GF}(p^n)$. Furthermore Z acts as field multiplication. Hence V and V_1 are equivalent if and only if the corresponding 1-spaces are equivalent. In particular we may assume that V, V_1 are 1-spaces and so Z acts irreducibly on both of them. Then the assertion follows with Lemma 2.25. $\square$

Lemma 2.27. *Suppose that p is a prime, G is a group, $L \leq G$ and V is a faithful $\mathrm{GF}(p)G$-module. Assume that $L \cong \mathrm{SL}_2(p^e)$ and V restricted to L is a natural $\mathrm{GF}(p^e)L$-module. If $\mathrm{Syl}_p(N_G(L)) \subseteq \mathrm{Syl}_p(G)$, then either*

(i) *L is normal in G; or*
(ii) *$p^e = 4$, $L \cong \mathrm{SL}_2(4)$ and $G \cong \mathrm{Alt}(7)$.*

Proof. Suppose that L is not normal in G. If $e = 1$, then $G \leq \mathrm{Aut}(V) \cong \mathrm{GL}_2(p)$ and L is normal in G, a contradiction. So suppose that $e > 1$. Let $S \in \mathrm{Syl}_p(L)$ and $S_0 \in \mathrm{Syl}_p(N_G(L))$ with $S_0 \geq S$. Since L acts irreducibly on V, $|C_G(L)|$ is coprime to p by Lemma 2.24 and so S_0/S embeds into $\mathrm{Out}(L)$. Theorem A.11 implies that S_0/S is cyclic and that the non-trivial elements of S_0/S are images of field automorphisms of L. As L is not normal in G, $S^G \neq \mathrm{Syl}_p(L)$ and so $|S^G| > 1 + p^e$. Let $P, Q \in S^G$ with $P \neq Q$. If $C_V(P) = C_V(Q)$, then $C_V(P) = C_V(Q) = [V,Q] = [V,P]$ and $\langle P,Q\rangle$ centralizes the series, $V > C_V(P) > 0$ which means that $\langle P,Q\rangle$ is a p-group as V is faithful. We may therefore assume that $\langle P,Q\rangle \leq S_0$. Since $P \neq Q$, and $\langle P,Q\rangle$ is generated by elements of order p, we have $p^{e+1} \leq |\langle P,Q\rangle| = |\Omega_1(S_0)| = p^{e+1}$. Hence, as $e > 1$, $P \cap S > 1 < Q \cap S$. Without loss of generality we may assume that $P \neq S$. Let $x \in P \cap S$, then $C_V(S) = C_V(x) = C_V(P)$. Let $y \in P\backslash S$. Then y centralizes $C_V(S)$, but this means that y centralizes $N_L(S)/S$ which acts faithfully on $C_V(S)$ and this is a contradiction. Hence, if $C_V(P) = C_V(Q)$, then $P = Q$. As an immediate consequence we have, for such P, Q, if $x \in (P\cap Q)^\#$, then $C_V(P) = C_V(x) = C_V(Q)$, which is impossible. Hence, for $P, Q \in S^G$ with $P \neq Q$, we have

$$P \cap Q = 1.$$

In particular, as $e > 1$, S is the unique conjugate of S contained in S_0.

Let $T \in S^G\backslash\{S\}$ and suppose that $S_0 \cap T > 1$. If p is odd, then as no field automorphism of L acts quadratically on V and the elements of T do act quadratically on S, we have $T \cap S_0 \leq S$, which is a contradiction. Thus $p = 2$. Furthermore, as S_0/S is cyclic, we have $T \cap S_0$ has order 2. In particular, we have $|T : N_G(S)| = p^e$ if p is odd otherwise $|T : N_G(S)| \geq 2^{e-1}$ and furthermore e is even. Suppose that $|T : N_G(S)| =$

2^{e-1} and pick $t \in (T \cap S_0)^{\#}$. Then t acts on $C_V(S)$ and $\dim[C_V(S), t] = e/2$. Now we see that $\dim C_V(S) \cap C_V(t) = \dim C_V(S) \cap C_V(T) = e/2$. Hence we have

(2.27.1) If $T \neq S$ is a conjugate of S, then either
(i) $|T : N_G(S)| = p^e$; or
(ii) $p = 2$, $|T : N_G(S)| = 2^{e-1}$, e is even, and $\dim C_V(S) \cap C_V(T) = e/2$.

From the members of $S^G \setminus \{S\}$, select T so that the $\dim C_V(S) \cap C_V(T)$ is maximal. Set $H = \langle S, T \rangle$, $U = C_V(S) \cap C_V(T)$ and $W = C_V(S) + C_V(T)$. Then $W/U \neq 0$ and, as $C_V(S) = [V, S]$ and $C_V(T) = [V, T]$, U and W are H-invariant and H acts non-trivially on W/U as H is not a 2-group. Also $\dim W/U = 2b$ for some $0 < b < e$. For $P, Q \in S^H$ with $P \neq Q$, the subspaces $C_V(P)/U$ and $C_V(Q)/U$ each have dimension b and, by the maximal choice of $\dim U$, intersect in zero. By (2.27.1) we have

$$|S^H| \geq \begin{cases} p^e + 1 & T \cap N_G(S) = 1 \\ 2^{e-1} + 1 & p = 2 \text{ and } T \cap N_G(S) \neq 1 \end{cases}.$$

Hence W/U has at least $(p^e + 1)(p^b - 1)$ non-trivial vectors if $T \cap N_G(S) = 1$. Thus

$$p^{2b} - 1 \geq (p^e + 1)(p^b - 1)$$

from which we conclude that $e = b$, a contradiction. Therefore $p = 2$, $T \cap N_G(S) \neq 1$ and

$$2^{2b} - 1 \geq (2^{e-1} + 1)(2^b - 1)$$

which yields $b = e - 1$ and $\dim C_V(P) \cap C_V(Q) \leq 1$ for every pair $P, Q \in S^G$ with $P \neq Q$. On the other hand, as $T \cap N_G(S) \neq 1$, we know by (2.27.1) that $\dim C_V(S) \cap C_V(T) = e/2$ and consequently $e = 2$. Now we have $\dim V = 4$ and $\mathrm{Aut}(V) \cong \mathrm{SL}_4(2)$. Furthermore, we see that $N_G(L)/C_G(L) \cong \mathrm{SL}_2(4){:}2 \cong \mathrm{Sym}(5)$ and $C_G(L)$ has order dividing 3. Using the fact that G has Sylow 2-subgroups which are dihedral of order 8, the list of maximal subgroups of $\mathrm{SL}_4(2)$ yields that $G \cong \mathrm{Alt}(7)$. This is (ii). $\square$

Later in Section 5, we shall introduce the notion of Sylow embedded subgroups and in Proposition 5.3 prove a significant generalization of Lemma 2.27 the proof of which requires that G is a $\mathcal{K}$-group.

THEOREM 2.28. *For any natural numbers $a > 1$ and $n > 1$ there is a prime number that divides $a^n - 1$ and does not divide $a^k - 1$ for any natural number $k < n$, with the following exceptions:*

(i) $a = 2$ *and* $n = 6$*; and*
(ii) $a+1$ *is a power of two, and* $n = 2$.

PROOF. This it the famous theorem Bang–Zsigmondy [**7, 82**]. $\square$

If r is a prime number that divides $a^n - 1$ and does not divide $a^k - 1$ for any natural number $k < n$, then we call r a *primitive prime divisor* of $a^n - 1$.

3. Identification theorems of some almost simple groups

In this section we provide various identification theorems which are required to prove the main theorems. We begin with amalgam type recognitions of Mat(22) and Mat(23).

LEMMA 3.1. *Suppose that X is a group and P, B and W are subgroups of X such that $P \cong \mathrm{PSL}_3(4)$, $P \cap B \cap W$ is a Borel subgroup of P and $P \cap B$ and $P \cap W$ are point and line stabilisers in P respectively. Assume that $B \sim 2^4{:}\mathrm{Alt}(6)$, $W \sim 2^4{:}\mathrm{Sym}(5)$ and $|W{:}W \cap B| = 5$. Then $\langle P, B, W\rangle \cong \mathrm{Mat}(22)$.*

PROOF. We may as well suppose that $X = \langle P, B, W\rangle$. We consider the graph Γ which has vertex set $\mathcal{P} \cup \mathcal{B}$ where

$$\mathcal{P} = \{Pg \mid g \in X\} \text{ and } \mathcal{B} = \{Bg \mid g \in X\}$$

and edge set consisting of the pairs

$$\{Pg, Bh\} \text{ such that } Pg \cap Bh \neq \emptyset.$$

Plainly Γ is a bipartite graph and X acts on Γ by right multiplication. For $\alpha \in \Gamma$, $\Gamma(\alpha)$ denotes the set of neighbours of α in Γ. The pointwise stabiliser in X of a subset Θ of Γ is written as X_Θ. Note that if $\alpha = P$, then $X_\alpha = P$ and, if $\beta = B$, then $X_\beta = B$ and the other vertex stabilisers are conjugates of these groups.

The kernel of the action of X on Γ is a normal subgroup of $X_P = P$ and is contained in $P \cap B < P$. Since P is a simple group, this means that X acts faithfully on Γ. As $W = (P \cap W)(W \cap B)$, we have $X = \langle P, B\rangle$ and so the stabiliser of the connected component containing P and B is X. This means that Γ is connected. If W normalizes P, then, as $B = \langle B \cap P, B \cap W\rangle$, we have that B also normalizes P. However $B \cap P$ is not normal in B and so this is impossible. Furthermore, as P acts on $\mathcal{P}$ and the minimal non-trivial permutation representation of P has degree 21, we have that

$$|\mathcal{P}| \geq 22.$$

Our next observation is elementary. Let $\alpha = P$ and $\beta = B$. Then X_α acts on $\Gamma(\alpha)$ as it acts on the points of the projective plane of order 4 and X_β acts as Alt(6) on $\Gamma(\beta)$. In particular, both actions are 2-transitive, $|\Gamma(\alpha)| = |P{:}P \cap B| = 21$ and $|\Gamma(\beta)| = |B{:}P \cap B| = 6$.

Since $X_{\alpha\beta} = P \cap B \sim 2^4{:}\mathrm{Alt}(5)$, $X_{\alpha\beta}$ acts transitively on $\Gamma(\beta) \setminus \{\alpha\}$ and $X_{\alpha\beta\gamma} \sim 2^4{:}\mathrm{Alt}(4)$ for any $\gamma \in \Gamma(\beta) \setminus \{\alpha\}$. In particular, X_α acts transitively on the vertices of distance 2 from α.

Let $x \in (W \cap B) \setminus P$ and set $\gamma = Px$. Then $Px \cap B = Px \cap Bx = (P \cap B)x$ is non-empty. Thus $\gamma \in \Gamma(\beta)$. Furthermore, as $P \cap W$ is normal in W and P is not normalized by W,

$$X_{\alpha\gamma} = P \cap P^x = P \cap W.$$

Now $X_{\alpha\gamma}$ acts on $\Gamma(\gamma)$ preserving the sets $\Gamma(\alpha)\cap\Gamma(\gamma)$ and $\Gamma(\gamma)\setminus(\Gamma(\alpha)\cap\Gamma(\gamma))$. Because $P \cap W$ is a line stabiliser in $X_\gamma = P^x$, it has orbits of lengths 5 and 16 on $\Gamma(\gamma)$. Since $|X_{\alpha\gamma}{:}X_{\alpha\beta\gamma}| = |P \cap W{:}P \cap W \cap B| = 5$, we deduce that $\Gamma(\alpha) \cap \Gamma(\gamma)$ has order 5 or 21. If the size is 21, then we have accounted for all the cosets of B in X and so $|\mathcal{B}| = 21$ and $|X| = 2^7 \cdot 3^3 \cdot 5 \cdot 7$ and this means $22 \le |\mathcal{P}| = 6$, which is absurd. Thus

$$|\Gamma(\alpha) \cap \Gamma(\gamma)| = 5 \text{ and } |\Gamma(\gamma) \setminus \Gamma(\alpha)| = 16.$$

Let $\theta \in \Gamma(\gamma)$ have distance 3 from α. Then $\theta^{X_{\alpha\gamma}} = \Gamma(\gamma) \setminus \Gamma(\alpha)$, and $X_{\alpha\gamma\theta} \cong \mathrm{Alt}(5)$ complements $O_2(X_{\alpha\gamma})$. Consequently $X_{\alpha\gamma\theta}$ acts on $\Gamma(\theta)$ fixing γ and with an orbit of length 5. In particular we have that X_α acts transitively on the set of vertices at distance 3 from α.

Now consider the path (β, α, τ) where $\tau \in \Gamma(\alpha) \setminus \{\beta\}$. Then $X_{\beta\alpha\tau}$ is the intersection of two point stabilisers in P. As P acts 2-transitively on the points of the projective plane, we see that $X_{\beta\alpha\tau}$ has index 20 in $X_{\beta\alpha}$ and so has shape $2^4 : 3$. Further X_β acts transitively on such paths. As $O_2(X_{\beta\alpha\tau})$ has to act trivially on the projective line through β and τ, we see that

$$O_2(X_{\beta\alpha\tau}) \ne O_2(X_\beta).$$

We now make such a path (β, α, Bx) where $x \in (P \cap W) \setminus B$ and note that the stabiliser of α and Bx contains $(B \cap W) \cap (B \cap W)^x \sim 2^4 : \mathrm{Sym}(3)$. It follows that $\Gamma(\beta)\cap\Gamma(Bx)$ contains at least 2 vertices and so also $|\Gamma(\beta) \cap \Gamma(\theta)| \ge 2$. Since $X_{\alpha\gamma\theta}$ acts on $\Gamma(\theta)$ with an orbit of length 1 and an orbit of length 5, we see that $\Gamma(\theta)\cap\Gamma(\beta)$ contains an element from the orbit of length 5 and so we deduce that every neighbour of θ is incident to some vertex at distance 2 from α. Thus there are no vertices at distance 4 from α, and, in particular, we have that $|\mathcal{P}| \le 22$. Since $|\mathcal{P}| \ge 22$, we have equality. Now $|X| = 22|P| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, which implies

$$|\mathcal{P}| = 22 \text{ and } |\mathcal{B}| = 77.$$

The fact that P acts 2-transitively on the 21 points of the projective plane of order 4 yields that X acts 3-transitively on $\mathcal{P}$. In particular, given any three members of $\mathcal{P}$ we may map them to three neighbours of the coset B. We now identify the members of $\mathcal{B}$ with their neighbours in $\mathcal{P}$. Thus $\mathcal{B}$ becomes a set of six element subsets of $\mathcal{P}$ which we call blocks. Since X acts 3-transitively on $\mathcal{P}$ we get that any three points are contained in a block. Suppose that β_1 and β_2 are blocks sharing a common point. Then, as we saw earlier, $|\Gamma(\beta_1) \cap \Gamma(\beta_2)| = 2$ which means that every subset of $\mathcal{P}$ of size 3 is contained in exactly one block. Thus $(\mathcal{P}, \mathcal{B})$ is a Steiner triple system with parameters $(3, 6, 22)$. Such systems are uniquely determined by [**81**] and therefore X is isomorphic to a subgroup of $\mathrm{Aut}((\mathcal{P}, \mathcal{B})) \cong \mathrm{Aut}(\mathrm{Mat}(22))$. As $X = \langle P, B\rangle$, we see $X = X'$. So $X \cong \mathrm{Mat}(22)$ and this completes the proof of the lemma. $\square$

Now we come to the identification of Mat(23). The proof of the next lemma is very similar to the previous one and so the proof is somewhat abbreviated.

LEMMA 3.2. *Suppose that X is a group and P, B and W are subgroups of X such that $P \cong \mathrm{Mat}(22)$, $B \sim 2^4{:}\mathrm{Alt}(7)$, $W \sim 2^4{:}(3 \times \mathrm{Alt}(5)).2$, $B \cap P \sim 2^4{:}\mathrm{Alt}(6)$, $W \cap P \sim 2^4{:}\mathrm{Sym}(5)$, $W \cap B \sim 2^4{:}(3 \times \mathrm{Alt}(4)).2$ and $P \cap B \cap W \sim 2^4{:}\mathrm{Sym}(4)$. Then $\langle P, B, W\rangle \cong \mathrm{Mat}(23)$.*

PROOF. We again suppose that $X = \langle P, B, W\rangle$ and, letting $\mathcal{P} = \{Pg \mid g \in X\}$ and $\mathcal{B} = \{Bg \mid g \in X\}$, we consider the graph Γ which has vertex set $\mathcal{P} \cup \mathcal{B}$ made into a graph as in Lemma 3.1. Again we have $W = (P \cap W)(W \cap B)$ and that W does not normalize P. In particular, we have Γ is connected and X acts faithfully on Γ. We also have that

$$|\mathcal{P}| \geq 23.$$

For $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{B}$ we know $|\Gamma(\alpha)| = 77$ and $|\Gamma(\beta)| = 7$.

Suppose that $\alpha = P$ and $\beta = B$. Then $X_{\alpha\beta} = P \cap B$ and so $X_{\alpha\beta}$ acts transitively on $\Gamma(\beta) \setminus \{\alpha\}$. In particular, X_α acts transitively on vertices at distance 2 from α. Let $\gamma \in \Gamma(\beta) \setminus \{\alpha\}$. Then $X_{\alpha\beta\gamma} \sim 2^4 : \mathrm{Alt}(5)$. Now we let $x \in (B \cap W) \setminus P$ and note that $\gamma = Px \in \Gamma(\beta)$ and that $X_\alpha \cap X_\gamma = P \cap P^x \geq (W \cap P)' \cong 2^4 : \mathrm{Alt}(5)$. Furthermore we have $X_{\alpha\beta\gamma} \cap (W \cap P)' \sim 2^4 : \mathrm{Alt}(4)$, as $(W \cap P)'$ has orbits of length $1, 1, 5$ on $\Gamma(\beta)$. So $X_{\alpha\beta\gamma} \cap (W \cap P)'$ has index 2 in $P \cap B \cap W$. We see that $\langle X_{\alpha\beta\gamma}, (W \cap P)'\rangle \cong \mathrm{PSL}_3(4)$ and thus $X_{\alpha\gamma} \cong \mathrm{PSL}_3(4)$ and, in particular, we have $|\Gamma(\alpha) \cap \Gamma(\gamma)| = 21$. Let $\theta \in \Gamma(\gamma)$ have distance 3 from α in Γ. Then as in Mat(22) the stabiliser of a point p has an orbit of length 21 on the blocks containing p and an orbit of length 56 on the blocks not containing p, we see $|\theta^{X_{\alpha\gamma}}| = 56$ and then $X_{\alpha\gamma\theta} \cong \mathrm{Alt}(6)$. Hence $X_{\alpha\gamma\theta}$ has orbits of length 1 and 6 on $\Gamma(\theta)$. In particular X_α acts transitively on the set of vertices of distance 3 from α.

Now consider the path (β, α, τ) where $\tau \in \Gamma(\alpha) \setminus \{\beta\}$. Then $X_{\beta\alpha\tau}$ is the intersection of two point stabilisers in P. We have that $B \cap P$ has orbits of length 16 and 60 on $\Gamma(\alpha) \setminus \{\beta\}$. We will choose τ in an orbit of length 60, then $X_{\beta\alpha\tau}$ has shape $2^4 : \mathrm{Sym}(3)$ and X_β acts transitively on such paths. We claim that we can make such a path with $\tau = Bx$ for some $x \in (P \cap W) \setminus B$. In fact if we choose such an element x, we see that Bx is centralized by $O_2(W)$. But the other orbit in $\Gamma(\alpha) \setminus \{\beta\}$ has length 16 and is centralized by $\mathrm{Alt}(6)$, which does not contain a subgroup of order 16. Now we can proceed as in the previous lemma. We have that $(B \cap W) \cap (B \cap W)^x$ contains $2^4 : (Z_3 \times \mathrm{Sym}(3))$. It follows that $|\Gamma(\beta) \cap \Gamma(Bx)| \geq 3$. In particular $|\Gamma(\theta) \cap \Gamma(\beta)| \geq 3$. Thus as in Lemma 3.1 we get

$$|\mathcal{P}| = 23 \text{ and } |\mathcal{B}| = 253.$$

As P acts 3-transitively on the 22 points of the $(3, 6, 22)$ Steiner system this yields that X acts 4-transitively on $\mathcal{P}$. In particular, given any four members of $\mathcal{P}$ we may map them to four neighbours of the coset B.

We now again identify the members of $\mathcal{B}$ with their neighbours in $\mathcal{P}$. Thus $\mathcal{B}$ becomes a set of seven element subsets of $\mathcal{P}$ which we call blocks. Since X acts 4-transitively on $\mathcal{P}$ we get that any four points are contained in a block. As in Lemma 3.1 we see that any four points are contained in exactly one block. Thus $(\mathcal{P}, \mathcal{B})$ is a Steiner system with parameters $(4, 7, 23)$. Application of [**81**] shows that this system is uniquely determined and so $X \cong \mathrm{Mat}(23)$. □

The remainder of this section is a compendium of statements of identification theorems that are required for the proofs of our main theorems. We start with 3-local characterisations.

Lemma 3.3. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z \leq S$ has order 3 and set $M = C_G(Z)$. Assume that the following conditions hold.*

(a) *$Q = F^*(M)$ is extraspecial of order 3^{1+4};*
(b) *M/Q contains a normal subgroup isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$; and*
(c) *Z is not weakly closed in S with respect to G.*

Then either $F^*(G) \cong \mathrm{F}_4(2)$ or $F^*(G) \cong \mathrm{PSU}_6(2)$.

Proof. This is [**60**, Theorem 1.3]. □

LEMMA 3.4. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z \le S$ has order 3 and set $M = C_G(Z)$. Assume that the following conditions hold.*

(a) *$Q = F^*(M)$ is extraspecial of order 3^{1+4};*
(b) *$F^*(M/Q) = O_2(M/Q)$ is extraspecial of order 2^{1+4};*
(c) *$M/O_{3,2}(M) \cong \mathrm{Alt}(5)$; and*
(d) *Z is not weakly closed in S with respect to G.*

Then $G \cong \mathrm{Co}_2$.

PROOF. This is taken from [**54**]. □

LEMMA 3.5. *Suppose that G is a finite group, $Z \le G$ has order 3 and set $M = C_G(Z)$. Let S is a Sylow 3-subgroup of M and J is an elementary abelian subgroup of S of order 3^4. Assume that the following conditions hold.*

(a) *$Q = F^*(M)$ is extraspecial of type 3_+^{1+4};*
(b) *$F^*(M/Q) \cong 2{\cdot}\mathrm{Alt}(5)$; and*
(c) *$J = F^*(N_G(J))$ and $O^{3'}(N_G(J)/J) \cong \mathrm{Alt}(6)$.*

Then $F^*(G) \cong \mathrm{McL}$.

PROOF. This is taken from [**61**]. □

LEMMA 3.6. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z \le S$ has order 3 and set $M = C_G(Z)$. Assume that the following conditions hold.*

(a) *$Q = F^*(M)$ is extraspecial of order 3^{1+6};*
(b) *$O_2(M/Q) \cong \mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8$; and*
(c) *Z is not weakly closed in S with respect to G.*

Then $F^*(G) \cong {}^2\mathrm{E}_6(2)$.

PROOF. See [**55**, Theorem 1.3]. □

LEMMA 3.7. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z \le S$ has order 3 and set $M = C_G(Z)$. Assume that the following conditions hold.*

(a) *$Q = F^*(M)$ is extraspecial of order 3^{1+6};*
(b) *$O_2(M/Q)$ acts on Q/Z as a subgroup of order 2^7 of $\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8$, which contains $Z(\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8)$; and*
(c) *Z is not weakly closed in S with respect to G.*

Then $F^*(G) \cong \mathrm{M}(22)$.

PROOF. See [**55**, Theorem 1.4]. □

LEMMA 3.8. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z \le S$ has order 3 and set $M = C_G(Z)$. Assume that the following conditions hold.*

(a) *$Q = F^*(M)$ is extraspecial of type 3_+^{1+8};*
(b) *$F^*(M/Q) = O_2(M/Q)$ is extraspecial of type 2_-^{1+6};*
(c) *$M/O_{3,2}(M)$ is isomorphic to the centralizer of a 3-central element in $\mathrm{PSp}_4(3) \cong \Omega_6^-(2)$; and*
(d) *Z is not weakly closed in S with respect to G.*

Then G is isomorphic to M(23).

PROOF. See [**59**, Theorem 1.3]. □

LEMMA 3.9. *Suppose that G is a finite group, $S \in \mathrm{Syl}_3(G)$, $Z \le S$ has order 3 and set $M = C_G(Z)$. Assume that the following conditions hold.*

(a) *$Q = F^*(M)$ is extraspecial of type 3_+^{1+8};*
(b) *$F^*(M/Q) = O_2(M/Q)$ is extraspecial of type 2_-^{1+6};*
(c) *$M/O_{3,2}(M) \cong \Omega_6^-(2)$; and*
(d) *Z is not weakly closed in S with respect to G.*

Then $G \cong \mathrm{F}_2$.

PROOF. This is [**59**, Theorem 1.4]. □

We also call on the following 5-local identification of the sporadic simple groups discovered by Richard Lyons.

LEMMA 3.10. *Suppose that G is a finite $\mathcal{K}_2$-group of local characteristic 5 which is generated by subgroups G_α and G_β with $G_\alpha \sim 5^{2+1+2}.2{\cdot}\mathrm{Alt}(5)$, $G_\beta \sim 5_+^{1+4}.2{\cdot}\mathrm{Alt}(6)$, $G_\alpha \cap G_\beta \in \mathrm{Syl}_5(G_\alpha) \cap \mathrm{Syl}_5(G_\beta) \subseteq \mathrm{Syl}_5(G)$ and no non-trivial subgroup of $G_\alpha \cap G_\beta$ is normalized by G. Then G is isomorphic to the Lyons sporadic simple group.*

PROOF. This is one way to phrase the main theorem of [**52**]. □

We now move on to more familiar 2-local identifications.

LEMMA 3.11. *Let G be a finite group with $G = O^2(G)$. Assume that t is an involution in G and $C_G(t) \cong \mathrm{GL}_2(3)$. Then $G \cong \mathrm{PSL}_3(3)$ or* Mat(11)*.*

PROOF. This is taken from [**12**, Theorem 1A]. □

LEMMA 3.12. *Let G be a finite group with subgroups H and M such that*

(a) *H has normal subgroups H_1 and H_2 such that $H_1 \cong H_2 \cong \mathrm{SL}_2(3)$, $|H : H_1H_2| = 2$, $H_1 \cap H_2 = Z(H_1) = Z(H_2)$ and $H = C_G(H_1 \cap H_2)$; and*

(b) $M/O_2(M) \cong \mathrm{SL}_3(2)$ *and* $O_2(M)$ *is elementary abelian of order* 8 *with* $O_2(M) \geq Z(H_1)$.

Then $G \cong \mathrm{G}_2(3)$.

Proof. See [**3**]. $\square$

Lemma 3.13. *Suppose that* G *is a finite group,* $G = O^2(G)$ *and* $O_2(G) = O(G) = 1$. *Let* $S \in \mathrm{Syl}_2(G)$. *Assume that* S *is isomorphic to a Sylow* 2*-subgroup of* $\mathrm{Alt}(8)$ *and that the centralizer of a central involution of* S *is soluble. Then* $G \cong \mathrm{Alt}(8)$, $\mathrm{Alt}(9)$ *or* $\mathrm{PSp}_4(3)$.

Proof. This is [**23**, Corollary A*]. $\square$

Lemma 3.14. *Suppose that* G *is a finite group and* $T \in \mathrm{Syl}_2(G)$. *Assume that* $G = O^2(G)$ *and that* $T \cong \mathrm{Dih}(8) \wr 2$. *Then* $G/O(G) \cong \mathrm{Alt}(10)$, $\mathrm{Alt}(11)$, $\mathrm{PSL}_4(q)$, $q \equiv 3 \pmod 4$ *or* $\mathrm{PSU}_4(q)$, $q \equiv 1 \pmod 4$.

Proof. See [**43**, Theorem 3.15]. $\square$

Lemma 3.15. *Suppose that* G *is a group of parabolic characteristic* 2 *and* H *a subgroup of* G *of odd index. If* $F^*(H) \cong \mathrm{P}\Omega_8^+(2)$ *and* $H = N_G(F^*(H))$, *then* $F^*(G) \cong \Omega_8^+(2)$ *or* $\mathrm{P}\Omega_8^+(3)$.

Proof. This is [**63**]. $\square$

Lemma 3.16. *Suppose that* G *is a finite group,* $F^*(G)$ *is simple and* G *has abelian Sylow* 2*-subgroups of order at least* 4. *Then* $G/F^*(G)$ *has odd order and* $F^*(G)$ *is isomorphic to one of* $\mathrm{SL}_2(2^e)$, $e \geq 2$, ${}^2\mathrm{G}_2(3^{2e+1})$, $e \geq 1$, J_1 *or* $\mathrm{PSL}_2(r^b)$ *where* r *is a prime with* $r^b \equiv 3, 5 \pmod 8$. *Furthermore, if* G *is simple and* $S \in \mathrm{Syl}_2(G)$ *has order* 2^a, *then* $N_G(S)$ *contains a cyclic subgroup of order* $2^a - 1$.

Proof. Let $S \in \mathrm{Syl}_2(G)$. For $|S| > 8$, this is proved by Walter [**77**, Theorem 1]. For $|S| = 8$, a combination of at least [**35**, Theorem], [**36**, Theorem A], [**78**, Theorem] and [**11**, Corollary] is needed. $\square$

4. Strongly p-embedded subgroups

Recall that for a prime p, a proper subgroup Y of a group X is *strongly* p*-embedded* in X if and only if Y has order divisible by p and $Y \cap Y^x$ has order coprime to p for all $x \in X \setminus Y$. Strongly 2-embedded subgroups are often referred to a *strongly embedded* subgroups.

We shall frequently call upon the following lemma which gives an easily checked criteria for a subgroup to be strongly p-embedded.

Lemma 4.1. *Suppose that* p *is a prime and* H *is a proper subgroup of* G. *Let* $S \in \mathrm{Syl}_p(H)$. *Then* H *is strongly* p*-embedded in* G *if and only if* $C_G(x) \leq H$ *for all* $x \in S$ *of order* p *and* $N_G(S) \leq H$.

PROOF. See [**26**, Proposition 17.11]. $\square$

The next lemma presents an even more simple check in the case that G has local characteristic p.

LEMMA 4.2. *Suppose that p is a prime, G is a group and H is a proper subgroup of G. Assume that there exists a p-central element $x \in H$ such that $C_G(x) \leq H$ and $x^G \cap H = x^H$. If E is a non-trivial p-subgroup of H with $C_G(O_p(N_G(E))) \leq O_p(N_G(E))$, then $N_G(E) \leq H$. If in particular G is of local characteristic p, then H is strongly p-embedded in G.*

PROOF. Let $S \in \mathrm{Syl}_p(H)$ and $x \in Z(S)^\#$ be such that $C_G(x) \leq H$ and $x^G \cap H = x^H$. Assume that $x \in T \leq H$. Then, for $y \in N_G(T)$, we have $x^y \in T$ and so $x^y \in x^G \cap H = x^H$. Hence $x^{yh} = x$ for some $h \in H$. Thus $yh \in C_G(x)$ and $y \in C_G(x)H \leq H$ by hypothesis. Thus

$$N_G(T) \leq H, \text{ for } T \leq H \text{ with } T \cap x^G \neq \emptyset.$$

For $1 \neq E \leq S$ we have $x \in N_S(E)$. By the remark before and by conjugation in H we may assume $N_S(E) \in \mathrm{Syl}_p(N_G(E))$. Hence $O_p(N_G(E)) \leq S$ and, if $C_G(O_p(N_G(E))) \leq O_p(N_G(E))$ we get $x \in C_G(O_p(N_G(E))) \leq O_p(N_G(E))$. But then $N_G(E) \leq N_G(O_p(N_G(E))) \leq H$, the assertion. If in addition G is of local characteristic p, then $N_G(E) \leq H$ for all $1 \neq E \leq S$ and so H is strongly p-embedded in G by Lemma 4.1. $\square$

When $p = 2$, we have a stronger result due to Holt [**32**] which does not require the group to have local characteristic 2.

THEOREM 4.3 (Holt). *Suppose that K is a simple group, P is a proper subgroup of K and r is a 2-central element of K. If $r^K \cap P = r^P$ and $C_K(r) \leq P$, then $K \cong \mathrm{PSL}_2(2^a)$ $(a \geq 2)$, $\mathrm{PSU}_3(2^a)$ $(a \geq 2)$, ${}^2\mathrm{B}_2(2^a)$ $(a \geq 3$ and odd$)$ or $\mathrm{Alt}(n)$ $(n \geq 5)$ where in the first three cases P is a Borel subgroup of K and in the last case $P \cong \mathrm{Alt}(n-1)$.*

PROOF. This formulation of Holt's Theorem can be found as stated here in [**60**]. $\square$

We mostly apply Holt's Theorem in the following way.

LEMMA 4.4. *Suppose that L is a group, P is a subgroup of L and r is a 2-central element of P with $r \in F^*(P)$. Assume that $C_G(r) \leq P$ and $r^L \cap P = r^{F^*(P)}$. If $O(L) = 1$ and $F^*(P)$ is a non-abelian simple group which is not an alternating group, then $F^*(G) = F^*(P)$.*

Proof. Since $O_2(P) = 1$, $O_2(L) = 1$ and, as $O(L) = 1$, we have $F^*(L) = E(L)$. Since $1 \neq E(L)$ is non-abelian $1 \neq C_{E(L)}(r) \leq C_L(r) \leq P$, hence $E(L) \cap P$ is a non-trivial normal subgroup of P. Therefore $F^*(P) \leq E(L)$ as $F^*(P)$ is simple. Now r normalizes every component of $E(L)$ and so $F^*(P)$ is contained in every component of $E(L)$. Hence $E(L)$ is simple. Now $r^{E(L)} \cap P \subset r^L \cap P = r^{F^*(H)}$ and so $r^{E(L)} \cap P = r^{F^*(P)}$. Since $C_{E(L)}(r) \leq E(L) \cap P$, if $F^*(P) \neq F^*(L)$, then $P \cap E(L)$ is a proper subgroup of $E(L)$ and we may apply Holt's Theorem 4.3 to get $F^*(P)$ is an alternating group, a contradiction. $\square$

The next two propositions of this section show in almost all cases we cannot have a strongly p-embedded subgroup H such that $F^*(H)$ is a simple groups of Lie type in characteristic p. Notice that Proposition 4.5 follows immediately from Theorem 4.3; however, in the proof of Theorem 4.3 the results from [**9**] are deployed.

Proposition 4.5. *Let G be a group with a strongly 2-embedded subgroup H. Then H is soluble.*

Proof. This is a consequence of Bender's famous theorem [**9**]. $\square$

Proposition 4.6. *Suppose that p is an odd prime, G is a finite $\mathcal{K}_2$-group and H is a strongly p-embedded subgroup of a group G. Then $F^*(H)$ is not a simple group of Lie type defined in characteristic p and of Lie rank at least 2, unless perhaps $F^*(H) \cong \mathrm{PSL}_3(p)$.*

Proof. This is a combination of [**56**, Corollary 1.4] and [**57**, Theorem 1.1]. $\square$

Theorem 4.7. *Suppose that p is a prime, G is a group of local characteristic p and H is a subgroup of G such that $F^*(H)$ is a simple group of Lie type defined in characteristic p. Assume that when p is odd $F^*(H)$ has Lie rank at least 3 and when $p = 2$ that $F^*(H)$ has Lie rank at least 2. If $N_G(X) \leq H$ for all non-trivial subgroups X which are normal in a Sylow p-subgroup of H, then either $G = H$ or H is strongly p-embedded in G. Furthermore, if G is a $\mathcal{K}_{\{2,p\}}$-group when p is odd, then we have $G = H$.*

Proof. This is [**66**, Theorem 1, Theorem 2]. $\square$

5. Sylow embedded subgroups of linear groups

Throughout this section we assume that all groups are $\mathcal{K}$-groups. Since the application of these results will always be made in the normalizers of p-groups, this is compatible with the $\mathcal{K}_p$ hypothesis in our main theorems.

Definition 5.1. Let p be a prime, V be a vector space over $\mathrm{GF}(p)$ and L and M be subgroups of $\mathrm{GL}(V)$. Then L is *Sylow embedded* in M if

(i) $L \le M$;
(ii) $C_M(L)$ is a p'-group; and
(iii) $N_M(L)$ contains a Sylow p-subgroup of M.

Furthermore, a subgroup L of $\mathrm{GL}(V)$ is a *Sylow maximal* subgroup of $\mathrm{GL}(V)$ if whenever L is Sylow embedded in a subgroup M of $\mathrm{GL}(V)$, then L is a normal subgroup of M.

As motivation for addressing this type of question, consider the following scenario. Suppose that G is a finite group, $S \in \mathrm{Syl}_p(G)$, $r \in Z(S)^{\#}$ and H is a subgroup of G containing S. Assume that H is a group of Lie type defined in characteristic p and set $Q = O_p(C_{F^*(H)}(r))$. Then, on the road to proving our Theorems 2 and 3, we would like to determine the structure of $N_G(Q)$ if $N_G(Q) \not\le H$. Let $L_0 = N_H(Q)/Q$ and $L = O^{p'}(L_0)$. Then, taking $V = Q/\Phi(Q)$, we typically have that L acts irreducibly on V. Hence for $M = N_G(Q)/Q$ we have that $C_M(L)$ is a p'-group and so L is Sylow embedded in M. In this section we show that, in this kind of situation, L is typically Sylow maximal in $\mathrm{GL}(V)$. Thus one of the purposes of this section is to determine the exceptions to this statement. Tools for doing this are provided in Appendix C.

We start with the case where L is quasisimple. In the primary hypothesis of this section, we consider certain cases related to the possibility in which H is an exceptional group. So assume that H is one of the groups $\mathrm{F}_4(q)$ with q odd, $\mathrm{E}_6(q)$, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$ or $\mathrm{E}_8(q)$, let R be a long root subgroup of H, put $P = N_G(R)$ and let $L = O^{p'}(L_0)$ for L_0 as above. Then $V = O_p(P)/R$ is a $\mathrm{GF}(p)L$-module. We use Lemma D.1 to extract the following data and at the same time we establish some specific notation.

- If $H \cong \mathrm{F}_4(q)$ with q odd, then V is denoted by V_{14}, $|V| = q^{14}$ and $L \cong \mathrm{Sp}_6(q)$;
- if $H \cong \mathrm{E}_6(q)$ or ${}^2\mathrm{E}_6(q)$, then V is denoted by V_{20}, $|V| = q^{20}$ and $L/Z(L) \cong \mathrm{PSL}_6(q)$ or $\mathrm{PSU}_6(q)$ respectively;
- if $H \cong \mathrm{E}_7(q)$, then V is denoted by V_{32}, $|V| = q^{32}$ and $L/Z(L) \cong \mathrm{P}\Omega^+_{12}(q)$; and,
- if $H \cong \mathrm{E}_8(q)$, then V is denoted by V_{56}, $|V| = q^{56}$ and $L/Z(L) \cong \mathrm{E}_7(q)$.

Recall that the notation V_{14}, V_{20}, V_{32} and V_{56} was fixed in Lemma D.1 and in particular these are modules obtained from certain parabolic subgroups in groups of Lie type.

If L is Sylow embedded in M then we will denote by S a Sylow p-subgroup of L and by S_0 a Sylow p-subgroup of $N_M(L)$ containing S.

Our primary hypothesis specifies modules V and groups L which are assumed to be Sylow embedded.

HYPOTHESIS 5.2. *Let p be a prime, $q = p^e$ and L be quasisimple such that one of the following holds*

(a) *L is one of $\mathrm{SL}_n(q)$ with $n \geq 2$, $\mathrm{SU}_n(q)$ with $n \geq 3$, $\mathrm{Sp}_{2n}(q)'$ with $n \geq 2$, $\Omega_n^{\pm}(q)$ with $n \geq 5$ and V is the corresponding natural $\mathrm{GF}(q)L$-module.*

(b) *$L \cong \mathrm{SL}_n(q)$, $n \geq 2$ and V is a direct sum of the natural $\mathrm{GF}(q)L$-modules W and W^*.*

(c) *$L \cong \mathrm{SL}_2(q)$ and V is an irreducible 8-dimensional $\mathrm{GF}(q^{1/3})L$-module.*

(d) *$L \cong \mathrm{SL}_2(q)$ and V is an irreducible 4-dimensional $\mathrm{GF}(q^{1/2})L$-module.*

(e) *$L \cong \mathrm{PSL}_2(q)$, q odd, and V is an irreducible 3-dimensional $\mathrm{GF}(q)L$-module.*

(f) *$L \cong \mathrm{SL}_2(q)$, $p > 3$, and V is an irreducible 4-dimensional $\mathrm{GF}(q)L$-module which is absolutely irreducible.*

(g) *$L \cong \mathrm{SL}_2(4)$ and V has $\mathrm{GF}(2)$-dimension 8 and V has two L-composition factors of $\mathrm{GF}(2)$-dimension 4.*

(h) *$L \cong \mathrm{Sp}_6(q)$, q odd, and $V = V_{14}$ is the 14-dimensional $\mathrm{GF}(q)L$-module.*

(i) *$L/Z(L) \cong \mathrm{PSL}_6(q)$ or $\mathrm{PSU}_6(q)$ and $V = V_{20}$ is the 20-dimensional $\mathrm{GF}(q)L$-module.*

(j) *$L/Z(L) \cong \mathrm{P}\Omega_{12}^{+}(q)$ and $V = V_{32}$ is the 32-dimensional irreducible $\mathrm{GF}(q)L$-module.*

(k) *$L/Z(L) \cong \mathrm{E}_7(q)$ and $V = V_{56}$ is the 56-dimensional $\mathrm{GF}(q)L$-module.*

In all cases, regard V as a $\mathrm{GF}(p)L$-module and L as a subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}_m(p)$ where $m = \dim_{\mathrm{GF}(p)}(V)$.

Our first objective in this section is to prove the following proposition.

PROPOSITION 5.3. *Assume that Hypothesis 5.2 holds and that L is not Sylow maximal in $\mathrm{GL}(V)$. If L is Sylow embedded in $M \leq \mathrm{GL}(V)$ and L not normal in M, then one of the following holds:*

(i) *$L \cong \mathrm{SL}_2(4)$, $E(M) \cong \mathrm{Alt}(7)$ and V is either a natural $\mathrm{GF}(4)L$-module or a direct sum of two natural $\mathrm{GF}(4)L$-modules.*

(ii) *$L \cong \mathrm{SL}_2(5)$, $E(M) \cong \mathrm{SL}_2(9)$ and V is an irreducible 4-dimensional $\mathrm{GF}(5)L$-module.*

(iii) $L \cong \mathrm{SL}_2(7)$, $E(M) \cong 2{\cdot}\mathrm{Alt}(7)$ *and* V *is an irreducible* 4-*dimensional* $\mathrm{GF}(7)L$-*module.*

(iv) $L \cong \mathrm{PSL}_2(9)$, $E(M) \cong 2{\cdot}\mathrm{PSL}_3(4)$ *and* V *is a* 3-*dimensional* $\mathrm{GF}(9)L$-*module.*

(v) $L \cong \mathrm{PSp}_4(2)'$, $E(M) \cong \mathrm{Alt}(7)$ *and* V *is a natural* $\mathrm{GF}(2)L$-*module.*

(vi) $L \cong \mathrm{SL}_2(5)$, $F(M) \cong 2^{1+4}_-$ *or* $4 \circ 2^{1+4}_-$ *and either*

  (a) $LF(M)$ *is normal in* M *and* $M/F(M) \cong \mathrm{Alt}(5)$ *or* $\mathrm{Sym}(5)$; *or*

  (b) $F(M) = 4 \circ 2^{1+4}_-$ *and* $M/F(M) \cong \mathrm{Alt}(6)$ *or* $\mathrm{Sym}(6)$.

  Furthermore, V *is a* 4-*dimensional irreducible* $\mathrm{GF}(5)L$-*module.*

Furthermore, if $E(M) \neq 1$, *then* $L \leq E(M)$.

Remark 5.4. *It is worth noting that in Proposition 5.3 (iv) for example, we have* $M \leq \mathrm{GL}_6(3)$ *and so* M *does not support the* $\mathrm{GF}(9)$ *vector space structure.*

For the proof of Proposition 5.3 the following little lemma will be of great importance as it reduces the possibilities for L substantially.

Lemma 5.5. *Assume Hypothesis* 5.2 *with* $p = 2$ *and* $q > 2$. *Then there is an elementary abelian subgroup* A *of order* 4 *in* L *such that* $C_V(a) = C_V(A)$ *for all* $a \in A^{\#}$.

Proof. If we have one of the classical groups, then a root group of order q acts in this way on the natural module. In the exceptional cases the same result is provided by Lemma D.17(i). □

Throughout the proof of Proposition 5.3 we consider subgroups M of $\mathrm{GL}(V)$ with L Sylow embedded in M. We begin with the cases where M is quasisimple. Our proof of Proposition 5.3 proceeds through a series of lemmas. Our first one deals with the possibility that M is a group of Lie type defined in characteristic p.

Lemma 5.6. *Suppose that Hypothesis* 5.2 *holds,* $M \leq \mathrm{GL}(V)$ *is quasisimple and* $M/Z(M)$ *is a simple group of Lie type in characteristic* p. *If* L *is Sylow embedded in* M, *then* $M = L$.

Proof. Since L is Sylow embedded in M, $N_M(L)$ contains a Sylow p-subgroup of M. Therefore Lemma A.17 implies that $N_M(L)/Z(M) = M/Z(M)$ and then as M is quasisimple, $M = L$ as claimed. □

Lemma 5.7. *Suppose that Hypothesis* 5.2 *holds with* $L/Z(L) \cong \mathrm{PSL}_2(q)$. *Assume that* $M \leq \mathrm{GL}(V)$, M *is quasisimple and* L *is Sylow embedded in* M. *Then either* $M = L$ *or one of the following holds:*

(i) $L \cong \mathrm{SL}_2(4)$, $M \cong \mathrm{Alt}(7)$ *and* V *is either a natural* $\mathrm{GF}(4)L$*-module or a direct sum of two natural* $\mathrm{GF}(4)L$*-modules.*

(ii) $L \cong \mathrm{SL}_2(5)$, $M \cong \mathrm{SL}_2(9) \cong 2{\cdot}\mathrm{Alt}(6)$ *and* V *is an irreducible* 4*-dimensional* $\mathrm{GF}(5)L$*-module.*

(iii) $L \cong \mathrm{SL}_2(7)$, $M \cong 2{\cdot}\mathrm{Alt}(7)$ *and* V *is an irreducible* 4*-dimensional* $\mathrm{GF}(7)L$*-module.*

(iv) $L \cong \mathrm{PSL}_2(9)$, $M \cong 2{\cdot}\mathrm{PSL}_3(4)$ *and* V *is a* 3*-dimensional* $\mathrm{GF}(9)L$*-module.*

Proof. We consider the possibilities for the simple group $M/Z(M)$ in turn. By Lemma 5.6 we know that $M/Z(M)$ is not a simple group of Lie type in characteristic p.

Fix $S \in \mathrm{Syl}_p(L)$. In the configurations described by Hypothesis 5.2 (a) or (b), S acts quadratically on V by Lemma C.14. Hence, in these cases, if p is odd, then Lemma C.12 yields $q = p = 3$ and this is impossible as $\mathrm{SL}_2(3)$ is not perfect. Therefore, if Hypothesis 5.2 (a) or (b) holds, then we additionally have $p = 2$.

(5.7.1) If $M/Z(M) \cong \mathrm{Alt}(m)$ with $m \geq 7$ and $m \neq 8$, then (i) or (iii) holds.

Assume that $M/Z(M) \cong \mathrm{Alt}(m)$ with $m \geq 7$, $m \neq 8$. Suppose first that $M/Z(M) \cong \mathrm{Alt}(7)$. Then we can only have

$$L/Z(L) \cong \mathrm{PSL}_2(4),\ \mathrm{PSL}_2(5),\ \mathrm{PSL}_2(9) \text{ or } \mathrm{PSL}_2(7).$$

If $q = p$, then we require that $M/Z(M)$ is a subgroup of $\mathrm{PSL}_4(p)$. Thus $q \neq 5$, as 7 does not divide $|\mathrm{PSL}_4(5)|$. If $p = 7$, then, as 5 does not divide the order of $\mathrm{PSL}_3(7)$, we have that V is a 4-dimensional $\mathrm{GF}(7)L$-module and $L \cong \mathrm{SL}_2(7)$. This yields $M \cong 2{\cdot}\mathrm{Alt}(7)$ or $6{\cdot}\mathrm{Alt}(7)$. As 3 does not divide $|Z(SL_4(7))|$, we get $M \cong 2{\cdot}\mathrm{Alt}(7)$. Assume that Hypothesis 5.2 (b) holds. Then V is a direct sum of two 2-dimensional L-modules and so a 7-element in M acts quadratically on V, which contradicts Lemma C.12. Hence Hypothesis 5.2 (b) does not hold and so we have the configuration listed in (iii). Thus we may assume that $q \in \{4, 9\}$. Assume $L \cong \mathrm{SL}_2(4)$. Then the $\mathrm{GF}(2)$-dimension of V is either 4 or 8 and all the composition factors for L have dimension 4. As again 3 does not divide $|Z(\mathrm{SL}_4(4))|$ and $\mathrm{Alt}(7)$ possesses no irreducible 8-dimensional module over $\mathrm{GF}(2)$ (see [**10**]), we may suppose that M leaves a 4-dimensional $\mathrm{GF}(2)$ subspace V_0 of V invariant. In particular, we have that V_0 is either the natural $\mathrm{GF}(4)L$-module or the second irreducible 4-dimensional $\mathrm{GF}(2)L$-module. Furthermore, as L is Sylow embedded in M, $|M : N_M(L)|$ is odd, so we have $N_M(L) \cong \mathrm{Sym}(5)$ and so by Lemma E.8 we see that V_0 is a natural $\mathrm{GF}(4)L$-module. This is listed in part (i).

Suppose that $M/Z(M) \cong \mathrm{Alt}(m)$ with $m \geq 9$. By Galois [**33**, Satz II.8.28], the minimal permutation degree of $L/Z(L) \cong \mathrm{PSL}_2(q)$ is at least q unless $q = 9$ in which case it is 6. Suppose that $q = 9$. Then $p = 3$ and plainly $N_M(L)$ does not contain some Sylow 3-subgroup of M. Therefore we have $q \neq 9$ and

$$p^e = q \leq m.$$

By combining Lemma C.4 with Hypothesis 5.2, we obtain

$$p^{p^e-2} \leq p^{m-2} \leq |V| \leq p^{4e}.$$

Since $p^e - 2 \leq 4e$ and $m \geq 9$, we have $e \geq 2$ and so $q \in \{2^2, 2^3, 2^4\}$ as $q \neq 9$. Because the Sylow 2-subgroups of $\mathrm{Alt}(9)$ have order 2^6 and L is Sylow embedded in M, we obtain $q = 2^4$ and $m \in \{9, 10\}$. But then $q > m \geq q$ which is absurd. This completes the consideration of the alternating groups when $m \neq 8$. ∎

We next consider the sporadic simple groups.

(5.7.2) We have $M/Z(M)$ is not a sporadic simple group.

Suppose that $M/Z(M)$ is a sporadic simple group and let $S_0 \in \mathrm{Syl}_p(N_M(L))$ with $S_0 \cap L = S$. Then, as $L \cong \mathrm{SL}_2(q)$, S is elementary abelian and, as $C_M(L)$ is a p'-group by Definition 5.1 (ii), S_0L/L is isomorphic to a subgroup of $\mathrm{Out}(L)$. Therefore, from [**27**, Theorem 2.5.12] we obtain S_0/S is cyclic of order dividing e_p, the p-part of e.

Suppose that $p = 2$. Then we have $|S_0| \leq 2^e e_2$. So as $e \leq m_2(S)$, we see that

$$|S_0| \leq 2^{m_2(S_0)} m_2(S_0).$$

Manipulating the data from [**27**, Table 5.3 and 5.6.1] yields $M \cong \mathrm{J}_1$ and so S_0 is elementary abelian of order 8. In particular, $L \cong \mathrm{PSL}_2(8)$ and so L has cyclic subgroups of order 9 whereas $|\mathrm{J}_1|_3 = 3$. Hence $p \neq 2$.

Suppose that p is odd. If $e \leq 2$, then $e_p = 1$ and so $S_0 = S$ is elementary abelian. Moreover, Hypothesis 5.2 yields $|V| \leq p^{4e}$ which means that $R_p(M) \leq 4e \leq 8$. Application of Lemma C.2 shows that $M/Z(M) \cong \mathrm{Mat}(11)$, $\mathrm{Mat}(12)$, $\mathrm{Mat}(22)$, J_1 or J_2 and $R_p(M) \geq 5$. In particular, $e = 2$ and this fact then shows $M/Z(M) \not\cong \mathrm{Mat}(12)$ or J_1 just by considering $|M|$. If $M \cong \mathrm{J}_2$, then the only possibility is that $p = 5$ and, as 13 divides $|\mathrm{SL}_2(25)|$ but not $|\mathrm{J}_2|$, we have a contradiction. Thus $M/Z(M) \cong \mathrm{Mat}(11)$ or $\mathrm{Mat}(22)$ and we have $p = 3$.

By Lemma C.12, S does not act quadratically on V and so, as $|V| \geq 3^5$, we see that Hypothesis 5.2 (e) holds. Thus V is a 3-dimensional $\mathrm{GF}(9)L$-module of order 3^6. By Lemma C.3 we have $M/Z(M) \not\cong \mathrm{Mat}(11)$ or $\mathrm{Mat}(22)$.

Hence $e \geq 3$. Suppose that S_0 is non-abelian. Then $p^e = |S| < |S_0| \leq p^e e_p$ and $m_p(S_0) \geq p$. Applying [**27**, Table 5.6.1] yields $p = 3$. Thus $e_3 \neq 1$ and

$$|S_0| \leq 3^e e_3 \leq 3^{m_3(S_0)} m_3(S_0)$$

with $m_3(S_0) \geq 3$. Now using [**27**, Table 5.3 and Table 5.6.1] again, we have $m_3(S) = 3$ and obtain a contradiction to $|S|$. Therefore, S_0 is abelian. We again consult [**27**, Tables 5.3 and 5.6.1] to obtain $p = 3$ and $M \cong$ O'N with $e = 4$. But $\mathrm{PSL}_2(81)$ has order divisible by 41 whereas O'N does not. We have demonstrated that M is not a sporadic group.■

(5.7.3) If $M/Z(M)$ is a group of Lie type defined in characteristic r with $r \neq p$, then parts (ii) and (iv) hold.

We start by using Lemma C.7 to obtain

$$e = m_p(L) \leq m_p(M/Z(M)) \leq R_r(M/Z(M)).$$

On the other hand, by Hypothesis 5.2, $R_{r'}(M/Z(M)) \leq 4e$ and therefore

$$R_{r'}(M/Z(M)) \leq 4R_r(M/Z(M)).$$

Now application of Lemma C.6 gives a list of candidates for $M/Z(M)$:

- $\mathrm{PSL}_2(r^f)$, $r^f \leq 17$ with r odd, $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_4(2)$.
- $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_3(4)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_5(2)$, $\mathrm{PSU}_6(2)$.
- $\mathrm{PSp}_4(2)'$, $\mathrm{PSp}_4(3)$, $\mathrm{PSp}_4(5)$, $\mathrm{PSp}_6(2)$, $\mathrm{PSp}_6(3)$, $\mathrm{P\Omega}_7(3)$, $\mathrm{P\Omega}_8^+(2)$, $\mathrm{P\Omega}_8^-(2)$.
- $\mathrm{F}_4(2)$, $\mathrm{G}_2(2)'$, $\mathrm{G}_2(3)$, $\mathrm{G}_2(4)$, ${}^3\mathrm{D}_4(2)$, ${}^2\mathrm{F}_4(2)'$, ${}^2\mathrm{B}_2(8)$, ${}^2\mathrm{G}_2(3)'$.

Since by Lemma E.4 $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(3)$, ${}^2\mathrm{G}_2(3)'$ and ${}^2\mathrm{B}_2(8)$ are minimal simple groups, we have $L = M$ in this case and so these groups are eliminated from our further considerations.

Suppose that $p = 2$. Then r is odd and the Sylow 2-subgroups of M must be extensions of an elementary abelian group of rank e by a cyclic group of order dividing e_2. By applying this remark to the candidates for $M/Z(M)$ above and using [**27**, Theorem 4.10.2], we only retain $M/Z(M) \cong \mathrm{PSL}_2(r^f)$ with $r^f \leq 17$. But then $m_2(S) = m_2(S_0) = e = 2$ and so $L \cong \mathrm{SL}_2(4) \cong \mathrm{Alt}(5)$ with $|V| \leq 2^8$ and $|S| \leq 8$. Thus, we further see that, we can only have $M/Z(M) \cong \mathrm{PSL}_2(11)$ as $\mathrm{SL}_2(4)$ is not Sylow maximal in $\mathrm{PSL}_2(9)$. Since 11 does not divide $|\mathrm{GL}_8(2)|$, this is impossible. Hence

$$p \text{ is odd.}$$

If $e = 1$, then $L/Z(L) \cong \mathrm{PSL}_2(p)$ with $p > 3$ and M has Sylow p-subgroups of order p. Furthermore, M is a subgroup of $\mathrm{SL}_4(p)$. As p is odd, we know Hypothesis 5.2 (a) and (b) do not hold and so, as $q = p$, we deduce that Hypothesis 5.2 (e) or (f) holds. Application of column two from Lemma C.5 shows that $M/Z(M) \cong \mathrm{PSL}_2(r^f)$ with $r^f \leq 9$, $\mathrm{PSp}_4(3)$ or $\mathrm{PSL}_3(4)$. In particular $p = 5$ or 7.

Suppose that $L/Z(L) \cong \mathrm{PSL}_2(5)$. If $M/Z(M) \cong \mathrm{PSL}_2(9)$, then $|V| = 5^4$ and this is possibility (ii) of the lemma. As $\mathrm{GL}_4(5)$ does not contain elementary abelian subgroups of order 27 and $\mathrm{PSp}_4(3)$ does, we have that $M/Z(M) \not\cong \mathrm{PSp}_4(3)$. Similarly, 7 does not divide $|\mathrm{GL}_4(5)|$ but does divide $|\mathrm{PSL}_3(4)|$ and so $M/Z(M) \cong \mathrm{PSL}_3(4)$ is impossible.

Suppose that $L/Z(L) \cong \mathrm{PSL}_2(7)$. Then $M/Z(M) \cong \mathrm{PSL}_3(4)$ and we have $|V| = 7^4$ by Lemma C.5. Since the group of scalars in $\mathrm{SL}_4(7)$ has order 2, we see that $M \cong \mathrm{PSL}_3(4)$ or $2{\cdot}\mathrm{PSL}_3(4)$ both of these groups have 2-rank at least 4 and this is greater than the 2-rank of $\mathrm{SL}_4(7)$ (which is 3) and so this case cannot occur. Therefore,

$$p \text{ is odd, } e \geq 2 \text{ and } p^e \geq 9.$$

Assume that p is odd, $e \geq 2$ and $p^e \geq 25$. We turn our considerations round and regard the embedding of L into M as a projective representation of L. Applying Lemma C.5 delivers $R_r(L) \geq 12$. Hence Lemma C.5 again yields either $M/Z(M) \cong \mathrm{F}_4(2)$ or ${}^2\mathrm{F}_4(2)'$ and $R_r(M) \geq 26$ (recall $r = 2$). Note that by [**27**, 4.10.3] and [**24**, Table 10.1 and 10.2] $m_3(\mathrm{F}_4(2)) = 4$ and $m_3({}^2\mathrm{F}_4(2)') = 2$. We deduce that $L/Z(L) \cong \mathrm{PSL}_2(25)$, $\mathrm{PSL}_2(3^3)$ or $\mathrm{PSL}_2(3^4)$. If $p = 5$, then $f = 2$ and we require $M/Z(M)$ to embed into $\mathrm{PSL}_8(5)$. Lemma C.5 shows that this is impossible for the candidate groups. Hence $p = 3$ and we have $M/Z(M) \cong \mathrm{F}_4(2)$. Since M has Sylow 3-subgroups of order 3^6 and $|\mathrm{Aut}(L)|_3 \leq 3^4$, we have a contradiction. Hence, for p odd and $e > 1$, we have $p^e = 9$. Now

$$L/Z(L) \cong \mathrm{PSL}_2(9), |S_0| = 9 \text{ and } M \leq \mathrm{SL}_8(3).$$

From the displayed list of candidates for $M/Z(M)$, we now only have to consider $M/Z(M) \cong \mathrm{PSL}_3(4)$, $\mathrm{PSL}_4(2)$, $\mathrm{PSp}_4(5)$. As $M \leq \mathrm{SL}_8(3)$, we see that $M/Z(M) \not\cong \mathrm{PSp}_4(5)$ as the Sylow 5-subgroup is too big. If $M/Z(M) \cong \mathrm{PSL}_4(2)$, then there exists $Y \leq M$ with L Sylow embedded in Y and $Y/Z(Y) \cong \mathrm{Alt}(7)$. We have already shown that this cannot occur in (5.7.1). Thus $M/Z(M) \cong \mathrm{PSL}_3(4)$. Since S has order 9, Lemma C.12 implies that Hypothesis 5.2 (d) or (e) holds. In the former case, M embeds into $\mathrm{SL}_4(3)$ which is impossible as 7 does not divide the order of the latter group. Thus Hypothesis 5.2 (e) holds and this is listed as case (iv). ■

Finally, we note that $\mathrm{Alt}(5) \cong \mathrm{SL}_2(4)$, $\mathrm{Alt}(6) \cong \mathrm{PSp}_4(2)'$, $\mathrm{Alt}(8) \cong \mathrm{PSL}_4(2)$ and so Lemma 5.6 implies that $p \neq 2$. Thus these groups fit in (5.7.3). Therefore combining (5.7.1), (5.7.2) and (5.7.3) we have the claimed result.

□

LEMMA 5.8. *Assume Hypothesis* 5.2 *holds with p odd and $L/Z(L) \not\cong \mathrm{PSL}_2(q)$. If $M \leq \mathrm{GL}(V)$, M is quasisimple and L is Sylow embedded in M, then $M = L$.*

PROOF. Assume that $M > L$ and let $S_0 \in \mathrm{Syl}_p(N_M(L)) \subseteq \mathrm{Syl}_p(M)$ which contains a Sylow p-subgroup S of L. By Lemma 5.6, M is not a group of Lie type in characteristic p.

Suppose first that Hypothesis 5.2 (a) holds. Then $L/Z(L)$ is a classical group and V is a natural module. In particular, L contains a non-trivial quadratic subgroup T which we select to have maximal possible order. By Lemma C.12, $|T| = 3$. Employing Lemma C.14 yields $L \cong \mathrm{SU}_3(3)$, $\Omega_5(3)$ or $\Omega_6^-(3)$ with V the corresponding natural module

For the remaining cases of Hypothesis 5.2, we may identify V with Q/R in Lemma D.17. In each case Lemma D.17 provides a non-trivial quadratically acting group. Lemma C.12 (i) shows that this group has order 3. Application of Lemma D.17 (ii) then yields $(L/Z(L), V) = (\mathrm{PSU}_6(3), V_{20})$, $(\mathrm{PSp}_6(3), V_{14})$ or $(\mathrm{E}_7(3), V_{56})$. Thus we have six cases to consider more deeply and in all cases we have $\dim V \leq 56$ and $S_0 = S$.

Since V admits a quadratic subgroup of order 3 in L, the possibilities for $M/Z(M)$ are enumerated in Lemma C.12. Thus $M/Z(M) \cong \mathrm{PSU}_n(2)$ with $n \geq 5$, $\mathrm{Alt}(n)$ with $n \geq 5$, $\mathrm{P}\Omega_8^+(2)$, $\mathrm{G}_2(4)$, $\mathrm{PSp}_6(2)$, Co_1, Suz or J_2.

By Lemma B.2 applied to L, we see that $M/Z(M) \not\cong \mathrm{Alt}(n)$ with $n \geq 5$.

If $M/Z(M) \cong \mathrm{PSU}_n(2)$ with $n \geq 5$, then Lemma C.5 implies that $n \leq 7$. Hence $|S| \in \{3^5, 3^6, 3^8\}$. Hence $L/Z(L) \cong \Omega_6^-(3)$, $M \cong \mathrm{PSU}_6(2)$ and $|V| = 3^6$. But $|\mathrm{PSU}(6,2)|$ does not divide $|\mathrm{PSL}_6(3)|$ and so we have a contradiction.

Suppose that M is one of the cases from Lemma C.12(iii). Then $|S| \leq 3^9$. So $L/Z(L) \not\cong \mathrm{PSU}_6(3)$, or $\mathrm{E}_7(3)$. If $L \cong \mathrm{PSU}_3(3)$, then $|S| = 27$. Hence $M/Z(M) \cong \mathrm{J}_2$ or $\mathrm{G}_2(4)$ and $|V| = 3^6$. But 5^2 does not divide $|\mathrm{GL}_6(3)|$ and divides $|J_2|$ and $|G_2(4)|$. If $L \cong \Omega_5(3)$, then $|S| = 3^4$. So we have $M \cong \mathrm{PSp}_6(2)$ and M is isomorphic to a subgroup of $\mathrm{GL}_5(3)$ which is also impossible as 7 does not divide $|\mathrm{GL}_5(3)|$. If $L \cong \Omega_6^-(3)$, then $|S| = 3^6$ and there are no candidates for $M/Z(M)$.

Finally, if $L/Z(L) \cong \mathrm{PSp}_6(3)$, then $M/Z(M) \cong \mathrm{Co}_1$ and $|V| = 3^{14}$. Since $|\mathrm{Co}_1|$ does not divide $|\mathrm{PSL}_{14}(3)|$, we have a contradiction.

Having considered all the possibilities, we have proved the lemma. $\square$

LEMMA 5.9. *Assume Hypothesis* 5.2 *holds with* $p = 2$ *and* $L/Z(L) \not\cong \mathrm{PSL}_2(q)$. *If* $M \le \mathrm{GL}(V)$, M *quasisimple and* $L \le M$ *is Sylow embedded in* M, *then either* $M = L$ *or* $L \cong \mathrm{Sp}_4(2)'$, $M \cong \mathrm{Alt}(7)$ *and* V *is the natural* $\mathrm{GF}(2)L$*-module.*

PROOF. Assume $L \neq M$. Again we consider the possibilities for $M/Z(M)$. Let $S_0 \in \mathrm{Syl}_2(N_M(L))$. By Lemma 5.6 we have that M is not a group of Lie type in characteristic 2.

As $L/Z(L) \neq \mathrm{PSL}_2(q)$, cases (c)-(g) of Hypothesis 5.2 do not hold. If L satisfies Hypothesis 5.2 (a) or (b), then Lemmas C.14 and C.16 show that L has a quadratic fours group whereas, if one of Hypothesis 5.2(h), (i), (j) or (k) holds, then Lemma D.17 (ii) provides the same result. Hence in each case L has a quadratic 4-subgroup, so Lemma C.13 provides the candidates for $M/Z(M)$.

Suppose $M/Z(M) \cong \mathrm{Alt}(n)$. Then Lemma B.1 to LS_0 yields $L/Z(L) \cong \mathrm{Alt}(m)$ where $m = n$, $n-1$, $n-2$ or $n-3$, or $n = 7$. Assume first that $n = 7$. Then $M \cong \mathrm{Alt}(7)$ or $3{}^{\cdot}\mathrm{Alt}(7)$ and

$$L \cong \mathrm{PSL}_3(2) \text{ or } \mathrm{Sp}_4(2)'. \text{ Furthermore, } 2^4 \le |V| \le 2^6.$$

If $L \cong \mathrm{SL}_3(2)$, then V is a direct sum of two irreducible modules. But letting $\langle x, f\rangle$ be a Frobenius subgroup of order 20 with f of order 4, [**22**, Chap. 11, Theorem 1.1] shows that $[V, f, f, f] \neq 0$ whereas the action of L shows that $[V, f, f, f] = 0$. This contradiction shows that $L \not\cong \mathrm{SL}_3(2)$.

If $L \cong \mathrm{Sp}_4(2)' \cong \mathrm{Alt}(6)$, we get $M \cong \mathrm{Alt}(7)$ acting in its 4-dimensional representation as described in the statement of the lemma.

Assume that $n \ge 8$. Then as the only isomorphisms between L and $\mathrm{Alt}(n)$ are $\mathrm{Alt}(8) \cong \mathrm{SL}_4(2) \cong \Omega_6^+(2)$, $|V| \le 2^8$ and $n \in \{9, 10, 11\}$. Since 11 does not divide $|\mathrm{GL}_8(2)|$, we have $9 \le n \le 10$. As $R_2(\mathrm{Alt}(10)) \ge R_2(\mathrm{Alt}(9)) \ge 7$ by Lemma C.4, we have $|V| = 2^8$ and V is a direct sum of two natural $\mathrm{SL}_4(2)$-modules. In particular, the elements corresponding to 3-cycles of $\mathrm{Alt}(8)$ act fixed-point-freely on V and the involutions of cycle type 2^4 centralize a 6-dimensional subspace (as such correspond to transvections in $\mathrm{SL}_4(2)$). But in $\mathrm{Alt}(9)$ these involutions invert a 3-cycle and so on the 8-dimensional module for $\mathrm{Alt}(9)$ they have centralizer of order 2^4, a contradiction. Therefore

$$M/Z(M) \text{ is not an alternating group of degree } n \ge 8.$$

If M is a group of Lie type in odd characteristic, then $M/Z(M) \cong \mathrm{PSU}_4(3)$, $|Z(M)|$ divides 9 and V has GF(2) dimension a multiple of 12. Furthermore, $|S_0| = 2^7$. Thus $L \cong \mathrm{SL}_3(4)$, or $\mathrm{SU}_3(4)$ and $|V| = 2^{12}$. Lemma C.5 shows that $\mathrm{PSU}_3(4)$ has no projective representation of dimension 4 over fields of characteristic 3. Therefore $L \cong \mathrm{SL}_3(4)$ and V is a direct sum of two natural L-modules. Since $|S| = 2^6$, we require $|N_M(L) : LZ(M)| = 2$. But we see that $\mathrm{PSU}_4(3)$ does not contain $\mathrm{PSL}_3(4).2$ as by [**14**, Table 8.11] $\mathrm{PSL}_3(4)$ is a maximal subgroup in $\mathrm{PSU}_4(3)$. Thus we have a contradiction. Therefore

M is not a group of Lie type defined in odd characteristic.

Suppose now that $M/Z(M)$ is a sporadic simple group. Then by Lemma C.13 we have $M/Z(M) \cong \mathrm{Mat}(12)$, $\mathrm{Mat}(22)$, $\mathrm{Mat}(24)$, J_2, Co_1, Co_2 or Suz. In particular, $2^5 \leq |S| \leq 2^{21}$ and the maximal order of a quadratically acting group is either 4 or $M/Z(M) \cong \mathrm{Mat}(22)$ and the maximal order is 16. In Lemma C.13 we also find the dimensions for the irreducible modules for M and, in particular, we see that there is no 20-dimensional irreducible representation. It follows that Hypothesis 5.2 (i), (j) and (k) cannot hold. Thus Hypothesis 5.2 (a) or (b) holds and so either L is a classical group acting naturally on V or V is a direct sum of two natural modules for $L \cong \mathrm{SL}_n(q)$.

Suppose that $L \cong \mathrm{SL}_n(q)$. Then, as $n \geq 3$ and $|S| \geq 2^5$, we have $L \cong \mathrm{SL}_3(4)$ or $\mathrm{SL}_4(2)$ by Lemma C.14. Since Lemma C.13 gives $|V| \geq 2^{10}$, we conclude that $L \cong \mathrm{SL}_3(4)$, $|V| = 4^6 = 2^{12}$ and $M/Z(M) \cong \mathrm{Mat}(22)$. Since $N_M(L)$ contains a Sylow 2-subgroup of M and M has no faithful permutation representation of degree $|M|/|Z(M)LS_0| = 11$, we have a contradiction.

Now we assume that Hypothesis 5.2 (a) holds and that $L \not\cong \mathrm{SL}_n(q)$. Then, as the maximal order of a quadratic subgroup in M is 16, Lemma C.14 yields L is isomorphic to one of $\mathrm{Sp}_4(2)'$, $\mathrm{Sp}_6(2)$, $\Omega_6^+(2)$, $\Omega_6^-(2)$, $\Omega_8^-(2)$, $\mathrm{SU}_4(2)$, $\mathrm{SU}_5(2)$ or $\mathrm{SU}_3(q)$ with $4 \leq q \leq 16$. Since $|V| \geq 2^{10}$ by Lemma C.13, we have $L \cong \mathrm{SU}_5(2)$ or $\mathrm{SU}_3(q)$ with $q \in \{4, 8, 16\}$. In the first case $|V| = 2^{10}$ and so we have $M \cong \mathrm{Mat}(12)$ or $\mathrm{Mat}(22)$ both of which have order smaller than the order of L. So $L \cong \mathrm{SU}_3(q)$ with $4 \leq q \leq 16$. This gives $|V| = 4^6 = 2^{12}$, $8^6 = 2^{18}$ or $16^6 = 2^{24}$ and L contains a quadratically acting group of order q. If $q \geq 8$, then $M/Z(M) \cong \mathrm{Mat}(22)$ and, if $q = 4$, then $|V| = 2^{12}$ and Lemma C.13 implies that $F^*(M) \cong 3{\cdot}\mathrm{Mat}(22)$ or J_2. But then in both cases $|L|$ does not divide $|M|$, and we have a contradiction. This completes the proof of the lemma. □

LEMMA 5.10. *Assume that Hypothesis* 5.2 *holds with* $L/Z(L) \cong \mathrm{PSL}_2(q)$*. If* r *is a prime with* $r \neq p$ *and* E *is an* r*-subgroup of* $\mathrm{GL}(V)$ *which is normalized by* L*, then either* L *centralizes* E *or* $L \cong \mathrm{SL}_2(5)$*,* $|V| = 5^4$ *and* $E \cong 2^{1+4}_-$ *or* $4 \circ 2^{1+4}_-$ *and* L *acts irreducibly on* V*.*

PROOF. Assume that L acts non-trivially on E. If V is a direct sum of at most two natural $\mathrm{SL}_2(q)$-modules, then Lemma C.11 yields $p = 2$ as S operates quadratically on V and $|S| \neq 3$. Hence $q = 2^f$. Furthermore, $C_V(s) = [V, S]$ for all $s \in S^\#$ and $|S| \geq 2^2$ and so $[E, L] = 1$ by Lemma 2.14. Thus Hypothesis 5.2 (a) and (b) cannot hold.

If L does not act irreducibly on V, then Hypothesis 5.2 (g) holds. So in this case $L \cong \mathrm{SL}_2(4)$ and $|V| = 2^8$. As EL is contained in $\mathrm{GL}_8(2)$ and since $|\mathrm{GL}_8(2)| = 2^{28} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 17 \cdot 31 \cdot 127$ and 3 divides $|L|$, we deduce that E is elementary abelian of order 3^4. But the minimal degree of a non-trivial permutation representation of $\mathrm{SL}_2(4)$ is 5 by Galois [**33**, Satz II.8.28] and $l(r, 2) = 2$, and therefore Lemma 2.23 delivers $|V| \geq 2^{10}$, which is a contradiction. Thus

$$L \text{ acts irreducibly on } V$$

and by Lemma 2.24

$$C_E(L) \text{ is cyclic.}$$

Assume that E is elementary abelian and suppose first that $q \neq 9$. Then by [**33**, Satz II.8.28] the minimum permutation degree of L is at least q, and so Lemma 2.23 implies that V has $\mathrm{GF}(p)$-dimension at least $ql(p, r)$. We consider the various possibilities for V.

If Hypothesis 5.2 (c) holds, then $|V| = q^{8/3}$. Thus $p^q = p^{p^e} \leq p^{8e/3}$, which means that $3p^e \leq 8e$. As 3 divides e, the only solution to this equation is $p = 2$ and $e = 3$. So $L \cong \mathrm{SL}_2(8)$ and $|V| = 2^8$. But in this case we should have $\dim V \geq 8l(r, 2) \geq 16$, a contradiction.

Thus one of Hypothesis 5.2 (d), (e) or (f) holds and so $|V| \leq p^{4e}$. Assume that p is odd. Then $p^q \leq p^{4e}$ so that $p^e \leq 4e$. Since $q \neq 9$, this is impossible. If $p = 2$, then $l(r, 2) \geq 2$ and so we require $2q \leq 4e$. This shows that $e = 2$, $|V| = 2^8$ and $L \cong \mathrm{SL}_2(4)$ contrary to L acting irreducibly on V.

Suppose now that $q = 9$ and $L/Z(L) \cong \mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$. In this case, $|V| \leq 9^3 = 3^6$ and so, as the minimal non-trivial permutation representation of L has degree 6 by [**33**, Satz II.8.28], we obtain $|V| = 3^6$ and $l(r, 3) = 2$ from Lemma 2.23. Therefore E is elementary abelian of order at least 2^4 as 5 divides $|L|$. Furthermore, Lemma 2.23 shows that L acts by transitively permuting the 6 maximal subgroups of E

which have non-trivial fixed vectors on V. Thus V is naturally the 6-dimensional GF(3)-permutation module for L whereas we know V an irreducible GF(3)L-module. This shows that no examples arise with E abelian. Moreover, we may deduce that

$$L \text{ centralizes any abelian subgroup which it normalizes}$$

and so such subgroups are cyclic.

Now assume that E is non-abelian. As every characteristic abelian subgroup of E is cyclic, E is of symplectic type. Hence [**2**, 23.9] implies that E contains an extraspecial normal subgroup E_0 of order r^{1+2w}. Furthermore, setting $R = C_E(E_0)$, we have $E = RE_0$ and R is either cyclic or $r = 2$ and R is dihedral, semidihedral or quaternion of order at least 16.

If R is cyclic, then $R = Z(E)$ and L acts on $E/Z(E)$ of order r^{2w}. On the other hand, if R is dihedral, semidihedral, or quaternion, then $Z(E) = Z(E_0)$ and $Z(E/Z(E)) = E_0/Z(E)Z(R/Z(E))$ which has order 2^{1+2w}. Since $Z(R/Z(E)) = E'/Z(E) \cap Z(E/Z(E))$, we obtain that $Z(R/Z(E))$ is characteristic in E. Thus, whatever the structure of R, L operates faithfully on an elementary abelian r-group of order r^{2w}. Therefore, by Lemma C.5, we have that

$$w \geq \begin{cases} (p^e - 1)/4 & \text{if } p \text{ is odd} \\ 2^{e-1} & \text{if } p = 2 \end{cases}$$

where the bound when $p = 2$ follows from the fact that w is an integer greater than $(2^e - 1)/2$. In addition, applying Lemma 2.12 we obtain $|V| \geq p^{l(p,r)r^w}$.

We consider the various possibilities for $|V|$ given in 5.2.

Suppose first that Hypothesis 5.2 (c) holds. Then $|V| = p^{8e/3}$. But then $8e \geq 3l(p,r)r^{(p^e-1)/4}$, which has no solution as e is a multiple of 3 (we found the inequality, which holds for integers $a \geq 2$ and $b \geq 3$, $a^b \geq 2(1 + b(a-1))$ useful to bound e). Hence Hypothesis 5.2 (d), (e) or (f) holds and we have $|V| \leq p^{4e}$. Thus

$$4e \geq l(p,r)r^w \geq l(p,r)r^{(p^e-1)/4}$$

if p is odd and otherwise we obtain

$$4e \geq l(2,r)r^{2^{e-1}}.$$

The first equation is only satisfied when $p = 3$, $e = 2$, $r = 2$ and $w = 2$, or $p = 5, 7$, $e = 1$, $r = 2$ and $w = 2$ whereas for the second equation we have no solutions.

So we have that $M/Z(M)$ is one of $\mathrm{PSL}_2(9)$, $\mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7)$ and in all cases $w = 2$ and $r = 2$. Since $\mathrm{PSL}_2(7)$ is not a subgroup of $\mathrm{Sp}_4(2)$, this case is impossible and so one of the first two cases occur. We recall that $E = E_0R$ where E_0 is extraspecial of order $2^{1+2w} = 2^5$. If $M/Z(M) \cong \mathrm{PSL}_2(9)$, then, as $M/Z(M)$ is not isomorphic to a subgroup of $\Omega_4^{\pm}(2)$, we have $|R| > 2$. If R is not cyclic, then E contains an extraspecial subgroup of order 2^7 and so, as Hypothesis 5.2 (d), (e) or (f) holds, we obtain $3^6 \geq |V| \geq 3^{2^3}$ from Lemma 2.12, which is ridiculous. Hence R is cyclic of order at least 4, but then E requires a representation of dimension 4 over a field of order at least 9 which is impossible as in this case we obtain $3^6 \geq |V| \geq 9^4$. Hence $p = 5$, $M/Z(M) \cong \mathrm{PSL}_2(5)$ and furthermore $|V| = 5^4$. Hence either $E \cong 2_-^{1+4}$ or $E \cong 4 \circ 2_-^{1+4}$ and these are the claimed exception to the statement that E and L commute. $\square$

Lemma 5.11. *Assume that Hypothesis* 5.2 *holds,* $L/Z(L) \ncong \mathrm{PSL}_2(q)$ *and* r *is a prime with* $r \neq p$*. Then* L *centralizes every* r*-subgroup of* $\mathrm{GL}(V)$ *which it normalizes.*

Proof. Assume that $E \leq \mathrm{GL}(V)$ is an r-group which is normalized but not centralized by L. Assume first that p is odd. By Lemmas C.14 and D.17, L contains a subgroup A which acts quadratically on V. As L does not centralize E, Lemma C.11 applies to the group $\langle A^{EL}\rangle$. This yields $p = 3$, $|A| = 3$ and that $r = 2$. Furthermore, either $L/Z(L)$ must be an alternating group of degree $2n + 1$ or $2n + 2$ or a simple group of Lie type which is naturally defined in both characteristics 2 and 3. Thus, as $L \ncong \mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$, $L/Z(L) \cong \Omega_5(3) \cong \mathrm{PSp}_4(3) \cong \Omega_6^-(2) \cong \mathrm{PSU}_4(2)$ with V one of the corresponding $\mathrm{GF}(3)L$-natural modules. Moreover, E contains a subgroup isomorphic to 2_+^{1+8} or 2_-^{1+6} in the respective cases. Since $|V| \leq 3^5$, this contradicts Lemma 2.12. Therefore

$$p = 2.$$

If $q = 2^e > 2$, then, by Lemma 5.5 L has an elementary abelian subgroup A of order 4 such that $C_V(a) = C_V(A)$ for all $a \in A^{\#}$. But then Lemma 2.14 shows that L centralizes E. Therefore

$$q = 2.$$

It is now obvious that $L \neq \mathrm{GL}(V)$, as then there is no candidate for E.

Assume that E is elementary abelian of order r^t. We first consider the special cases given in Hypothesis 5.2 (i), (j) and (k). Thus we have one of the following situations $L \cong \mathrm{SL}_6(2)$ or $\mathrm{SU}_6(2)$ with $|V| = 2^{20}$, $L \cong \Omega^+_{12}(2)$ with $|V| = 2^{32}$ or $L \cong \mathrm{E}_7(2)$ with $|V| = 2^{56}$. Then, as $l(2,r) \geq 2$, we have $t \leq 10$ in the first two cases, $t \leq 16$ in the third and $t \leq 28$ in the last. But Lemma C.5 shows that in each case $L/Z(L)$ has no projective representation in odd characteristic of dimension at most t and therefore L cannot act on E.

So we may assume that Hypothesis 5.2 (a) or (b) holds. Thus $L \cong \mathrm{SL}_m(2)$, $\mathrm{Sp}_{2m}(2)'$, $\mathrm{SU}_m(2)$ or $\Omega^{\pm}_{2m}(2)$ and $|V| = 2^{2m}$. As $l(2,r) \geq 2$, $|E| \leq r^m$ by Lemma 2.22. Since $m \leq R_2(L)$, application of Lemma C.6 yields that $L/Z(L) \cong \mathrm{SL}_3(2)$, $\mathrm{Sp}_4(2)'$ or $\mathrm{SU}_4(2)$. In particular, L is contained in $\mathrm{GL}_8(2)$ and so $r \leq 4$. Thus $L \not\cong \mathrm{Sp}_4(2)'$. If $L \cong \mathrm{SL}_3(2)$, then $|E| \leq 3^3$ which is impossible as 7 does not divide $|\mathrm{GL}_3(3)|$. As $|\mathrm{GL}_8(2)|_3 = 3^5$, we see that $\mathrm{SU}_4(2)$ cannot normalize a group of order 3^4 in $\mathrm{GL}_8(2)$.

Hence L centralizes every elementary abelian subgroup of $\mathrm{GL}(V)$, which it normalizes and consequently the same is true for any abelian group which L normalizes.

If L does not act irreducibly on V, we have $L \cong \mathrm{SL}_m(2)$ and the centralizer in $\mathrm{GL}(V)$ of L is either trivial or is isomorphic to $\mathrm{SL}_2(2)$. Hence any abelian subgroup of $\mathrm{GL}(V)$ which is centralized by L is trivial if $L \cong \mathrm{Sp}_{2n}(2)$, $\Omega^{\pm}_{2m}(2)$ or has order 3. As $Z(E) \neq 1$, we have $|Z(E)|$ is cyclic of order 3 and $L \cong \mathrm{SL}_m(2)$ or $\mathrm{SU}_m(2)$. Furthermore, any characteristic subgroup of E is cyclic, and so E is of symplectic type and $Z(E)$ has order 3. Thus E is extraspecial. By Lemma 2.12, $|V| \geq 4^{3^w}$ which gives $3^w \leq m$. In particular $R_{p'}(L) \leq m$ and Lemma C.6 yields $L \cong \mathrm{SL}_3(2)$ or $\mathrm{SU}_4(2)$. But then we have $w = 1$, a contradiction as neither of these groups can act non-trivially on an extraspecial group of order 27. $\square$

We can now gather the previous lemmas of this section together and present a proof of Proposition 5.3.

Proof of Proposition 5.3. Suppose that Hypothesis 5.2 holds and that L is not Sylow maximal in $\mathrm{GL}(V)$. Then there exists $M \leq \mathrm{GL}(V)$ such that L is Sylow embedded in M and L is not normal in M. Recall that in this situation $S \in \mathrm{Syl}_p(L)$ and $S_0 \geq S$ is such that $S_0 \in \mathrm{Syl}_p(N_M(L)) \subseteq \mathrm{Syl}_p(M)$.

Suppose that $F(M)$ is not centralized by L. Then by Lemmas 5.10

and 5.11, $L \cong \mathrm{SL}_2(5)$ and $[F(M), L]$ is a 2-group with L acting irreducibly on V. Thus $C_M(L)$ is cyclic of order at most 4 and $F(M) \cong 2_-^{1+4}$ or $4 \circ 2_-^{1+4}$. Moreover, $F(M)$ acts irreducibly on V. Therefore, Lemma 2.24 implies that $C_M(F(M))$ is cyclic and $E(M) = 1$. Hence $M/Z(F(M)) \leq \mathrm{Aut}(F(M))$. If $F(M) \cong 2_-^{1+4}$, we obtain $F(M)L$ is normal in M. In the second case we have $\mathrm{Out}(F(M)) \cong \mathrm{Sp}_4(2)$, and these two cases together give part (vi) of the proposition. Therefore, we may assume that

$$L \text{ centralizes } F(M).$$

Since $[F(M), L] = 1$, we have $E(M) \neq 1$. Suppose that $E(M)$ is a p'-group. Then p is odd and Lemma C.10 implies that L contains no non-trivial elements which operate quadratically on V. Therefore $L/Z(L) \cong \mathrm{PSL}_2(q)$ by Lemma D.17 and Hypothesis 5.2. Furthermore, as L centralizes $F(M)$, $C_L(E(M)) \leq Z(L)$.

If L normalizes all the components of $E(M)$, then L induces by conjugation automorphisms of each component. Since L has order divisible by p and $E(M)$ does not, we deduce that L operates as a group of outer automorphisms and this contradicts the Schreier property [**27**, Theorem 7.1.1 (a)]. Hence

$$L \text{ permutes the components of } E(M) \text{ non-trivially.}$$

Let K be a component which is not normalized by L. Then, by Galois [**33**, Satz II.8.28], either K^L contains at least q components or $L/Z(L) \cong \mathrm{PSL}_2(9)$ and K^L has at least 6 components. By Lemma 2.15, there exists $x \in K \setminus Z(K)$ of odd prime order $r \neq p$. Suppose that $L \ncong \mathrm{PSL}_2(9)$. Then $\langle x^L \rangle$ contains an elementary abelian group of order r^q. Now we have $|V| \leq p^{4e}$ from Hypothesis 5.2 and $|V| \geq l(p,r)q$ by Lemma 2.22 and so

$$p^e \leq l(p,r)p^e \leq 4e.$$

Because p is odd, this yields $e = 1$ and $p = 3$ which is impossible as L is perfect. If $L/Z(L) \cong \mathrm{PSL}_2(9)$, $|V| = 3^4$ or 3^6. Hence $E(M)L \leq \mathrm{GL}_6(3)$. However, r is odd and so $l(3,r) > 2$, and we have $6 \geq 6l(3,r)$, a contradiction. Therefore p divides $|E(M)|$.

Now we have $S_0 \cap E(M) \in \mathrm{Syl}_p(E(M))$ and $1 \neq [S_0 \cap E(M), L] \leq L \cap E(M)$ which is normal in L. This means that $L \leq E(M)$. Hence there is a component K_1 in $E(M)$, whose order is divisible by p. Since K_1 is normalized by L and $S_0 \cap K_1 \in \mathrm{Syl}_p(K_1)$, we get $L \leq K_1$ as $C_M(L)$ is a p'-group. As $C_M(L)$ is a p'-group, so is $C_M(K_1)$ and so we see that K_1 is the unique component of M of order divisible by p. In particular,

K_1 is normal in M. Now we note that $S_0 \cap K_1$ normalizes L and so L is Sylow embedded in K_1. Therefore $L < K_1$ as L is not normal in M. If $K_1 = E(M)$ then Lemmas 5.7, 5.8 and 5.9 imply Proposition 5.3. Thus it remains to show that $K_1 = E(M)$. Suppose that $K_1 \neq E(M)$. Then, as $C_M(K_1)$ is a p'-group, p is odd and by Lemma 2.24 V is not an irreducible K_1-module. This contradicts Lemmas 5.7, 5.8 and 5.9 and so $K_1 = E(M)$. This completes the proof of the proposition. □

We now move on to study situations where the acting group is not quasisimple. These sort of configurations tie in closely with the Levi complements of the normalizer of a root subgroup in orthogonal groups. We start with the small rank cases.

LEMMA 5.12. *Let V be a vector space over* $\mathrm{GF}(p)$ *and* $L \leq \mathrm{GL}(V)$. *Assume that one of the following holds.*

(a) $L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$, $q = p^e \geq 4$ *and V is the tensor product of three natural* $\mathrm{SL}_2(q)$*-modules.*
(b) $L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$, $q = p^e \geq 4$ *and V is the tensor product module of two natural* $\mathrm{SL}_2(q)$*-modules.*
(c) $L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q^2)$, $q = p^e$ *and V is the tensor product of a natural* $\mathrm{SL}_2(q)$*-module and the 4-dimensional* $\Omega_4^-(q)$*-module.*
(d) $L \cong \mathrm{SL}_2(q) \times \mathrm{PSL}_2(q)$, $q = p^e > 3$, *p odd, and V is the tensor product module of the natural* $\mathrm{SL}_2(q)$*-module and the 3-dimensional* $\Omega_3(q)$*-module.*

Then L is not Sylow embedded in a quasisimple subgroup of $M \leq$ $\mathrm{GL}(V)$, *$M \neq L$.*

PROOF. Suppose the statement is false. Then $L < M$ and $N_M(L)$ contains a Sylow p-subgroup S_0 of M. Furthermore, by Lemma A.17, M is not a group of Lie type in characteristic p.

In all cases under consideration, define L_1 to be the first factor in the description of L. Then V_{L_1} is a direct sum of natural $\mathrm{SL}_2(q)$-modules. Hence $S_1 = S_0 \cap L_1$ acts quadratically on V.

Suppose first that p is odd. Then, by Lemma C.12, we have $|S_1| = 3$. It follows that (c) holds. Therefore $L \cong \mathrm{SL}_2(3) \times \mathrm{SL}_2(9)$, $|V| = 3^8$ and M has elementary abelian Sylow 3-subgroups of order 27. Furthermore, Lemma C.12 also yields $M/Z(M) \cong \mathrm{PSU}_n(2)$, $n \geq 5$, $\mathrm{Alt}(n)$, $n \geq 5$ or a collection of exceptional examples $\Omega_8^+(2)$, $\mathrm{G}_2(4)$, $\mathrm{PSp}_6(2)$, Co_1, Suz or J_2. By considering the orders of the Sylow 3-subgroups of the candidates for M, we obtain that $M/Z(M) \cong \mathrm{G}_2(4)$ or J_2. However, by [**27**, Table 5.3g] J_2 contains $\mathrm{PSU}_3(3)$ and the same applies for $\mathrm{G}_2(4)$ as $\mathrm{PSU}_3(3) \cong G_2(2)' \leq G_2(4)$ by [**1**], and so we see that these groups have extraspecial Sylow 3-subgroups. We conclude that there are no candidates for M. Hence

$$p = 2.$$

Assume that $q = 2^e \geq 4$. Then $|S_1| = q \geq 4$ and so M contains a quadratic fours group. As L and so also M has to contain an elementary abelian r-group of order at least r^2 for some prime $r > 3$, using Lemma C.13 and considering the orders of M yields

$$\begin{aligned} &M/Z(M) \cong \mathrm{Alt}(m) \text{ for some } m \geq 10 \\ &\text{or } q = 4 \text{ and } M/Z(M) \cong \mathrm{J}_2, \mathrm{Co}_1, \mathrm{Co}_2 \text{ or Suz.} \end{aligned}$$

Suppose that $M/Z(M)$ is a sporadic simple group. Then $q = 4$ and $|V| = 4^8 = 2^{16}$ or $4^4 = 2^8$ by assumptions (a), (b) and (c). Since M has no trivial composition factors on V, this contradicts the data provided in Lemma C.13 (i).

So we have that $M/Z(M) \cong \mathrm{Alt}(m)$ with $m \geq 10$. Since S_0 normalizes L, we see that S_0' normalizes each component of L and $S_0'' \leq L$. Therefore S_0'' is abelian. As $\mathrm{Alt}(18)$ contains $\mathrm{Sym}(16)$ which has Sylow 2-subgroups $\mathrm{Dih}(8) \wr \mathrm{Dih}(8)$, we see that the second commutator group of a Sylow 2-subgroup of $\mathrm{Alt}(18)$ is non-abelian. Hence $10 \leq m \leq 17$. In particular a Sylow 2-subgroup of M does not contain elementary abelian subgroups of order 2^9. Thus, if $q \geq 8$, then (b) holds with $q = 8$ or 16. If $q = 16$, then 17^2 divides $|M|$, a contradiction. Suppose that $q = 8$. Then $L \cong \mathrm{SL}_2(8) \times \mathrm{SL}_2(8)$, $14 \leq m \leq 17$ and L_1 centralizes an element σ of order 7. Since $m \leq 17$, it follows that σ is a 7-cycle and that L_1 embeds into $\mathrm{Sym}(m-7)$. Using [**33**, Satz II.8.28], we have $m - 7 \geq 8 + 1 = 9$. Thus $m \in \{16, 17\}$. Now Lemma C.4 implies $2^{12} = |V| \geq 2^{14}$, a contradiction. Hence

$$q = 4.$$

Suppose that (a) holds. Then $L \cong \mathrm{SL}_2(4) \times \mathrm{SL}_2(4) \times \mathrm{SL}_2(4)$ and so M contains an elementary abelian subgroup of order 5^3. Let $\tau \in L_1$ have order 3. Then as V_{L_1} is a direct sum of natural L_1-modules, we have $C_V(\tau) = 0$. Now τ commutes with a subgroup of L isomorphic to $\mathrm{Alt}(5) \times \mathrm{Alt}(5)$ and so we deduce that τ is conjugate to either $(1, 2, 3)$ or $(1, 2, 3)(4, 5, 6)$. Neither of these elements act fixed-point-freely on the natural $\mathrm{Alt}(m)$-module. Hence V must be the spin module for M by Lemma C.13. We therefore have $16 = \dim V \geq \frac{1}{2} 2^{\lfloor \frac{m-1}{2} \rfloor}$, which is a contradiction as $m \geq 15$.

Suppose that (b) holds. Then $L \cong \mathrm{SL}_2(4) \times \mathrm{SL}_2(4)$ and $|V| = 2^8$.

Thus $m \leq 11$, and, as 5^2 divides $|M|$, we get that $m = 10$ or 11. Lemma C.4 then gives $m = 10$. Since $|N_M(L)|_2 = |M|_2$, we deduce that $N_M(L) \cong \mathrm{Sym}(5) \wr 2$, but $\mathrm{Alt}(10)$ does not contain such a subgroup. Hence (b) does not hold.

Finally suppose that (c) holds. Then $L \cong \mathrm{SL}_2(4) \times \mathrm{SL}_2(16)$ and so as 17 divides $|L|$ we get $m = 17$. Since L_1 commutes with an element of order 17, we have a contradiction. This contradiction shows that

$$q \neq 4.$$

Since $q = 2$, only case (c) is possible. So $L \cong \mathrm{Sym}(3) \times \mathrm{Alt}(5)$ and $|V| = 2^8$. Furthermore, $|M|_2 \leq 16$ and M contains an elementary abelian subgroup of order 8. By Lemma 2.16, $2 \times \mathrm{Dih}(8)$ cannot be the Sylow 2-subgroup of a simple group, and so M has elementary abelian Sylow 2-subgroups of order 8. Thus $M \cong \mathrm{J}_1$ or ${}^2\mathrm{G}_2(3^a)$ by Lemma 3.16. However 11 divides $|\mathrm{J}_1|$, and not $|\mathrm{GL}_8(2)|$ and ${}^2\mathrm{G}_2(3^a)$ has order coprime to 5 and so $q \neq 2$. This proves the lemma. □

Lemma 5.13. *Suppose that p is a prime, G is a finite group with $G = E(G)$, $O_p(G) = 1$ and $S \in \mathrm{Syl}_p(G)$. Assume that $K \leq G$, K_1 is a component of K and $K = \langle K_1^S \rangle$. Then either K centralizes S or K is contained in a component of G.*

Proof. Suppose that S does not centralize K. Let $J_1, \dots, J_m$ be the components of G and set $S_i = S \cap J_i$. As $G = E(G)$, $J_i \trianglelefteq G$. Since $S = S_1 \cdots S_m$, and K is not centralized by S, we may choose S_1 such that $[S_1, K] \neq 1$. Therefore $1 \neq [S_1, K] \leq J_1 \cap K$. Since $[S_1, K]$ is normalized by $K = E(K)$, we have that $[S_1, K]$ contains a component of K or $[S_1, K] \leq Z(K)$. In the latter case, we know $1 = [S_1, K, K] = [S_1, K] \neq 1$, a contradiction. Thus J_1 contains a component of K and, as S normalizes J_1 and permutes the components of K transitively by conjugation, we have $K \leq J_1$ as claimed. □

In the next lemma we complete the investigation of groups satisfying the hypothesis of Lemma 5.12.

Lemma 5.14. *Let V be a vector space over $\mathrm{GF}(p)$ and $L \leq \mathrm{GL}(V)$. Assume that one of the following holds.*

(a) *$L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$, $q = p^e \geq 4$ and V is the tensor product of three natural $\mathrm{SL}_2(q)$-modules.*
(b) *$L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$, $q = p^e \geq 4$ and V is the tensor product module of two natural $\mathrm{SL}_2(q)$-modules.*
(c) *$L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q^2)$, $q = p^e$ and V is the tensor product of a natural $\mathrm{SL}_2(q)$-module and the 4-dimensional $\Omega_4^-(q)$-module.*

(d) $L \cong \mathrm{SL}_2(q) \times \mathrm{PSL}_2(q)$, $q = p^e > 3$, *p odd, and V is the tensor product module of the natural* $\mathrm{SL}_2(q)$*-module and the* 3*-dimensional* $\Omega_3(q)$*-module.*

Then L is Sylow maximal in $\mathrm{GL}(V)$.

PROOF. Suppose that L is Sylow embedded in $M \leq \mathrm{GL}(V)$, L not normal in M, and let $S_0 \in \mathrm{Syl}_p(N_M(L)) \subseteq \mathrm{Syl}_p(M)$. Furthermore, assume that M is chosen of minimal order with the above properties. When studying case (a), we shall assume that the proposition has already been proved for case (b).

Note that $|V| \leq q^8$ and so, writing $q = p^e$, we have $|V| \leq p^{8e}$. If possibility (a) holds, then we write $L = L_1L_2L_3$ with $L_i \cong \mathrm{SL}_2(q)$ normal in L and in the other cases we write $L = L_1L_2$ and we always assume that $L_1 \cong \mathrm{SL}_2(q)$. Let $S_i = S_0 \cap L_i \in \mathrm{Syl}_p(L_i)$. Then, in particular, S_1 acts quadratically on V. As a consequence of the structure of L and its action on V, we obtain the following statement.

(5.14.1)

(i) In case (a), for $\{i, j, k\} = \{1, 2, 3\}$, $C_V(S_i)$ is the tensor product of two natural $\mathrm{SL}_2(q)$-modules for L_jL_k and $C_V(S_iS_j)$ is a natural $\mathrm{SL}_2(q)$-module for L_k.

(ii) In case (b), for $\{i, j\} = \{1, 2\}$, $C_V(S_i)$ is a natural $\mathrm{SL}_2(q)$-module for L_j;

(iii) In case (c), $C_V(S_1)$ is a 4-dimensional orthogonal $\mathrm{SL}_2(q^2)$-module for L_2 and $C_V(S_2)$ is a natural $\mathrm{SL}_2(q)$-module for L_1; and

(iv) In case (d), $C_V(S_1)$ is a 3-dimensional orthogonal $\mathrm{PSL}_2(q)$-module for L_2 and $C_V(S_2)$ is a natural $\mathrm{SL}_2(q)$-module for L_1. ■

(5.14.2) Assume r is a prime with $r \neq p$ and that $E \leq \mathrm{GL}(V)$ is an r-group which is normalized by L. Then L is normal in EL. In particular, if $q > 3$, then E is centralized by L.

Suppose that L acts non-trivially on E and that L is not normal in EL. Choose E with $|E|$ be maximal. From the at most three choices, if possible, select L_1 so that it operates non-trivially on E.

Suppose that p is odd. Suppose that L_1 does not centralize E. As S_1 acts quadratically Lemma C.11 yields $q = |S_1| = 3$ and $r = 2$. The only possibility is that (c) holds. So $L = L_1 \circ L_2 \cong \mathrm{SL}_2(3) \circ \mathrm{SL}_2(9)$ and $|V| = 3^8$. Note that $|\mathrm{GL}_8(3)|_2 = 2^{19}$ and $|\mathrm{GL}_8(3)|_s \leq s^2$ for $s > 3$ a prime. If L_2 centralizes E, then $EL_1 \leq C_{\mathrm{GL}(V)}(L_2) \cong \mathrm{GL}_2(3)$, $E = O_2(L_1) \leq L$ and so $EL = L$, a contradiction. So $L_1 \circ L_2$ acts on E with L_2 operating non-trivially. It follows that E is a 2-group and $L/O_2(L) \cong 3 \times \mathrm{PSL}_2(9)$ operates faithfully on $E/\Phi(E)$. Let E_0 be a critical subgroup of E. Then E_0 admits $S_1L_2 \cong 3 \times \mathrm{PSL}_2(9)$ faithfully. Since S_1 acts quadratically on V, E_1 is not elementary abelian and $Z(E_1)$ commutes with S_1L_2. Therefore $V = C_V(Z(E_1)) \oplus [V, Z(E_1)]$ is a S_1L_2 invariant decomposition. Since L_2 has just two composition factors on V, $V = [V, Z(E_1)]$ and $Z(E_1)$ acts as scalar matrices. Thus $Z(E_1) = Z(L_2)$. Thus E_1 is extraspecial. Since $|V| = 3^8$, we have $|E_1| \leq 2^7$. Now $\Omega_6^+(2) \cong \mathrm{Alt}(8)$ has no subgroup of order 27 and $\Omega_6^-(2) \cong \mathrm{PSp}_4(3)$ has no non-soluble 3-local subgroup. Hence this case cannot occur. This argument shows that L_1 centralizes E. In particular, we have proved that, if p is odd, then (c) or (d) holds as otherwise we may change the choice of L_1.

So suppose that (c) and (d) with $[E, L_1] = 1$. Then EL_2 centralizes L_1. We have that EL_2 acts faithfully on $V/C_V(S_1) \cong [V, S_1]$ which is either a 3- or 4-dimensional module over $\mathrm{GF}(q)$. But then Lemma 5.10 yields $L_2 \cong \mathrm{SL}_2(5)$ a contradiction as $L_2 \cong \mathrm{SL}_2(q^2)$ or $\mathrm{PSL}_2(q)$ in this case. This proves (5.14.2) for p odd.

Suppose that $p = 2$. We only need to consider cases (a), (b) and (c). We have $C_V(s) = C_V(S_1)$ for all $s \in S_1^{\#}$. If $q > 2$, then by Lemma 2.14 we get that $[E, S_1] = 1$ and so $[E, L_1] = 1$. Again, by changing the choice of L_1, we have a contradiction unless (c) holds. Hence E is centralized by L_1 and $L_2 \cong \mathrm{SL}_2(q^2) \cong \Omega_4^-(q)$ acts faithfully on E. Furthermore, EL_2 acts faithfully on $[V, S_1]$ of order q^4. Again Lemma 5.10 provides a contradiction. So $q = 2$, and again (c) holds this time with $L_2 \cong \mathrm{SL}_2(4) \cong \Omega_4^-(2)$ and $|V| = 2^8$. If L_2 centralizes E, then EL_1 embeds in $\mathrm{GL}_2(2)$ and so $E \leq L_1$ and $L = EL$, a contradiction. So suppose that $[E, L_2] \neq 1$. Since L_2 is not isomorphic to a subgroup of $\mathrm{GL}_2(5)$ or $\mathrm{GL}_2(7)$, by considering $|\mathrm{GL}_8(3)|$ we have that $r = 3$. Then $|E| \geq 3^4$ as 5 does not divide $|\mathrm{GL}_3(3)|$. It follows that $EO_3(L_1)L_2$ has Sylow 3-subgroups of order at least 3^6, a contradiction as $|\mathrm{GL}_8(2)|_3 = 3^5$. This completes the proof of (5.14.2).

(5.14.3) $q \in \{2, 3\}$.

Assume that $q > 3$. Then L is a product of components of LS_0 By (5.14.2), $L \leq C_M(F(M))$ and so $E(M) \neq 1$ and $[E(M), L_i] \neq 1$ for each L_i. Suppose that L is normal in $M_1 = E(M)L$. Then each L_i is a component of M_1 and so $L_i \leq E(M)$ and is a component of M. Hence L is a product of components of M and as $C_M(L)$ is coprime to p, we have L is normal in M, another contradiction. Hence by the minimality of M, $M = E(M)L$.

Let E_1 be the product of all the components of M which are not divisible by p and assume that $E_1 \neq 1$. Then L certainly does not centralize E_1. Furthermore, L is Sylow embedded in E_1L. Since S_1 acts quadratically on V, Lemma C.10 implies that L_1 centralizes E_1. It follows that L_2E_1 or, if (i) holds, $L_2L_3E_1$ acts on $C_V(S_1)$. By Thompson's $A \times B$-Lemma, this action is faithful and so we may apply (5.14.1) with Proposition 5.3 (or case (b) of this lemma assuming that it has already been proven) to obtain a contradiction. Thus every component of M has order divisible by p.

Let $S_E = S \cap E(M)$. Then $[S_E, L] \leq L \cap E(M)$. If $L \cap E(M) \leq Z(L)$, then we have $[S_E, L, L] = [S_E, L] = 1$ and we deduce that $S_E = 1$ and $C_M(L)$ has order coprime to p. Hence $L \cap E(M)$ contains at least one component L_i of L. Now $\langle L_i^{S_E} \rangle \leq J_1$, a component of $E(M)$, by Lemma 5.13. Notice that for $j \neq i$, $[S_i, L_j] = 1$, L_j normalizes J_1. In particular, L normalizes J_1 and so also $C_{E(M)}(J_1)$. Suppose that $L \not\leq E(M)$. Choose k as small as possible so that $L \cap E(M) \leq J_1 \cdots, J_m$ with $J_1, \ldots, J_m$ components of M. Then, by Lemma 5.13, $m = 1, 2$ as $L \not\leq E(M)$. Suppose that $E(M) \neq J_1J_m$. Then $[C_{E(M)}(J_1J_m) \cap S_E, L] \leq L \cap C_{E(M)}(J_1J_m) \leq Z(L)$ and so is $C_{E(M)}(J_1J_m) \cap S_E$ is centralized by L, a contradiction as every component of M has order divisible by p and $C_M(L)$ is a p'-group. Hence $E(M) = J_1J_m$ and as L must act faithfully on $E(M)$, we have $E(M)L = E(M)$, a contradiction as we have assumed $L \not\leq E(M)$. Therefore $L \leq E(M)$ and so

$$M = E(M).$$

By Lemma 5.12, $L \not\leq J_1$. Now assume that $J_1, \ldots, J_m$, where $2 \leq m \leq 3$, are the components of M. Let $k \leq m$, $C_M(J_k)$ contains $C_L(L \cap J_k)$ and acts upon $C_V(S \cap L \cap J_k)$. This action is faithful by the Thompson $A \times B$-Lemma. Thus $C_L(L \cap J_k)$ is Sylow embedded in $C_M(J_k)$ (with respect to $\mathrm{GL}(C_V(S \cap L \cap J_k))$). If $C_L(L \cap J_k)$ has two components, then (5.14.1)(i) shows that the case (b) of this lemma holds. Since we are assuming that this is true when case (a) is considered, we have $C_L(L \cap J_k)$ is Sylow maximal in $C_M(J_k)$. Thus $C_L(L \cap J_k) = C_M(J_k)$ in this case. In particular, if $m = 3$, then we have a contradiction as in this case $E(M) = L$. Hence

$$m = 2.$$

Assume that (a) holds. Then we may as well also assume that $L_2L_3 \leq J_2$. The above paragraph, then shows that $J_2 = L_2L_3$, a contradiction.

Thus

(a) does not hold.

We now choose notation so that $L_1 \leq J_1$ and $L_2 \leq J_2$. Furthermore, we have L_1 is Sylow embedded in J_1 (with respect to $\mathrm{GL}(C_V(S_2))$) and L_2 is Sylow embedded in J_2 (with respect to $\mathrm{GL}(C_V(S_1))$). (5.14.1) provides the hypothesis of Lemma 5.7.

Since $\mathrm{GF}(q)$ is a splitting field for the action of L_2 on V, we have $C_M(J_2)$ supports a $\mathrm{GF}(q)$ structure on $C_V(S_2)$. It follows that $J_1 = L_1$. Similarly J_2 supports a $\mathrm{GF}(q)$ structure on $C_V(S_1)$. Thus if (b) holds, we also have $J_2 = L_2$ and we have a contradiction. In cases (c) and (d), we deploy Lemma 5.7 to see that (c) holds with $J_2 \cong 2{}^{\cdot}\mathrm{PSL}_3(4)$. Since this group is not contained in $\mathrm{PSL}_3(9)$, we have a contradiction.

This proves the claim. ■

By hypothesis and the last claim we have (c) holds and $|V| = p^8$ with $p = 2$ or 3 and $L_2 \cong \Omega_4^-(2) \cong \mathrm{PSL}_2(4)$ or $L_2 \cong \Omega_4^-(3) \cong \mathrm{PSL}_2(9)$ respectively. Furthermore, $S_0 \cap L$ is elementary abelian of order p^3 and either $S_0 = S_0 \cap L$ or $S_0 \cong 2 \times \mathrm{Dih}(8)$. From (5.14.2), L_2 centralizes $F(M)$ and so $E(M) \neq 1$ and as $M \leq \mathrm{GL}_8(p)$, L_2 normalizes every component of M. Hence $L_2 \leq E(M)$. By Lemma 5.13, $L_2 \leq K$ where K is a component of M. Suppose that $L_2 = K$. Then L_1 centralizes K and $C_M(K)$ embeds into $\mathrm{GL}_2(p)$. But then L is normal in M and we have a contradiction.

If K does not act irreducibly on V, then K embeds into $\mathrm{GL}_4(p)$ and L_2 is Sylow embedded in K. Furthermore, L_2 acts on the submodule as $\Omega_4^-(p)$. Application of Lemma 5.7 provides a contradiction. Hence we have that K acts irreducibly on V and, in particular $[K, L_1] \neq 1$. So either $L_1 \leq K$ or at least the element of order p in L_1 induces outer automorphism on K.

We now consider the possibilities for K. By Lemma A.17 we see that K is not of Lie type defined in characteristic p. Assume that K is a sporadic simple group. Then Lemma C.2 shows that

$$K/Z(K) \cong \mathrm{Mat}(11), \mathrm{Mat}(12), \mathrm{Mat}(22), \mathrm{J}_1, \mathrm{J}_2.$$

We have that $|\mathrm{Mat}(11)|_3 = 3^2$, $|\mathrm{Mat}(12)|_3 = 3^3$, $|\mathrm{Mat}(22)|_3 = 3^2$, $|\mathrm{J}_1|_3 = 3$ and $|\mathrm{J}_2|_3 = 3^3$. Assume $p = 3$. As no sporadic group has an outer automorphism of order 3 (see [**27**, Table 5.3]), we have $L_1 \leq K$ and so K has elementary abelian Sylow 3-subgroups of order 27. By [**27**, Table 5.6.1] this is not true for any of these groups. So we have $p = 2$. As 11 does not divide $|\mathrm{GL}_8(2)|$ we have $K \cong \mathrm{J}_1$ or J_2. Since $|\mathrm{J}_2|_2 = 2^7$ and $|S| \leq 2^4$, we have $K \cong \mathrm{J}_1$ and $L \leq \mathrm{J}_1$. As $|\mathrm{J}_1|_3 = 3$, we have a contradiction. So K is not a sporadic simple group.

Suppose that K is of Lie type defined in characteristic r, $r \neq p$. Then $R_{p'}(K) \leq 8$. Now Lemma C.6 shows that $K/Z(K)$ is on the following list

- $\mathrm{PSL}_2(r)$, $r \leq 17$ with r odd, $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_4(2)$.
- $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$.
- $\mathrm{PSp}_4(2)'$, $\mathrm{PSp}_4(3)$, $\mathrm{PSp}_6(2)$, $\mathrm{P}\Omega_8^+(2)$.
- $\mathrm{G}_2(2)'$, ${}^2\mathrm{B}_2(8)$, ${}^2\mathrm{G}_2(3)'$.

When $p = 2$, we additionally know that $|S| \leq 16$ and 15 divides $|L_2|$ and so also $|K|$. Just $\mathrm{PSL}_2(r)$ with r odd remains. Since additionally 15 divides $|K|$, we have $r \leq 9$. Since $L_2 \leq K$, we get $r = 9$. We treat this case below as an alternating group. So $p = 3$. Again we treat $\mathrm{Alt}(6) \cong \mathrm{PSp}_4(2)'$ later. Since S is elementary abelian of order 9 or 27, and as $\mathrm{SL}_3(4)$ has non-abelian Sylow 3-subgroups, from the candidates above we only need to consider $K \cong \mathrm{PSL}_3(4)$ or $\mathrm{SL}_4(2)$ with $|S_0 \cap K| = 9$. Hence S_0 must induce an outer automorphism on K and therefore $K \cong \mathrm{PSL}_3(4)$. Since $\mathrm{PGL}_3(4) \geq \mathrm{PGU}_3(2)$ which has non-abelian Sylow 3-subgroups, we have a contradiction.

Finally consider $K/Z(K) \cong \mathrm{Alt}(m)$ for some $m \geq 5$. If $p = 3$, $S_0 \leq K$ and $|S_0| = 27$, a contradiction. So $p = 2$ and as $|S_0| \leq 16$, $n = 6, 7$. Since $L_2 \cong \mathrm{Alt}(5)$ and L_1' has order 3, we see that L_1' is not contained in K and so we conclude that L_1' centralizes K and L_1 induces an outer automorphism of K centralizing L_2. It follows that $K \cong \mathrm{Alt}(7)$ and $M = (3 \times \mathrm{Alt}(7)){:}2$. In particular in K we have $\mathrm{Sym}(5)$ containing L_2, but then by Lemma E.8 L_2 is not the orthogonal group. This final contradiction proves the lemma. □

Proposition 5.15. *Let p be a prime, V be a vector space over $\mathrm{GF}(p)$ and $L \leq \mathrm{GL}(V)$ with $L \cong \mathrm{SL}_2(q) \circ \Omega_t^\epsilon(q)$, $t \geq 5$, $q = p^e$ and $\epsilon = \pm$ if t is even and otherwise $\epsilon = 0$, and q is odd. Suppose that, as a $\mathrm{GF}(q)L$-module, V is the tensor product of the natural orthogonal module of dimension t for $\Omega_t^\epsilon(q)$ with the natural 2-dimensional module for $\mathrm{SL}_2(q)$. Then L is Sylow maximal in $\mathrm{GL}(V)$.*

Proof. Assume that the claim is false. Thus there exists $M \leq \mathrm{GL}(V)$ such that L is Sylow embedded in M and L is not normal in M. In particular, we have $\mathrm{Syl}_p(N_M(L)) \subseteq \mathrm{Syl}_p(M)$. We choose M of minimal order with this property and let $S_0 \in \mathrm{Syl}_p(N_M(L))$ and $S = S_0 \cap L$.

Decompose L as $L = L_1L_2$, with $L_1 \cong \mathrm{SL}_2(q)$ and $L_2 \cong \Omega_t(q)$. Notice that, as a $\mathrm{GF}(q)L_1$-module, V is a direct sum of natural $\mathrm{SL}_2(q)$-modules and so $S_1 = S \cap L_1$ acts quadratically on V and $C_V(s) = C_V(S_1)$ for all $s \in S_1^{\#}$. From the point of view of L_2, we have V_{L_2} is a direct sum of two natural modules. Since the splitting field of these representations is $\mathrm{GF}(q)$, we have $C_{\mathrm{GL}(V)}(L_2) \cong \mathrm{GL}_2(q)$ and, in particular, we observe that $L_1 = O^{p'}(C_{\mathrm{GL}(V)}(L_2))$ is normal in $C_{\mathrm{GL}(V)}(L_2)$.

Suppose that L_2 does not centralize $F(M)$. As L_2 is quasisimple, this is the case if $E(M) = 1$. Choose r such that L_2 does not centralize $R = O_r(M)$ and recall that $r \neq p$ as L acts irreducibly on V. Then $M = RL$ by the minimal choice of M. By Lemma C.14, L_2 contains elements which act quadratically on V and so $[R, L_2]L_2$ is the normal closure of such elements. If p is odd, Lemma C.11 applies to show that $L_2/Z(L_2) \cong \Omega_5(3) \cong \mathrm{PSU}_4(2)$ and $[R, L_2]$ contains an extraspecial subgroup of order 2^{1+8}. Lemma 2.12 implies that

$$q^{2t} = 3^{10} = |V| \geq 3^{2^4},$$

which is a contradiction. Hence

$$p = 2.$$

Assume that $q \geq 4$. Then, as $C_V(S_1) = C_V(s)$ for all $s \in S_1^{\#}$, Lemma 2.14 shows that S_1 centralizes R. Hence RL_2 embed into $\mathrm{GL}(C_V(S_1))$ by the Thompson $\mathrm{A} \times B$-Lemma. Since L_2 is Sylow embedded in RL_2 and $C_V(S_1)$ is the orthogonal $\mathrm{GF}(q)L_2$-module, Lemma 5.11 applies to yield a contradiction. Therefore

$$q = p = 2, |V| = 2^{2t}, t = 2m \text{ is even and } \epsilon = \pm.$$

Suppose that $E \leq R$ is elementary abelian and normalized by L_2. Assume further that L_2 acts faithfully on E. Then, as $p = 2$, $l(r, 2) \geq 2$ and so $|E| \leq r^t$ by Lemma 2.23. Furthermore, we may assume $C_V(E) = 1$ and so V is a direct sum of centralizers of hyperplanes. This now shows that there are at most t of them. If L_2 fixes all these hyperplanes F, then as L_2 is simple it centralises E/F for all hyperplanes and then E, a contradiction. Hence L_2 acts faithfully on theses hyperplanes and so L_2 must embed into $\mathrm{Sym}(t)$. Since the 2-rank of $\mathrm{Sym}(t)$ is m and the 2-rank of $\Omega_t^{\pm}(2)$ is at least $(m-1)(m-2)/2$ (see Lemma C.14), we have $t = 2m \leq 8$. Since $\Omega_6^{\pm}(2)$ is not a subgroup of $\mathrm{Sym}(6)$ and $\Omega_8^{\pm}(2)$ is not contained in $\mathrm{Sym}(8)$, we have a contradiction. We conclude that L_2 centralizes every characteristic elementary abelian subgroup of R. Since $C_{\mathrm{GL}(V)}(L_2) = L_1$, we must have $r = 3$. Now suppose that E be a critical subgroup of R. Then E is special and $Z(E)$ has order 3.

Thus E is extraspecial of order 3^{1+2w}. Now Lemma 2.12 yields that $2^{2t} = |V| \geq 4^{3^w}$ which gives

$$t \geq 3^w.$$

If $w = 1$, then $\mathrm{Aut}(E)$ is soluble and we have a contradiction. Similarly, if $w = 2$, then L_2 embeds into $\mathrm{GL}_2(9)$ which it does not. Therefore $t \geq 3^3 = 27 > 8$. Now L_2 contains a subgroup isomorphic to $\mathrm{Alt}(t-1)$ and so Lemma C.4 implies that $2w \geq t-3$ which means that $2w \geq t-2$ as t is even. However this gives

$$t \geq 3^w \geq 3^{(t-2)/2}$$

so that, by the binomial theorem,

$$t^2 \geq 3^{t-2} \geq 1 + 2(t-2) + 2(t-2)(t-3)$$

which yields the contradiction $27 \leq t \leq 6$. We have proven that

(5.15.1) $F(M)$ is centralized by L_2. In particular, $F(M) \neq F^*(M)$.

Suppose that $E(M)$ has order coprime to p. Then p is odd. Since L_2 contains an element x which acts quadratically on V, Lemma C.10 implies that x centralizes $E(M)$ and so x centralizes $F^*(M)$ by (5.15.1), a contradiction as $x \in L_2 \setminus Z(L_2)$. Hence p divides $|E(M)|$. Let $S_E = S_0 \cap E(M) \in \mathrm{Syl}_p(E(M))$. Then, as $E(M)$ is normalized by L and $C_S(L) = 1$, we have $[S_E, L] \neq 1$ and so $\langle S_E^L \rangle = L_1, L_2$ or L.

Suppose that $L_2 \not\leq E(M)$. Then $L_1 = \langle S_E^L \rangle \leq E(M)$. Furthermore, if K is a component of M and p divides $|K|$, we have that $\langle (S_E \cap K)^L \rangle = L_1$ and so now we have $L_1 \leq K$ and K is the unique component of M which has order divisible by p. Moreover, L_2 normalizes K. If L_2 centralizes K, then $K = L_1 \cong \mathrm{SL}_2(q)$ and $q \geq 4$. Furthermore, as L_2 centralizes $F(M)$ and $L_2 \not\leq E(M)$, we have $E(M) > K$. Let $K_1 = C_{E(M)}(K)$. Then p does not divide $|K_1|$ and L_2K_1 acts faithfully on $C_V(S_1)$ by the $A \times B$-Lemma. Since L_2 is Sylow embedded in K_1L_2 with respect to $\mathrm{GL}(C_V(S_1))$, we may apply Proposition 5.3 to see that $[L_2, K_1] = 1$, but then L_2 centralizes $E(M)$ and we have a contradiction. Hence L_2 does not centralize K. But now $L_2 \leq KC_M(K)$ by the Schreier property [**27**, Theorem 7.1.1 (a)]. By minimality we may assume that $M = KL_2$. But then we get that $E(M) = KL_2$ and so $L_2 \leq E(M)$, we have a contradiction. Therefore

$$L_2 \leq E(M).$$

Let X be a component of M which does not commute with L_2. Then as S does not contain a subgroup isomorphic to $S_2 \times S_2$, we have $L_2 \leq X$ and X is normalized by L_1. Suppose that $L_2 = X$. Then L_2 is normal in M and so is $C_M(L_2)$ and thus $L_1 = O^{p'}(C_M(L_2))$ is normalized by M.

But then L is normal in M, a contradiction. Hence $L_2 \neq X$. Since X is normal in M, if $C_{S_1}(X) \neq 1$, then L_1 centralizes X. Thus, in this case, X embeds into $\mathrm{GL}(C_V(S_1))$ and, as S_E normalizes L_2, L_2 is Sylow embedded in X with $C_V(S_1)$ the natural L_2-module. Now Proposition 5.3 provides a contradiction to $L_2 \neq X$. Hence $C_{S_1}(X) = 1$. Suppose that $C_S(X) \neq 1$. Then as $L_2 \leq X$, $C_S(X) \leq S \cap O^{p'}(C_M(L_2)) = S \cap L_1 = S_1$. Since no element of S_1 centralizes X, we conclude that $C_M(X)$ is a p'-group. Now we see that if $q > 3$, then $L \leq K$ and otherwise either $L_1' \leq K$ or $L' \leq K$.

We now consider the possibilities for the quasisimple group X. As usual, since $(S \cap X)L_2$ contains a Sylow p-subgroup of X, we have X is not a group of Lie type in characteristic p by Lemma A.17.

Assume that p is odd. Then Lemma C.12 applied to XS_1 shows that $q = 3 = p$ and there is no quadratic group of order 9 on V. Since the restriction V_{L_2} is a direct sum of two natural modules $\mathrm{GF}(q)L_2$-modules, Lemma C.14 implies that

$$L_2 \cong \Omega_5(3) \text{ or } \Omega_6^-(3).$$

Suppose that $L_2 \cong \Omega_5(3)$. Then $3^4 \leq |X|_3 = 3^5$. Applying Lemma C.12 yields $X/Z(X) \cong \mathrm{PSU}_5(2)$, or $\mathrm{Alt}(n)$, $n \leq 14$, $\Omega_8^+(2)$ or $\mathrm{PSp}_6(2)$. As L_2 has no permutation representation of degree less than 27 by [**20**, Theorem 71], $X/Z(X) \not\cong \mathrm{Alt}(n)$ with $n \leq 14$. Suppose that $X/Z(X) \cong \mathrm{PSU}_5(2)$ or $\Omega_8^+(2)$. Then $L_1 \cong \mathrm{SL}_2(3)$ and $L_1 \leq X$. Now L_2 is contained in a parabolic subgroup of X which is impossible. Finally, if $X/Z(X) \cong \mathrm{Sp}_6(2)$, then S_1 induces outer automorphisms on X and this is impossible. Thus $L_2 \not\cong \Omega_5(3)$.

Suppose that $L_2/Z(L_2) \cong \mathrm{P}\Omega_6^-(3) \cong \mathrm{PSU}_4(3)$. Then $3^6 \leq |X|_3 \leq 3^7$ and, as $Z(L_1)$ acts as scalars on V, we have that $Z(L_1) \leq Z(X)$. It follows from Lemma C.12 that the candidates for $X/Z(X)$ are $\mathrm{PSU}_6(2)$, $\mathrm{Alt}(n)$, $15 \leq n \leq 17$, or Suz. If $X/Z(X)$ is an alternating group, then $|S \cap X| = 3^6$ and so $S_1 \not\leq X$. Since $C_M(X)$ is a $3'$-group, we infer that X has an outer automorphism of order 3, a contradiction. Suppose that $X/Z(X) \cong \mathrm{PSU}_6(2)$. Then $R_3(X) \leq 12$, and this contradicts Lemma C.5. Finally suppose that $X/Z(X) \cong \mathrm{Suz}$. Then $L_1 \leq X$, $X \cong 2{}^{\cdot}\mathrm{Suz}$ and taking $x \in O_2(L_1) \setminus Z(L_1)$, we see that $C_X(x)$ involves $\Omega_6^-(3)$ and this contradicts the data provided in [**27**, Table 5.3]. We have shown that

$$p = 2.$$

So we now consider $q = 2^e$. By Lemma C.14 and C.16 L contains a quadratic fours group A on V. If $X/Z(X)$ is a group of Lie type in odd characteristic, then Lemma C.13 implies $X \cong 3{\cdot}\mathrm{PSU}_4(3)$. In particular $|S_2| \leq 2^7$ and this shows that $L_2 \cong \Omega_6^{\pm}(2)$ and $L_1' = Z(X)$. In particular, $|V| = 2^{12}$ and, as S inverts L_1', V_X is a $\mathrm{GF}(4)X$-module. Hence XS_1 is a subgroup of $\Gamma\mathrm{L}_6(4)$ and S_1 induces a field automorphism which centralizes $\Omega_6^-(2)$ in X. We have $\Omega_6^-(2) \cong \mathrm{PSp}_4(3)$ and so S_1 induces the field automorphism on $X/Z(X)$, centralizing $\mathrm{PSp}_4(3)$. But then by [**14**, Table 8.10] we have that L is a maximal subgroup of XS_1, which does not contain a Sylow 2-subgroup of XS_1 and so we have a contradiction.

So assume now that $X/Z(X) \cong \mathrm{Alt}(m)$. We apply Lemma C.4 which shows that $2te \geq m-2$. Thus the 2-rank of X is at most $2se+1$ where $t = 2s$. On the other hand, the 2-rank of L_2 is at least $e(s-1)(s-2)/2$ by Lemma C.14. Thus $t \leq 14$ and $e = 1$. By comparing the orders of $\mathrm{Alt}(2t+2)$ and $\Omega_t^{\pm}(2)$, yields $2t = 6$ and $X = L_2 \cong \Omega_6^+(2)$, a contradiction as $X > L_2$.

Next consider the case when $X/Z(X)$ is a sporadic simple group, again with A operating quadratically. Suppose that $X/Z(X) \not\cong \mathrm{Mat}(22)$. Then $|A| \leq 4$ by Lemma C.13. Hence $L_2 \cong \Omega_6^-(2)$ or $\Omega_8^-(2)$ by Lemma C.14. Thus $|V| = 2^{12}$ or 2^{16}. It follows from Lemma C.13 that $X \cong \mathrm{J}_2$ and $L_2 \cong \Omega_6^-(2)$, which is impossible as $|\mathrm{J}_2|$ is not divisible by 3^4. Hence $X/Z(X) \cong \mathrm{Mat}(22)$ and $|S_2| \leq 2^7$. Since 3^4 does not divide $|X/Z(X)|$, we have $L_2 \cong \Omega_6^+(2)$ and $|V| = 2^{12}$. Since the centralizer of a 3 element in L_2 is non-soluble, we have a contradiction to the data presented in [**27**, Table 5.3c].

This final contradiction shows that it is impossible for $L_2 < X$ and so we have completed the proof of the lemma. □

6. Main hypothesis and notation for the proof of the main theorems

In this brief section we establish the notation and hypotheses that will hold sway for the remainder of this work.

HYPOTHESIS 6.1. *We have p is a prime, G is a finite group, and H is a subgroup of G which contains a Sylow p-subgroup of G. Furthermore,*

(i) *G is of parabolic characteristic p;*
(ii) *$F^*(H)$ is a simple group of Lie type of rank at least 2 defined over a field of order p^e;*
(iii) *$H = N_G(F^*(H))$; and*

(iv) *G is a $\mathcal{K}_p$-group or $p = 2$ and $C_H(z)$ is soluble for some non-trivial 2-central element of H.*

Take
$$S_0 \in \mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(G)$$
and set
$$S = S_0 \cap F^*(H).$$

In the case $F^*(H) \not\cong \mathrm{Sp}_4(2)'$, $F^*(H)$ contains a long root subgroup and we define

R to be a long root subgroup contained in $Z(S)$

and put
$$Q = O_p(C_{F^*(H)}(R)),\ C = C_G(R)$$
and
$$L = O^{p'}(N_{F^*(H)}(Q)) = O^{p'}(C \cap F^*(H)).$$

We emphasise that Hypothesis 6.1 (iv) means that if $p = 2$ and $C_H(z)$ is soluble for some $z \in Z(S_0)^\#$, then G is not assumed to be a $\mathcal{K}_2$-group.

Define $P(S, L)$ to be the parabolic subgroup of $F^*(H)$ containing S of maximal order such that
$$P(S, L) \cap N_{F^*(H)}(R) = N_{F^*(H)}(S).$$

We also define
$$V(Q, S) = Z(C_S(C_Q(Z_2(S)))).$$

This notation will be fixed for the remainder of this work.

The generic case occurs when $F^*(H)$ is a genuine group of Lie type, $Z(S) = R$ is a long root subgroup of $F^*(H)$ and $F^*(H) \not\cong \mathrm{PSL}_3(p^a)$ or $\mathrm{SL}_4(2)$. Thus by Lemma A.3 the generic case is as described below.

HYPOTHESIS 6.2. *Hypothesis 6.1 holds with $F^*(H)$ isomorphic to one of*

- $\mathrm{PSL}_n(p^e)$, *$n \geq 4$, but not* $\mathrm{SL}_4(2)$;
- $\mathrm{PSU}_n(p^e)$, *$n \geq 4$;*
- $\mathrm{PSp}_{2n}(p^e)$, *$n \geq 2$, p odd;*
- $\mathrm{P\Omega}^{\pm}_{2n}(p^e)$, *$n \geq 4$;*
- $\mathrm{P\Omega}_{2n+1}(p^e)$, *$n \geq 3$, p odd;*
- $\mathrm{F}_4(p^e)$, *p odd;*
- $\mathrm{G}_2(p^e)$, *$p \neq 3$ and $p^e \neq 2$;*
- $\mathrm{E}_n(p^e)$, *$n = 6, 7, 8$;*
- ${}^3\mathrm{D}_4(p^e)$*; or*
- ${}^2\mathrm{E}_6(p^e)$.

We use the next two lemmas frequently and without quotation.

Lemma 6.3. *Assume Hypothesis* 6.1 *holds. Then either* $Z(S_0) \leq Z(S)$ *or* $F^*(H) \cong \mathrm{Sp}_4(2)'$.

Proof. Set $X = F^*(H)$. If X is a genuine group of Lie type this comes from Lemma D.25. If $X \cong {}^2\mathrm{F}_4(2)'$, then $\mathrm{Aut}(X) \cong {}^2\mathrm{F}_4(2)$ by Lemma A.13. Noting that $X \cong \mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$ by [**1**] and ${}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8)$ by [**37**, Proposition 2.9.1]). We can use Theorem A.11 to see that $\mathrm{Aut}(X) \cong \mathrm{G}_2(2)$ or ${}^2\mathrm{G}_2(3)$ respectively in these cases. Now application of Lemma A.3 yields $|Z(S_0)| = p$. $\square$

Lemma 6.4. *When Hypothesis* 6.2 *holds,* Q *is semi-extraspecial and* $Z(S) = Z(Q) = R$.

Proof. This comes from Lemma D.16. $\square$

7. The embedding of Q in G under Hypothesis 6.2

In this section we assume that Hypothesis 6.2 holds and in addition include $F^*(H) \cong \mathrm{PSL}_3(p^e)$ when p is odd. In the case when $F^*(H) \cong \mathrm{PSL}_3(p^e)$, we do not assume Hypothesis 6.1(iv), that is we do not assume that G is a $\mathcal{K}_p$-group. This will become important in the application in Section 9. Thus $Q = O_p(C_{F^*(H)}(R))$ is semi-extraspecial and, if $F^*(H) \cong \mathrm{PSL}_3(p^e)$, then $Q = S$.

Proposition 7.1. *We have that* $O_p(C_G(r)) = Q$ *for all* $r \in R^\#$.

Proof. As in Lemma D.1 we set
$$\widetilde{L} = O^{p'}(N_{F^*(H)}(R)/Q).$$
Then $L \geq Q$ is the preimage of $\widetilde{L}$ and centralizes R. Select $r \in (R \cap Z(S_0))^\#$. If $O_p(C_G(r)) = Q$ for all such r, then, as by Lemma A.4 any element in R is conjugate into $Z(S_0)$ under $F^*(H)$, the proposition will be proved.

We know that L normalizes $O_p(C_G(r)) \cap C_{F^*(H)}(r)$. Since $O_p(L) = Q$, we have $O_p(C_G(r)) \cap C_{F^*(H)}(r) \leq Q$. In particular, $[L, O_p(C_G(r))] \leq Q$ and so $O_p(C_G(r))$ centralizes L/Q and consequently the elements of $O_p(C_G(r))$ induce trivial automorphisms on L/Q.

Assume that $\alpha \in O_p(C_G(r)) \setminus S$ is such that α^p acts as an inner automorphism of $F^*(H)$. Then, by Theorem A.11 (ii), α acts as either a graph, graph-field or a field automorphism of $F^*(H)$.

If α operates as a graph-field automorphism, then $F^*(H)$ is not a twisted group by Theorem A.11 (iv). Also, in this case, $F^*(H) \not\cong \mathrm{PSL}_3(p^a)$ with p odd and so L/Q is non-trivial. Now α normalizes L and $\widetilde{L}$ is also not a twisted group. If $\widetilde{L}/Z(\widetilde{L}) \not\cong \mathrm{PSL}_2(p^e)$, α acts as a graph-field automorphism on $\widetilde{L}$, a contradiction. If $\widetilde{L}/Z(\widetilde{L}) \cong \mathrm{PSL}_2(p^e)$, then α induces a field automorphism of $\widetilde{L}$, which is also impossible.

If α is a field automorphism, then if $F^*(H)$ is an untwisted group it acts as the same type of automorphism on $\widetilde{L}$ which is impossible as α centralizes $\widetilde{L}$ (here note that if $F^*(H) \cong \mathrm{PSL}_3(p^a)$, then α acts non-trivially on $C_{F^*(H)}(R)/S$ which is a cyclic group of order $(p^e-1)/\gcd(3,p^e-1)$). Hence $F^*(H)$ is a twisted group and we argue that $\widetilde{L}$ is defined over $\mathrm{GF}(p^{e/p})$. But, using Lemma D.1, in case of $F^*(H) \cong {}^2\mathrm{E}_6(p^e)$ we have $\widetilde{L}/Z(\widetilde{L}) \cong \mathrm{PSU}_6(p^e)$, in case of $F^*(H) \cong {}^3\mathrm{D}_4(3)$ we have $\widetilde{L}/Z(\widetilde{L}) \cong \mathrm{PSL}_2(p^{3e})$ and in case of $F^*(H) \cong \mathrm{PSU}_n(p^e)$, $n \geq 5$, we have $\widetilde{L}/Z(\widetilde{L}) \cong \mathrm{PSU}_{n-2}(p^e)$. Hence in any case α acts non-trivially on $\widetilde{L}$, a contradiction. If $F^*(H) \cong \mathrm{PSU}_4(p^e)$, then if p is odd, we get a contradiction again. If $p=2$, then α is trivially on $L/Z(\widetilde{L}) \cong L_2(p^e)$. Hence α is not a field automorphism, besides $F^*(H) \cong \mathrm{PSU}_4(2^e)$. This case we will handle later.

So α is a graph automorphism and, moreover, $p=2$ or 3. Suppose that $p=3$. Then $F^*(H) \cong \mathrm{P}\Omega_8^+(3^e)$ and $\widetilde{L} \cong \mathrm{SL}_2(3^e) \circ \mathrm{SL}_2(3^e) \circ \mathrm{SL}_2(3^e)$. Now α permutes the three $\mathrm{SL}_2(3^e)$-subgroups of $\widetilde{L}$, which is impossible as α acts trivially.

Hence $p=2$. Since L/Q is centralized by α, we deduce that the action of α on the subgraph of the Dynkin diagram for $F^*(H)$ which corresponds to L is trivial. Then, using Lemma D.1 we see L has to be $\mathrm{SL}_2(2^e)$ (recall $F^*(H) \not\cong \mathrm{PSL}_3(2^e)$ or $\mathrm{PSp}_4(2^e)$) and so $F^*(H) \cong \mathrm{PSL}_4(2^e)$. By Hypothesis 6.2 we have $e \geq 2$.

Now we deal with $F^*(H) \cong \mathrm{PSL}_4(2^e)$ or $\mathrm{PSU}_4(2^e)$. Let T be a complement to S in $N_{F^*(H)}(S)$. Then by [**27**, Theorem 1.12.1 e,f], [**27**, Theorem 2.4.7] and [**27**, Table 2.4], $T \cong (2^e-1)^3$ in the first case and $T \cong (2^{2e}-1)(2^e-1)$ in the second case and α acts on TS/S. If α induces the graph automorphism on $F^*(H)$, we get that $[TS,\alpha]S/S$ has order 2^e-1 by Definition A.9 (iii). If α induces a field automorphism on $F^*(H)$, we get that $[TS,\alpha]S/S$ has order 2^e+1 by Definition A.9(ii). Since α centralizes R, $[TS,\alpha]S$ centralizes R. But then $[TS,\alpha]$ normalizes $O_2(C_G(r))$ and this is a contradiction as the former group is not a 2-group. This proves

$$O_p(C_G(r)) \leq Q.$$

As $C_G(r)$ has characteristic p, $O_p(C_G(r))$ is not centralized by $O^{p'}(L)$. Therefore $O_p(C_G(r)) \not\leq R$ and, because $O_p(C_G(r)) \leq Q$, we have $[O_p(C_G(r)), Q] = R$ by Lemma D.16. Hence

$$Q > O_p(C_G(r)) > R.$$

Now Lemmas D.1 and D.10 (v) show that $F^*(H) \cong \mathrm{PSL}_n(q)$, $n \geq 3$, $\mathrm{PSU}_4(3)$ or $\mathrm{G}_2(4)$.

We consider them in reverse order. Suppose that $F^*(H) \cong \mathrm{G}_2(4)$. Because $R \leq O_2(C_G(r))$, we get $|O_2(C_G(r))| = 2^6$ and $O_2(C_G(r))$ is non-abelian by Lemma D.10 (v). Since Q/R is a direct sum of Alt(5)-permutation modules for L, we have that $O_2(C_G(r))/R$ is an irreducible L-module. It follows that $Z(O_2(C_G(r))) = R$ and then, as Q centralizes $O_2(C_G(r))/R$, that by [**22**, Chap. 5, Theorem 3.2] $Q \leq O_2(C_G(r))$ which is absurd.

Suppose that $F^*(H) \cong \mathrm{PSU}_4(3)$. Then $R = \langle r \rangle$ and so Q centralizes $O_3(C_G(r))/R$. This means that $Q \leq O_3(C_G(r))$, a contradiction.

So we come to the main business $F^*(H) \cong \mathrm{PSL}_n(q)$, $q = p^e$. We have $n \geq 3$. Let P_1 and P_2 be the maximal subgroups of $F^*(H)$ containing $C_{F^*(H)}(r)$ and set $E_i = O_p(P_i)$. We have $|E_1| = |E_2| = q^{n-1}$. Define $M_i = N_G(E_i)$. Assume $O_p(C_G(r)) = E_1$. Then $C_G(r) \leq M_1$. Since P_1 acts transitively on the elements of E_1, we conclude that $O^{p'}(P_1)C_G(r) = M_1$. If $e = 1$, then $P_1/O_3(P_1) \cong \mathrm{GL}_{n-1}(p)$ and so $C_G(r) \leq P_1$. If $n = 3$ and $e > 1$, we apply Lemma 2.27 to obtain $C_{O^{p'}(P_1)}(r)$ is normal in $C_G(r)$ while, if $n > 3$, G is a $\mathcal{K}_p$-group by assumption and we may apply Proposition 5.3 obtain the same statement. Since $O_p(C_{O^{p'}(P_1)}(r)) = Q$, [**22**, Chap 5, Theorem 3.2] implies that Q is normal in $C_G(r)$, which is a contradiction. Hence

$$O_p(C_G(r)) \text{ is not equal to either } E_1 \text{ or } E_2.$$

Assume first $n \geq 5$. Then we have that $\widetilde{L} \cong \mathrm{SL}_{n-2}(p^e)$ and, by Lemma D.1 Q/R is a direct sum of a natural $\widetilde{L}$-module and its dual. For $n \geq 3$, these modules are not isomorphic and so $O_p(C_G(r)) \in \{E_1, E_2\}$, a contradiction. We get that

$$n \leq 4.$$

Suppose first that $e = 1$. Then $R = \langle r \rangle$ and so $[Q, O_p(C_G(r))] \leq R$, which using [**22**, Chap. 5, Theorem 3.2] gives $Q = O_p(C_G(r))$, a contradiction. So we may assume that $e \geq 2$. If $n = 4$, then we consider a basis $\{v_1, v_2, v_3, v_4\}$ of the 4-dimensional vector space V over $\mathrm{GF}(p^e)$. We choose notation such that r centralizes $\langle v_1, v_2, v_3 \rangle$ and $r(v_4) = v_1 + v_4$. Then with the notation from above we have that E_1 corresponds to the transvections to $\langle v_1\rangle$ and E_2 corresponds to the transvections to $\langle v_1, v_2, v_3\rangle$. In particular $Z_2(S) \cap E_1$ is the centralizer in E_1 of $\langle v_1, v_2\rangle$ and $Z_2 \cap E_2$ are the transvections to $\langle v_1, v_2\rangle$. We consider the element $\delta = \mathrm{diag}(\alpha, \alpha^{-3}, \alpha, \alpha) \in \mathrm{SL}_4(p^e)$ where $\alpha \in \mathrm{GF}(p^e)$ has order $p^e - 1$. Then $\delta \notin Z(\mathrm{SL}_4(p^e))$ as $p^e \notin \{3, 5\}$. Now we calculate directly that $E_1 \cap Z_2(S)$ is centralized by δ and $E_2 \cap Z_2(S)$ is not. It follows that $Z_2(S)/Z(S)$ has exactly two $N_{C_H(R)}(S)$-invariant subgroups. Thus as $Z_2(S)/R \cap O_p(C_G(r))/R \neq 1$, we deduce that $O_p(C_G(r)) \cap E_1 \not\leq R$ or $O_p(C_G(r)) \cap E_2 \not\leq R$, and this is a contradiction. Thus $F^*(H) \cong \mathrm{PSL}_3(p^e)$ with p odd.

Let now $\{v_1, v_2, v_3\}$ be a basis of the 3-dimensional vector space over $\mathrm{GF}(p^e)$. Then as above r is the transvection $r(v_3) = v_1 + v_3$ and E_1 is the group of transvection to $\langle v_1\rangle$ and E_2 the group of transvections to $\langle v_1, v_2\rangle$. Let $\omega \in C_{F^*(H)}(R)$ be the image of $\mathrm{diag}(\alpha, \alpha^{-2}, \alpha) \in \mathrm{SL}_3(p^e)$ where α has order $p^a - 1$. Then ω has order $(p^e - 1)/\gcd(p^e - 1, 3)$. In fact $C_H(r) = S\langle\omega\rangle$.

We have that ω acts on E_1/R as multiplication by α^3 and on E_2/R as α^{-3}. As by assumption $O_2(C_G(r))$ is neither E_1 nor E_2 both modules have to be equivalent. If $p^e - 1$ is not divisible by 3, then α^3 has order $p^e - 1$ and so p has order e modulo $p^e - 1$, whereas, if 3 divides $p^e - 1$, then α^3 has order $(p^e - 1)/3$ and we calculate that p has order e modulo $(p^e - 1)/3$ (for if it is less than e, then $p = 2$ and $e = 2$). Now application of Lemma 2.26 shows that α^3 is conjugate to α^{-3} by a Galois automorphism of $\mathrm{GF}(p^e)$. Hence we have that

$$\alpha^{3p^a+3} = 1 \text{ for some } 0 < a \leq e.$$

As α has order $p^e - 1$, $p^e - 1$ divides $3p^a + 3$. If $a = e$, then $\alpha^6 = 1$, which means that $\mathrm{GF}(p^e) = \mathrm{GF}(7)$, a contradiction as $e > 1$. Hence $3p^a + 3 = k(p^e - 1)$, $0 < a < e$ and k is a natural number. As $p \geq 3$, we have

$$k + 3 = kp^e - 3p^a = p^a(kp^{e-a} - 3) \geq 3(3k - 3) = 9(k - 1)$$

and so $k = 1$. But then 4 is divisible by p, a contradiction. This shows that $O_p(C_G(r)) = Q$ and concludes the proof of the proposition. $\square$

Lemma 7.2. *We have*

(L1) $F^*(N_G(Q)) = Q$*; and*

(L2) *if* $1 \neq U \leq G$ *and* $[U, Q] = 1$*, then* $N_G(U) \leq N_G(Q)$*.*

In particular, Q *is large and* Q *is weakly closed in* S_0 *with respect to* G*.*

Proof. Let $r \in \Omega_1(Z(S_0))^\#$. By Lemma 6.3 and Lemma 6.4, $r \in R$. By Proposition 7.1, $O_p(C_G(r)) = Q$. Hence $C_G(r) \leq N_G(Q)$. Next

$O_p(N_G(Q)) \leq S_0$, so $O_p(N_G(Q)) \leq C_G(r)$. We have $Q = O_p(N_G(Q))$. As Q is normal in S_0, (L1) comes from Hypothesis 6.1(i).

Suppose that $1 \neq U \leq C_G(Q)$. Then $U \leq R$ by Lemma 6.4 and (L1). Let $x \in N_G(U)$ and $u \in U^{\#}$. Then $u^x \in U^{\#}$ and so Proposition 7.1 implies

$$Q^x = O_p(C_G(u))^x = O_p(C_G(u^x)) = Q.$$

Hence $N_G(U) \leq N_G(Q)$ as claimed in (L2).

That Q is large now follows from Definition 2.2 and Lemma 2.3 (iii) implies that Q is weakly closed in S_0 with respect to G. $\square$

LEMMA 7.3. *For all $g \in G \setminus N_G(Q)$, we have $R \cap R^g = 1$.*

PROOF. As $N_G(Q) \leq N_G(R)$, (L2) implies that $N_G(R) = N_G(Q)$. Suppose that $g \in G \setminus N_G(R)$ and assume that $x \in R \cap R^g$. Then there exists $y \in R$ such that $x = y^g$ and by Proposition 7.1 we have

$$Q = O_p(C_G(x)) = O_p(C_G(y^g)) = O_p(C_G(y))^g = Q^g,$$

which is a contradiction. $\square$

Recall the definition of $P(S, L)$ and $V(Q, S)$ as given in Section 6.

LEMMA 7.4. *We have $V(Q, S) = Z(C_S(C_Q(Z_2(S))))$ is normalized by $N_G(S)$.*

PROOF. By Lemma 7.2, we have Q is weakly closed in S with respect to G. In particular, this means that $N_G(S) \leq N_G(Q)$. Hence all the operations in the construction of $V(Q, S)$ are invariant under $N_G(S)$ and hence $N_G(S)$ normalizes $V(Q, S)$. $\square$

LEMMA 7.5. *Assume that $F^*(H) \not\cong \mathrm{PSL}_3(p^e)$ and that p is odd. Set $P = P(S, L)$ and $V = \Omega_1(Z(O_p(P)))$. Then $R^{N_G(V)} = R^P$. In particular, $\langle Q^P \rangle$ is normalized by $N_G(V)$ and $N_G(V) = PN_{N_G(V)}(Q)$.*

PROOF. Set $\widetilde{P} = O^{p'}(P/O_p(P))$. Suppose first that $V = Z_2(S)$. Then Lemma D.22(i) shows that V is a natural $O^{p'}(\widetilde{P})$-module. In particular, $V^{\#} = \bigcup_{x \in P} R^x$. Suppose that $g \in N_G(V)$. Then $R^g \subset V^{\#}$ and so $R^x \cap R^g \neq 1$ for some $x \in P$. Since $R^x \cap R^g \neq 1$, Lemma 7.3 implies that $R^g = R^x$ and thus $R^{N_G(V)} = R^{N_H(V)}$ in this case. Therefore, we must assume that $V \neq Z_2(S)$. In this case Lemma D.22 (ii) applies to yield

$$\widetilde{P} \cong \begin{cases} \Omega_4^{\pm}(p^e) \text{ and } V \text{ is the natural orthogonal module of order } p^{4e}, \\ \Omega_3(q) \text{ and } V \text{ is the natural orthogonal module of order } p^{3e}. \end{cases}$$

Assume that $g \in N_G(V)$ is such that $t = r^g$ and t is not conjugate to R in P. Then t is $N_G(V)$-conjugate to r but not $N_H(V)$-conjugate to r. Then t corresponds to a non-singular vector in V. Therefore

$$O^{p'}(C_{\widetilde{P}}(t)) \cong \Omega_3(p^e) \cong \mathrm{SL}_2(p^e).$$

Assume first that $|V| = p^{4e}$. In particular, $|S : C_S(t)| = p^e$. Set

$$Q_t = O_p(C_G(t)) = Q^g.$$

Then, as $g \in N_G(V)$ and $Q \leq N_G(V)$, we see that $Q_t \leq N_G(V)$. Furthermore, $C_{N_H(V)}(t) \leq N_G(Q_t)$. Suppose that $Q_t \cap S_0 \neq Q_t \cap S$. Then there is some element $s \in C_{S_0}(t) \setminus F^*(H)$ such that $[O^{p'}(C_{\tilde{P}}(t)), s] = 1$. Application of Theorem A.11 and Lemma A.15 shows that s cannot induce a field automorphism on $\tilde{P}$. This implies $p = 2$. Furthermore we have that s induces a $\mathrm{GF}(2^e)$-transvection on V. But $Q \leq P$ and so no element of Q induces a $\mathrm{GF}(2^e)$-transvection on V, a contradiction. This shows

$$Q_t \cap N_H(V) = Q_t \cap S.$$

As $Q_t \cap N_H(V)$ is normal in $C_{N_H(V)}(t)$, we conclude from the structure of $\widetilde{P}$ that

$$Q_t \cap C_S(t) = Q_t \cap N_H(V) \leq O_p(N_G(V)) \leq C_H(V).$$

But $Q_tC_S(t)$ is a p-group in $C_G(t)$ and so $|Q_t : Q_t \cap C_S(t)| \leq p^e$. Now $V \not\leq Q$ and so $V = V^g \not\leq Q^g = Q_t$, and V centralizes a subgroup of index at most p^e in Q_t whereas in V it centralizes a subgroup of index p^{2e}. Thus we have a contradiction. Hence $r^{N_G(V)} = r^P$ and so by Lemma 7.3 we have $R^{N_G(V)} = R^P$.

Assume next that $|V| = p^{3e}$. Then $N_{N_H(V)}(\langle t\rangle)$ preserves the decomposition

$$V = V_1 \times V_2$$

where V_1 has order p^e and corresponds to the non-singular 1-space containing t and V_2 has order p^{2e} and is the non-degenerate space perpendicular to V_1. Furthermore, unless $N_{N_H(V)}(\langle t\rangle)$ acts as a subgroup of $\mathrm{O}_2^{\pm}(3)$ or $\mathrm{O}_2^-(5)$ on V_2, V_2 is an irreducible $\mathrm{GF}(p)N_{N_H(V)}(\langle t\rangle)$-module. Now

$$|V \cap Q| = p^{2e}$$

and furthermore $|V \cap Q_t| = |(V \cap Q)^g|$. In particular, as $|V : V_2| = p^e$, we must have $V_2 \cap Q_t \neq 1$. But $N_{N_H(V)}(\langle t\rangle)$ normalizes both Q_t and V and so as it acts irreducibly on V_2, we have $V_2 \leq Q_t$. Hence $V_2 = Q_t \cap V$; however, $t \in Q_t$ and $t \notin V_2$. Hence we are left with the two exceptional cases when

$$p^e = 5 \text{ or } p^e = 3.$$

Assume first $p^e = 5$. Then $N_H(V)$ has orbits of length 6, 10 and 15 on the subgroups of V of order 5. As R is not conjugate to the element, which is normalized by a dihedral group of order 6, we see that R must have 21 conjugates under $N_G(V)$. Since 7 does not divide $|\mathrm{GL}_3(5)|$, we have a contradiction. Hence $p^e = 3$. Thus $N_H(V)$ has orbits of length $3, 4$ and 6 on subgroup of order 3 in V. Since neither 5 nor 7 divides $|\mathrm{GL}_3(3)|$, this time we see that $|R^{N_G(V)}| = 13$. In particular, 13 divides $|N_G(V)/C_G(V)|$ and $N_H(V)/C_H(V)$ contains $\mathrm{Sym}(4)$. As $|N_G(V)/C_G(V)|$ is not divisible by 9, the order of $\mathrm{GL}_3(3)$ implies that $|N_G(V)/C_G(V)| = 2^x \cdot 3 \cdot 13$, $x \leq 5$. Application of Sylow's theorem to the prime 13, shows that we have a normal Sylow 13-subgroup. Then the existence of $\mathrm{Sym}(4)$, shows that an element of order 13 normalizes S and so centralizes R, which contradicts the assumption that R has 13 conjugates. This contradiction proves the main clause of the lemma. Since $Q = O_p(N_G(R))$, we also obtain $Q^P = Q^{N_G(V)}$ and this yields $\langle Q^P \rangle$ is normalized by $N_G(V)$. □

Proposition 7.6. *Assume that $F^*(H) \not\cong \mathrm{PSL}_3(p^e)$ with p odd. If L is normal in $N_G(Q)$, then $N_G(Q) \leq H$.*

Proof. Set $M = N_G(Q)$ and suppose that L is normal in M. Then, as $S \in \mathrm{Syl}_p(L)$, the Frattini Argument shows that $M = LN_M(S)$. Combining Lemmas 7.4, D.22 and D.23 shows that

$$N_M(S) \leq N_M(V).$$

Therefore

$$M = LN_M(V).$$

Now, by Lemma 7.5, $N_M(V)$ normalizes $\langle Q^P \rangle$. Hence M normalizes $\langle L, Q^P \rangle = F^*(H)$. But then $M \leq H$ and we are done. □

Proposition 7.7. *Assume that $F^*(H) \not\cong \mathrm{PSL}_3(p^e)$ and that p is odd. If $N_G(Q) \leq H$, then $N_G(V(Q,S)) \leq H$.*

Proof. Set $P = P(S, L)$. By Lemma D.23 $V = \Omega_1(Z(O_p(P))) = V(Q,S)$. By Lemma 7.5 $N_G(V) = PN_{N_G(V)}(Q)$. By assumption we know $N_G(Q) \leq H$, hence $N_G(V) \leq H$. □

8. The groups which satisfy Hypothesis 6.2 with $\mathbf{N_{F^*(H)}(Q)}$ not soluble and $\mathbf{N_G(Q) \not\leq H}$

In this section we assume

Hypothesis 8.1. *Hypothesis* 6.2 *holds. In addition we assume that*

$$N_{F^*(H)}(Q) \text{ is not soluble.}$$

Define

$$M = N_G(Q),$$

and recall from Hypothesis 6.1 that

$$L = O^{p'}(N_{F^*(H)}(Q)) = O^{p'}(M \cap F^*(H))$$

and set $\widetilde{M} = M/Q$.

We emphasise that Hypothesis 8.1 means in particular that $Z(S) = R$ and $F^*(H)$ is *not* isomorphic to one of the groups $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_5(2)$, $\mathrm{P\Omega}_8^+(2)$, $\mathrm{PSp}_4(3)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSU}_4(3)$, $\mathrm{P\Omega}_7(3)$ or $\mathrm{P\Omega}_8^+(3)$ these being the groups which satisfy Hypothesis 6.2 but have $N_{F^*(H)}(Q)$ soluble by Lemma D.15.

We also recall that, by Proposition 7.1, we have that

$$Q = O_p(C_G(r)) = O_p(C_{F^*(H)}(r))$$

for all $r \in R^{\#}$. We will prove the following two propositions.

Proposition 8.2. *Suppose that Hypothesis* 8.1 *holds. If* $N_G(Q) \not\leq H$*, then* $H = F^*(H) \cong \mathrm{G}_2(5)$ *and* $E(N_G(Q)/Q) \cong \mathrm{SL}_2(9)$.

Proposition 8.3. *Suppose Hypothesis* 6.1 *holds with* $p = 5$ *and* $F^*(H) \cong \mathrm{G}_2(5)$. *Assume that* G *has local characteristic* 5 *and is a* $\mathcal{K}_2$*-group. If* $N_G(Q) \not\leq H$*, then* $G \cong \mathrm{LyS}$.

Lemma 8.4. *Suppose that* $F^*(H) \not\cong \mathrm{G}_2(p^e)$*. If* L *is not soluble, then* L *is normal in* $N_G(Q)$.

Proof. As G has parabolic charactersitic p we have that $\tilde{M}$ acts faithfully on Q/R. As $N_H(L) \geq S_0 \in \mathrm{Syl}_p(G)$, when we consider $N_G(Q)$ acting on Q/R, we have that $\widetilde{L}$ is Sylow embedded in $\widetilde{M}$.

Suppose that $F^*(H) \cong \mathrm{P\Omega}_n^\epsilon(p^e)$, $n \geq 7$. Then using Lemma D.1 we see that $\widetilde{L}$ satisfies the hypothesis of Lemma 5.14 or of Proposition 5.15. These results assert that $\widetilde{L}$ is Sylow maximal in $\widetilde{M}$ and so L is normal in M. The remaining cases are $F^*(H) \cong \mathrm{PSL}_n(p^e)$, $n \geq 4$, $\mathrm{PSU}_n(p^e)$, $n \geq 4$, $\mathrm{PSp}_{2n}(p^e)$, $n \geq 2$, p^e odd, $\mathrm{E}_n(p^e)$, $\mathrm{F}_4(p^e)$, p^e odd, ${}^3\mathrm{D}_4(p^e)$, or ${}^2\mathrm{E}_6(p^e)$. Recall that $\mathrm{PSL}_4(p^e) \cong \mathrm{P\Omega}_6^+(p^e)$ and $\mathrm{PSU}_4(p^e) \cong \mathrm{P\Omega}_6^-(p^e)$. By Lemma D.1 we have that Hypothesis 5.2 is satisfied. Application of Proposition 5.3 shows that either L is normal in M or one of the following holds:

(i) $\widetilde{L} \cong \mathrm{SL}_2(4)$, $E(\widetilde{M}) \cong \mathrm{Alt}(7)$ and Q/R is either a natural $\mathrm{GF}(4)\widetilde{L}$-module or a direct sum of two natural $\mathrm{GF}(4)\widetilde{L}$-modules.

(ii) $\widetilde{L} \cong \mathrm{SL}_2(5)$, $E(\widetilde{M}) \cong \mathrm{SL}_2(9)$ and Q/R is an irreducible 4-dimensional $\mathrm{GF}(5)\widetilde{L}$-module.

(iii) $\widetilde{L} \cong \mathrm{SL}_2(7)$, $E(\widetilde{M}) \cong 2{\cdot}\mathrm{Alt}(7)$ and Q/R is an irreducible 4-dimensional $\mathrm{GF}(7)\widetilde{L}$-module.

(iv) $\widetilde{L} \cong \mathrm{PSL}_2(9)$, $E(\widetilde{M}) \cong 2{\cdot}\mathrm{PSL}_3(4)$ and Q/R is a 3-dimensional $\mathrm{GF}(9)\widetilde{L}$-module.

(v) $\widetilde{L} \cong \mathrm{PSp}_4(2)'$, $E(\widetilde{M}) \cong \mathrm{Alt}(7)$ and Q/R is a natural $\mathrm{GF}(2)\widetilde{L}$-module.

(vi) $\widetilde{L} \cong \mathrm{SL}_2(5)$, $F(\widetilde{M}) \cong 2^{1+4}_-$ or $4 \circ 2^{1+4}_-$ and either

  (a) $\widetilde{L}F(\widetilde{M})$ is normal in $\widetilde{M}$ and $\widetilde{M}/F(\widetilde{M}) \cong \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$; or

  (b) $F(\widetilde{M}) = 4 \circ 2^{1+4}_-$ and $\widetilde{M}/F(\widetilde{M}) \cong \mathrm{Alt}(6)$ or $\mathrm{Sym}(6)$.

  Furthermore, Q/R is a 4-dimensional irreducible $\mathrm{GF}(5)\widetilde{L}$-module.

As $F^*(H) \not\cong \mathrm{G}_2(p^e)$, we see from Lemma D.1 that $\widetilde{L} \not\cong \mathrm{SL}_2(p^e)$ with $\widetilde{L}$ acting irreducibly on Q/R. Hence (ii), (iii) and (vi) do not arise in this case. Furthermore, we do not have $|Q/Z(Q)| = p^{3e}$, so (iv) does not show up either. Since $\mathrm{Sp}_4(2)'$ does not act on an extraspecial group of order 32, we also do not have case (v). This leaves (i). Hence we have

$$p^e = 4,\ \widetilde{L} \cong \mathrm{SL}_2(4) \text{ and } E(\widetilde{M}) \cong \mathrm{Alt}(7).$$

In addition, Q/R is either the natural $\mathrm{SL}_2(4)$-module or a direct sum of two natural modules. As $\mathrm{Alt}(7)$ is not isomorphic to a subgroup of $\Omega_4^{\pm}(2)$, the latter option is what occurs. Hence $|Q/Z(Q)| = 2^8$. In particular, using Lemma D.1, we obtain $F^*(H) \cong \mathrm{PSL}_4(4)$ or $\mathrm{PSU}_4(4)$. However $\mathrm{SL}_2(4){:}2 \cong \mathrm{Sym}(5)$ is a subgroup of $\mathrm{Alt}(7)$, as a Sylow 2-subgroup of $E(\widetilde{M}) \cong \mathrm{Alt}(7)$ is contained in $\widetilde{N_H(Q)}$. This shows that some element α in the preimage of $E(\widetilde{M})$ induces an outer automorphism on $F^*(H)$ and has $\widetilde{L}\langle\widetilde{\alpha}\rangle \cong \mathrm{Sym}(5)$. Since $E(\widetilde{M})$ is perfect, its preimage centralizes R. Thus α centralizes R. Suppose that $F^*(H) \cong \mathrm{PSL}_4(4)$. Then α must act as a graph automorphism on $F^*(H)$. This means that α induces an inner automorphism on $\widetilde{L}$ contrary to $\widetilde{L}\langle\widetilde{\alpha}\rangle \cong \mathrm{Sym}(5)$. Similarly, if $F^*(H) \cong \mathrm{PSU}_4(4)$ then α induces a field automorphism and so again it induces an inner automorphism on $\widetilde{L}$. This proves the lemma. □

Lemma 8.5. *Suppose that $F^*(H) \cong \mathrm{G}_2(p^e)$, $p \neq 3$, $p^e \geq 4$. Then either L is normal in $N_G(Q)$ or $F^*(H) \cong \mathrm{G}_2(5)$ and $E(\widetilde{M}) \cong \mathrm{SL}_2(9)$.*

Proof. We first consider the special case when $p^e = 4$. Then $F^*(H) \cong \mathrm{G}_2(4)$ and, by Theorem A.11, $H \cong \mathrm{G}_2(4)$ or $\mathrm{Aut}(\mathrm{G}_2(4)) \sim \mathrm{G}_2(4){:}2$. Further $\widetilde{L} \cong \mathrm{SL}_2(4)$ and, by Lemma D.10, Q/R is a direct sum of two permutation modules for $\widetilde{L} \cong \mathrm{Alt}(5)$. In particular we are in the situation of Hypothesis 5.2(g). Application of Proposition 5.3 shows that $\widetilde{L}$ is normal in $\widetilde{M}$ in this case. So $p^e \neq 4$.

As $p^e \neq 4$, Lemma D.10 implies that Q/R is an irreducible $\mathrm{SL}_2(p^e)$-module. Now we have Hypothesis 5.2(d). Again $\widetilde{L}$ is Sylow embedded in $\widetilde{M}$ and so by Proposition 5.3 we get that either $\widetilde{L}$ is normal in $\widetilde{M}$ or one of the following holds:

(i) $\widetilde{L} \cong \mathrm{SL}_2(5)$, $E(\widetilde{M}) \cong \mathrm{SL}_2(9)$ and Q/R is an irreducible 4-dimensional $\mathrm{GF}(5)\widetilde{L}$-module.

(ii) $\widetilde{L} \cong \mathrm{SL}_2(7)$, $E(\widetilde{M}) \cong 2{}^{\cdot}\mathrm{Alt}(7)$ and Q/R is an irreducible 4-dimensional $\mathrm{GF}(7)\widetilde{L}$-module.

(iii) $\widetilde{L} \cong \mathrm{SL}_2(5)$, $F(\widetilde{M}) \cong 2^{1+4}_-$ or $4 \circ 2^{1+4}_-$ and either

  (a) $\widetilde{L}F(\widetilde{M})$ is normal in $\widetilde{M}$ and $\widetilde{M}/F(\widetilde{M}) \cong \mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$; or

  (b) $F(\widetilde{M}) = 4 \circ 2^{1+4}_-$ and $\widetilde{M}/F(\widetilde{M}) \cong \mathrm{Alt}(6)$ or $\mathrm{Sym}(6)$.

  Furthermore, Q/R is a 4-dimensional irreducible $\mathrm{GF}(5)\widetilde{L}$-module.

If (ii) holds, then $E(\widetilde{M}) \cong 2{}^{\cdot}\mathrm{Alt}(7)$. We calculate $|\mathrm{G}_2(7)| = 2^8 \cdot 3^3 \cdot 7^6 \cdot 19 \cdot 43$. Hence by Sylow's Theorem we get that a Sylow 7-subgroup is normalized by a group of order 36 and so in $M \cap H$ we have that $\mathrm{PGL}_2(7)$ is involved. But no automorphism group of $\mathrm{Alt}(7)$ involves $\mathrm{PGL}_2(7)$.

So we are left with $p^e = 5$. That is $F^*(H) = H \cong \mathrm{G}_2(5)$ by Theorem A.11. Now $\widetilde{M}$ embeds into $\mathrm{Out}(Q) \cong \mathrm{GSp}_4(5)$ and $\widetilde{L} \leq \widetilde{M}'$ embeds into $\mathrm{Sp}_4(5)$. But in this case we may apply [**53**, Lemma 4.19] to obtain that any proper over-group of $\widetilde{L} \cong \mathrm{SL}_2(5)$, has a normal subgroup $\mathrm{SL}_2(9)$. Hence (iii) does not occur and we are left with

$$p^e = 5, \widetilde{L} \cong \mathrm{SL}_2(5) \text{ and } E(\widetilde{M}) \cong \mathrm{SL}_2(9).$$

This proves the lemma. □

Proof of Proposition 8.2. This is a combination of Proposition 7.6 and Lemmas 8.4 and 8.5. □

Proof of Proposition 8.3. Since $F^*(H)$ has no non-trivial outer automorphisms by Theorem A.11, we have $F^*(H) = H$ and so $S \in \mathrm{Syl}_5(G)$. By Proposition 8.2, we have $E(\widetilde{M}) \cong \mathrm{SL}_2(9)$. Let M_1 be the preimage of $E(\widetilde{M})$. Set

$$G_0 = \langle M_1, O^{5'}(N_H(Z_2(S)))\rangle.$$

As $O^{5'}(N_H(Z_2(S)))$ is contained in a parabolic subgroup of $\mathrm{G}_2(5)$ it has structure $5^{2+1+2}.\mathrm{SL}_2(5)$, and so we see that G_0 satisfies the hypothesis of Lemma 3.10. Thus $G_0 \cong \mathrm{LyS}$.

Suppose that $G \neq G_0$ and set $M_0 = M \cap G_0$. We intend to show that G_0 is strongly 5-embedded in G. For this it suffices by Lemma 4.1 to show that $N_G(S) \le G_0$ and $C_G(x) \le G_0$ for all elements of elements of order 5 in G_0. By [**27**, Table 5.3 q], G_0 has exactly two conjugacy classes of elements of order 5. We let $r, s \in S$ be representatives of these classes chosen so that $r \in R^{\#}$ is 5-central and $C_S(r) \in \mathrm{Syl}_5(C_{M_0}(r))$. From [**27**, Table 5.3 q] and using the fact that there is just one 5-central class, we have

$$M_0 \sim 5_+^{1+4}.(4 \circ \mathrm{SL}_2(9)).2 \sim 5_+^{1+4}.\mathrm{SL}_2(9).4$$

Now, as $\widetilde{M_1}$ acts irreducibly on Q/R, we have $C_{\widetilde{M}}(\widetilde{M_1})$ is cyclic of order at most 4 and so it has order 4 and is central in $\widetilde{M}$. It follows that $\widetilde{M}/C_{\widetilde{M}}(\widetilde{M_1})$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{PSL}_2(9))$. This shows that $|M : M_0| \le 2$. If $M > M_0$, then all the elements of order 3 in M are conjugate as they are in $\mathrm{Aut}(\mathrm{PSL}_2(9))$. However, this cannot be the cases as a group of order 9 acting on a vector space of dimension 4 over GF(5) cannot have all its elements conjugate by Lemma 2.23. Hence $M = M_0 \le G_0$. Now, as Q is weakly closed in G by Proposition 7.1, we also have $N_G(S) \le M \le M_0$.

We now consider $C_G(s)$. By [**27**, Table 5.3 1], we have

$$W = O_5(N_{G_0}(\langle s \rangle)) \cong 5 \times 5_+^{1+2}$$

and $W \in \mathrm{Syl}_5(N_{G_0}(\langle s \rangle)$. Since $C_S(x)$ has order at most 5^3 for $x \in S \backslash Q$, we see that $s \in Q$ and so $W \le Q$ and $W' = R$. Let S^* be a 5-group in $N_G(\langle s \rangle) \cap N_G(W)$. Then S^* normalizes R and hence Q. Thus $S^* \le N_G(Q) \le G_0$ and so $S^* \le W$. In particular $W \in \mathrm{Syl}_5(N_G(\langle s \rangle))$. Now $O_5(N_G(\langle s \rangle)) \le W$. Consider the case that $O_5(N_G(\langle s \rangle))$ is abelian. Since $N_G(W)$ acts irreducibly on $W/Z(W)$ by [**41**, Proposition 2,6], we must have $O_5(N_G(\langle s \rangle)) \le Z(W)$, but this contradicts G of local characteristic 5. Hence $O_5(N_G(\langle s \rangle))$ is non-abelian. But then $O_5(N_G(\langle s \rangle))' \le W' = R$ and $N_G(\langle s \rangle) \le M = M_0 \le G_0$. We have shown that G_0 is strongly 5-embedded in G. Now application of [**56**, Theorem 1.2] yields a contradiction to $G > G_0$. Thus $G = G_0 \cong \mathrm{LyS}$. $\square$

9. The groups with $\mathbf{F^*(H) \cong PSL_3(p^e)}$, p odd

In this section we continue with the proof of Main Theorem 1. We have already seen in Section 7 that $\mathrm{PSL}_3(p^e)$, p odd, plays an unusual role. This is also reflected in the statements of both main theorems. For this section, we assume Hypothesis 6.1(i) - (iii) together with $F^*(H) \cong \mathrm{PSL}_3(p^e)$ with p odd. We also continue with the notation introduced in Section 6.

From the structure of $F^*(H)$ we have $S = Q$ is a Sylow p-subgroup of $F^*(H)$,

$$N_{F^*(H)}(Q) = N_{F^*(H)}(S) = QT,$$

where

$$T \cong (p^e - 1) \times (p^e - 1)/\gcd(p^e - 1, 3).$$

So, if $N_G(Q) \leq H$, then $N_G(Q) = N_G(QT)$.

Furthermore, by Theorem A.11, we have S_0/Q is cyclic and the non-trivial elements of $S_0 \setminus Q$, if there are any, are in the cosets represented by a field automorphism of $F^*(H)$. Let $E_1 \leq Q$ correspond to the transvection group to a point and $E_2 \leq Q$ to the transvection group to a hyperplane in the natural representation of $\mathrm{SL}_3(p^e)$. Then $N_{F^*(H)}(E_1)$ and $N_{F^*(H)}(E_2)$ are the maximal parabolic subgroups of $F^*(H)$ which contain Q. For $i = 1, 2$, set

$$H_i = O^{p'}(N_{F^*(H)}(E_i)).$$

By Proposition 7.1 we have that $Q = O_p(C_G(r))$ for all $r \in R^\#$ and, by Lemma 7.2, $N_G(R) = N_G(Q)$ and Q is large.

Our main result is

Proposition 9.1. *Suppose that Hypothesis* 6.1*(i), (ii) and (iii) hold with $F^*(H) \cong \mathrm{PSL}_3(p^e)$. Suppose that $N_G(Q) \leq H < G$ and $p^e \notin \{3, 7\}$. Then $N_G(E) \leq H$ for all E normal in S_0. Furthermore, if G has local characteristic p, then H is strongly p-embedded in G.*

As we mentioned in the introduction the groups ${}^2\mathrm{F}_4(2)'$ with $p = 3$ and O'N with $p = 7$ satisfy the assumptions of the theorem but do not have a strongly p-embedded subgroup.

Lemma 9.2. *For $i = 1, 2$, we have $H_i = \langle Q^g \mid g \in N_G(E_i)\rangle$ is normal in $N_G(E_i)$. In particular $N_G(E_i) = N_{N_G(E_i)}(Q)H_i$.*

Proof. Recall that for $i = 1, 2$, $H_i/E_i \cong \mathrm{SL}_2(p^e)$ and H_i acts transitively on $E_i^\#$. Therefore, as $Q = O_p(C_G(r))$ for $r \in R^\#$ by Proposition 7.1, for $i = 1, 2$, we get that

$$H_i = \langle O_p(C_G(x)) \mid 1 \neq x \in E_i\rangle.$$

Since $N_G(E_i)$ normalizes E_i, we see that H_i is normal in $N_G(E_i)$. The Frattini Argument and Lemma 7.2 now show that $N_G(E_i) = N_{N_G(E_i)}(Q)H_i$. $\square$

LEMMA 9.3. *Suppose that* $N_G(Q) \le H$. *Then* $N_G(E_i) = N_H(E_i)$, $i = 1, 2$.

PROOF. By Lemma 9.2 we have $N_G(E_i) = H_i N_G(Q) \le H$. $\square$

LEMMA 9.4. *Suppose that* $N_G(Q) \le H$ *and* $p^e \notin \{3, 7\}$. *Then, for* $i = 1, 2$ *we have* $|N_H(Q) : N_{N_H(Q)}(E_i)| \le 2$, $i = 1, 2$.

PROOF. Since T normalizes both E_1 and E_2 but no other subgroups of order p^{2e} and $N_G(Q) = N_H(TQ)$, we have $N_G(Q)$ permutes the set $\{E_1, E_2\}$. This provides the result. $\square$

LEMMA 9.5. *Suppose that* $N_G(Q) \le H$. *If* $U \le Q$ *and* $|U| = p^{2e}$, *then either* $N_G(U) = N_H(U)$ *or* $p^e \in \{3, 7\}$.

PROOF. By Lemma 7.2 we have Q is large. Suppose that $p^e \notin \{3, 7\}$. If U is not elementary abelian, then $1 \ne \Phi(U) \le R$. Thus as Q is large gives

$$N_G(U) \le N_G(\Phi(U)) \le N_G(Q) \le H,$$

we are done.

So we may assume that U is elementary abelian. As Q is semi-extraspecial, maximal elementary abelian subgroups of Q have order p^{2e}. Therefore $R \le U$ and so $Q \le N_G(U)$ and

$$C_G(U) = U.$$

Suppose $N_G(U) \not\le H$. Choose $K \le N_G(U)$ with $Q \le K$ and K minimal with respect to $K \not\le H$. Let $Q \le S_1 \in \mathrm{Syl}_p(K)$. Then, as Q is large $N_G(S_1) \le N_G(Q)$ by Lemma 2.3(i) and $N_G(Q) \le H$ by assumption, the minimal choice of K yields $K = \langle S_1^K \rangle$ and $K \cap H$ is the unique maximal subgroup of K which contains S_1. Thus

$$K \text{ is a } p\text{-minimal group.}$$

Note that $O_p(K) \le S_1 \le H$. We have $Z(O_p(K))$ centralizes $U \ge R$ and so, as S_1/Q induces only field automorphisms on Q, $Z(O_p(K)) \le Q$. If $Z(O_p(K)) \le R$, then $K \le N_G(Q) \le H$ as Q is large, a contradiction. Thus $Z(O_p(K)) \not\le R$, then $[Z(O_p(K)), Q] = R$ as Q is semi-extraspecial. Thus $O_p(K) \le Q$ and $\Phi(O_p(K)) = 1$ as otherwise $K \le H$. As $U \trianglelefteq K$ it follows that $O_p(K) \le C_G(U) = U$. Thus

$$O_p(K) = U.$$

We are going to apply Lemma C.20 to $\overline{K} = K/U$. The group A there then will be $\overline{Q}$, which is of order p^e.

Suppose that $x \in K \setminus H$. If $\langle Q, Q^x \rangle < K$, then by the choice of K we have $\langle Q, Q^x \rangle \le H$ and so $Q^{xh} = Q$ for some $h \in H$. But then

$xh \in H$ and this means $x \in H$, a contradiction. Hence $K = \langle Q, Q^x \rangle$ for all $x \in K \setminus H$, which is C.20(iii). Now suppose that $y \in Q \setminus U$. Then, as Q is semi-extraspecial, $U = C_Q(u)$ for all $u \in U \setminus R$. Then $C_U(Q) = C_U(y)$ for all $y \in Q$ with $\overline{y} \neq 1$ in particular Lemma C.20(i) holds. Furthermore, $C_U(K) \leq C_U(Q) = R$ and so $C_U(K) = 1$ as otherwise $K \leq N_G(Q)$, which is Lemma C.20(ii). Now Lemma C.20 implies that $\overline{K} \cong \mathrm{SL}_2(p^e)$ and U is its natural module.

Now $N_K(Q)$ contains a cyclic group C of order $p^e - 1$, which is therefore contained in H. By Lemma 9.4 there is a subgroup W in C of index at most two, which normalizes both E_1 and E_2. Then we have that W normalizes E_1, E_2 and U. By Lemma 9.3 all three groups are pairwise different. In K we calculate that a generator w of W acts on R as multiplication with a $o(w)$-th root of unity λ, on U/R as λ^{-1} and on Q/U as λ^2. Hence these two last representations must be equivalent. By Lemma 2.26 we get that λ^{-1} and λ^2 must be conjugate under the Galois group of $\mathrm{GF}(p^e)$ (recall that we have $e = n$ in the Lemma 2.26). This means that there is some a such that $(\lambda^{-1})^{p^a} = \lambda^2$, which implies that $o(\lambda)$ divides $p^a + 2$. On the other hand we have that $o(\lambda) = (p^e - 1)/2$ or $p^e - 1$. Hence in both cases

$$p^e - 1 \text{ divides } 2p^a + 4.$$

Recall that $a \leq e$. If $e = a$, then $p^e - 1$ divides $2p^e + 4 - 2(p^e - 1) = 6$, which is impossible as $p^e \notin \{3, 7\}$. So $a \leq e - 1$ and then $p^e - 1 \leq 2p^{e-1} + 4$ and so $p^{e-1}(p - 2) \leq 5$, which implies that $p^e = 9$. But plainly $p^e - 1 = 8$ does not divide $2p + 4 = 10$. This contradiction proves the lemma. □

Our next objective is to show that, if G is of local characteristic p and $G \neq H$, then H is strongly p-embedded in G. However, we note that the next lemma does not require the hypothesis that G is of local characteristic p.

Lemma 9.6. *Suppose that $N_G(Q) \leq H$ and $p^e \notin \{3, 7\}$. If $r \in R^\#$, then $r^G \cap H = r^H$.*

Proof. Recall that by Proposition 7.1 we have that $r^G \cap R = r^H \cap R$ and H is transitive on $R^\#$. First we show that $r^G \cap F^*(H) = r^H$. For this we may assume that $r^g \in Q \setminus R$, for some $g \in G \setminus H$. In particular $|C_Q(r^g)| = q^2$. By [**53**, Proposition 5.3] we have that $\mathrm{Aut}(Q/Z(Q))$ involves $\mathrm{GL}_2(q)$. Therefore all elements in $Q \setminus R$ are conjugate in $\mathrm{Aut}(Q)$. As there is an elementary abelian group of order q^2 (E_1 for example) in Q, we have that $U = C_Q(r^g)$ is elementary abelian and contains R. As every elementary abelian subgroup of order

p^{2e} in S_0 is contained in Q and U normalizes $Q^g \leq O_p(C_G(r^g))$, we have that $U \leq Q^g$. Therefore $Q^g \leq N_G(U) \leq H$ by Lemma 9.5. But then $Q^{gh} = Q$ for some $h \in H$ and this means that $gh \in N_G(Q) \leq H$. Hence $g \in H$, a contradiction. This shows

$$r^G \cap F^*(H) = r^H.$$

Suppose that $r^g \in H \setminus F^*(H)$ for some $g \in G$. By Theorem A.10 we have that r^g induces a field automorphism on $F^*(H)$ and by Lemma A.15 all elements of order p in the coset $F^*(H)r^g$ are conjugate under the inner and diagonal automorphisms of $F^*(H)$. So r^g centralizes some subgroup X of $F^*(H)$, $X \cong \mathrm{PSL}_3(p^e)$. However, $C_G(r) \leq N_G(Q) \leq H$ and this group is soluble, a contradiction. This proves the lemma. □

Proof of Proposition 9.1. This follows from Lemma 9.6 and Lemma 4.2. □

We next determine under which circumstances the assumption of Proposition 9.1 that $N_G(Q) = N_H(Q)$ holds.

Proposition 9.7. *One of the following holds*

(i) $N_G(Q) = N_H(Q) = N_H(TQ)$;
(ii) $p = 3$ *and* $N_G(Q)/Q \sim 2.\mathrm{Dih}(8)$;
(iii) $p = 5$ *and* $N_G(Q)/Q \sim 4.\mathrm{Sym}(4)$;
(iv) $p = 7$ *and* $N_G(Q)/Q \sim 6.\mathrm{Dih}(8)$, $6.\mathrm{Dih}(16)$, $6.\mathrm{Sym}(3)$ *or* $6.\mathrm{Dih}(12)$;
(v) $p = 13$ *and* $N_G(Q)/Q \sim 12.\mathrm{Sym}(4)$.

In particular, in case (i) we have $C_G(r) \leq H$ *for all* $r \in R^{\#}$.

Proof. We aim for a contradiction and so assume that $N_G(Q) > N_H(Q)$. By [**53**, Proposition 5.3], we have that $\mathrm{Aut}(Q)$ is an extension of a p-group by $\Gamma\mathrm{L}_2(p^e)$. Hence $N_G(Q)/Q$ is isomorphic to a subgroup of $\Gamma\mathrm{L}_2(p^e)$, which contains $T \cong (p^e-1)\times(p^e-1)/\gcd(p^e-1, 3)$ properly. Set $w = (p^e - 1)/\gcd(p^e - 1, 3)$.

We consider $F^*(H)$ as the image of $\mathrm{SL}_3(p^e)$ and in doing this identify Q with the lower unitriangular matrices and T with the image of the diagonal subgroup. Then the image D of $\langle\delta\rangle$, $\delta = \mathrm{diag}(\lambda, 1, \lambda^{-1})$ in T acts as field multiplication on Q/R. In particular, DQ is normalized by $N_G(Q)$. So we have to search for a subgroup X of $\mathrm{P}\Gamma\mathrm{L}_2(p^e)$, whose order is not divisible by p and which contains a cyclic group $U = TQ/DQ$ of order w properly as these are precisely the candidates for $N_G(Q)/DQ$.

Since T is not cyclic and D acts as scalars on Q/R, we see that the only T-invariant subgroups of Q properly containing R are E_1 and E_2. Hence $N_{N_G(Q)}(T)$ permutes $\{E_1, E_2\}$ and so normalizes $\langle H_1, H_2\rangle =$

$F^*(H)$. Hence we may additionally assume that U is not normal in X. Since U commutes with D, we have that $U \leq \mathrm{PGL}_2(p^e) \trianglelefteq P\Gamma\mathrm{L}_2(p^e)$. We first consider X_0 the normal subgroup of X which acts as a subgroup of $\mathrm{PGL}_2(p^e)$ and contains U.

We remark that $\mathrm{PGL}_2(p^e)$ is a subgroup of $\mathrm{PSL}_2(p^{2e})$ according to [**33**, Satz 8.27]. Hence the Dickson's list of subgroup of two dimensional linear groups given in [**33**, Satz 8.27] provides all the candidates for X_0. Assume that $w > 4$. Then we see that X_0 is not isomorphic to $\mathrm{Alt}(5)$ and also not a subgroup of $\mathrm{Sym}(4)$. So we have that X_0 is contained in a dihedral group. But then U is contained in the cyclic normal subgroup of X_0 and so U is characteristic in X_0 and so normal in X, a contradiction. So we have shown that in any case $N_G(Q) = N_H(Q)$. In addition, we have shown that U is normal in $N_G(Q)/DQ$ and so, as $TQ/DQ = U$, $N_G(Q)$ normalizes TQ. Now suppose that $w = 4$. In this case $p^e \in \{5, 13\}$. In $\mathrm{PGL}_2(5)$ or $\mathrm{PGL}_2(13)$, the over-groups of a cyclic group of order 4 which do not have order divisible by 5 or 13 respectively, are isomorphic to $\mathrm{Dih}(8)$ or $\mathrm{Sym}(4)$. Both are uniquely determined. Since U is normal in $\mathrm{Dih}(8)$, we have (iii) and (vi). The last remaining cases is that $w = 2$ and $p^e \in \{3, 7\}$. If $p^e = 3$, then $\mathrm{PGL}_2(3) \cong \mathrm{Sym}(4)$ and U is normal unless $G \cong \mathrm{Dih}(8)$. This gives (ii). So suppose that $p^e = 7$. Notice that an element projecting to a generator of U can be chosen to centralize Q/E_1 and invert $E_1/Z(Q)$. Thus U acts as an element of $\mathrm{PGL}_2(7)$ not contained in $\mathrm{PSL}_2(7)$. It follows that the candidates for X are subgroups of $\mathrm{Dih}(12)$ and $\mathrm{Dih}(16)$ of order greater than 4 and not contained in $\mathrm{PSL}_2(7)$. This gives $\mathrm{Dih}(8)$, $\mathrm{Dih}(16)$, $\mathrm{Dih}(12)$ and $\mathrm{Sym}(3)$. This gives part (v).

Now we conclude that whenever $p^e \notin \{3, 5, 7, 13\}$, then, as Q is a large subgroup of G by Lemma 7.2, we also have $C_G(r) \leq N_G(Q) \leq H$ for all $r \in R^{\#}$. $\square$

10. The groups with $\mathbf{F^*(H) \cong PSL_3(2^e)}$ or $\mathbf{Sp_4(2^e)'}$

In this section, we treat two further exceptional configurations which our generic arguments do not handle, as the 2-local structure is very restricted. These are the cases with $F^*(H) \cong \mathrm{PSL}_3(2^e)$ or $\mathrm{Sp}_4(2^e)'$. We are going to prove:

PROPOSITION 10.1. *Suppose Hypothesis* 6.1 *holds with* $F^*(H) \cong \mathrm{PSL}_3(2^e)$ *or* $\mathrm{Sp}_4(2^e)'$ *for some* $e \geq 1$. *Then one of the following holds:*

(i) $G = H$;
(ii) $F^*(H) \cong \mathrm{Sp}_4(2)'$ *and* $G \cong \mathrm{Mat}(11)$; *or*
(iii) $F^*(H) \cong \mathrm{PSL}_3(4)$ *and* $G \cong \mathrm{Mat}(23)$.

Throughout this section we assume the notation as described in Section 6 with $F^*(H) \cong \mathrm{PSL}_3(2^e)$ or $\mathrm{Sp}_{2n}(2^e)$.

We first examine the cases which arise when $e = 1$.

LEMMA 10.2. *If* $F^*(H) \cong \mathrm{PSL}_3(2)$, *then* $G = H$.

PROOF. Since $F^*(H) \cong \mathrm{PSL}_3(2)$, $H \cong \mathrm{PSL}_3(2)$ or $\mathrm{PGL}_3(2)$. Then $S_0 \cong \mathrm{Dih}(8)$ or $\mathrm{Dih}(16)$. Hence, if z is an involution in $Z(S_0)$, then, as G has parabolic characteristic 2, $O_2(C_G(z))$ is either a dihedral group of order at most 16 or a cyclic group. As the automorphism group of a cyclic group and of a dihedral group of order at least 8 is a 2-group, we infer that $N_G(O_2(C_G(z)))$ is a 2-group. Therefore $C_G(z) = S_0 \leq H$. Suppose that $z^G \cap H \neq z^H$. Then, as $F^*(H) \cong \mathrm{PSL}_3(2)$ has just one conjugacy class of involutions, we have $F^*(H) < H$ and z is conjugate to an involution $y \in S_0 \setminus F^*(H)$. As $S_0 \cong \mathrm{Dih}(16)$ all involutions in $S_0 \setminus F^*(H)$ are conjugate. By the Frattini Argument we see that $C_{F^*(H)}(y)$ has order divisible by 3, contrary to $C_G(z) = S_0$. Thus $z^G \cap H = z^H = z^{F^*(H)}$. By the Thompson Transfer Lemma [**26**, Lemma 15.16] we see $O^2(G) \cap H = F^*(H)$. Now $F^*(H)$ is strongly 2-embedded in $O^2(G)$ and by Proposition 4.5, if $H < G$, H is soluble. We conclude that $G = H$ as claimed. $\square$

LEMMA 10.3. *If* $F^*(H) \cong \mathrm{Sp}_4(2)'$, *then* $G = H$ *or* $G \cong \mathrm{Mat}(11)$ *and* $H \cong \mathrm{Mat}(10)$.

PROOF. Let z be an involution in $Z(S)$. By Lemma 2.4 we may assume inductively that G has no subgroup of index two. As $\mathrm{Sym}(6)$ has a Sylow 2-subgroup isomorphic to $2 \times \mathrm{Dih}(8)$, we get $H \not\cong \mathrm{Sym}(6)$ by Lemma 2.16.

Suppose $H \cong \mathrm{Aut}(\mathrm{Alt}(6))$ and let H_1 be a normal subgroup of H such $H_1 \cong \mathrm{Mat}(10)$. As all involutions in $\mathrm{Mat}(10)$ are in $\mathrm{Alt}(6)$ all the involutions in H_1 are conjugate to z. Let t correspond to a transposition in the subgroup of H isomorphic to $\mathrm{Sym}(6)$. Then $C_{S_0}(t) = \langle t\rangle \times D$, $D \cong \mathrm{Dih}(8)$, with $z \in Z(D)$. Assume t is G-conjugate to z in G. Let $T \leq C_G(t)$ with $|T : C_{S_0}(t)| = 2$. Then T is a Sylow 2-subgroup of G. Since $\langle z\rangle = C_{S_0}(t)'$ is normal in T, we have $z \in Z(T)$. As $|Z(T)| = |Z(S)| = 2$, we see that $t \notin Z(T)$, a contradiction. Hence t is not G-conjugate to z and the Thompson Transfer Lemma [**26**, Lemma 15.16] implies that G has a normal subgroup G_1 of index two, a contradiction. Therefore, we may assume that

$$H = F^*(H) \text{ or } H \cong \mathrm{PGL}_2(9) \text{ or } \mathrm{Mat}(10).$$

Suppose $C_G(z) \leq H$. Then $C_G(z)$ is a 2-group. If $t \in S_0 \setminus F^*(H)$ is an involution, then $S_0 \cong \mathrm{Dih}(16)$ and $H \cong \mathrm{PGL}_2(9)$. Thus $|C_H(t)|$

is divisible by 5. In particular $z^G = z^H = z^{F^*(H)}$. By the Thompson Transfer Lemma again we see that $O^2(G) \cap H = F^*(H)$ and so $F^*(H)$ is strongly 2-embedded in $O^2(G)$, which by Proposition 4.5 shows $G = H$.

We will now assume $C_G(z) \not\leq H$. In particular, as S_0 is either dihedral of order 8 or 16 or semidihedral of order 16, the only normal subgroups of S_0 are elementary abelian of order at most 4, cyclic, dihedral, quaternion or semidihedral. As $O_2(C_G(z))$ must admit a nontrivial automorphism of odd order centralizing z, we deduce $O_2(C_G(z))$ is a quaternion group of order 8 and therefore

$$H \cong \mathrm{Mat}(10).$$

This means $C_G(z) \cong \mathrm{GL}_2(3)$ and consequently $G \cong \mathrm{Mat}(11)$ or $\mathrm{PSL}_3(3)$ by Lemma 3.11. Since 5 divides the order of H but not the order of $\mathrm{PSL}_3(3)$, we conclude that $G \cong \mathrm{Mat}(11)$. As $\mathrm{Aut}(\mathrm{Mat}(11)) = \mathrm{Mat}(11)$ by [**27**, Table 5.3a], we now get $G \cong \mathrm{Mat}(11)$ and the lemma holds. □

Because of Lemmas 10.2 and 10.3 from now on we assume that $e \geq 2$. We fix notation as in Lemmas D.2 and D.3 for certain subgroups of $F^*(H)$. Thus we have elementary abelian subgroups E_1, E_2 of S with $S = E_1E_2$ and $E_1 \cap E_2 = Z(S)$. We have that

$$|E_1| = |E_2| = \begin{cases} 2^{2e} & \text{if } F^*(H) \cong \mathrm{PSL}_3(2^e) \\ 2^{3e} & \text{if } F^*(H) \cong \mathrm{PSp}_4(2^e). \end{cases}$$

Also, for $i = 1, 2$, set

$$L_1 = O^{2'}(N_{F^*(H)}(E_i))$$

and

$$\overline{L}_i = L_i/E_i.$$

We also recall

Lemma 10.4. (i) *$E_1 \cup E_2$ contains all of the involutions of S; and*

(ii) *for $i = 1, 2$, $\overline{L}_i \cong \mathrm{SL}_2(2^e)$ and $E_i/C_{E_i}(L_i)$ is a natural $\overline{L}_i$-module.*

(iii) *$|H : N_H(E_i)|_2 \leq 2$ and H contains a Sylow 2-subgroup of $N_G(E_i)$, $i = 1, 2$.*

Lemma 10.5. *One of the following holds:*

(i) *$N_G(E_i) = N_H(E_i)$ for $i = 1, 2$; or*

(ii) *$F^*(H) \cong \mathrm{PSL}_3(4)$ and, up to notation, $N_G(E_1)/E_1 \cong \mathrm{Alt}(7)$ and $N_G(E_2)/E_2 \cong (\mathrm{Alt}(5) \times 3){:}2$.*

Proof. As E_i contains 2-central involutions, Lemma 2.1 yields $N_G(E_i)$ is of characteristic 2. For the remainder of the proof we fix $i = 1$ and focus on showing that $N_G(E_1) \leq H$ unless we have the configuration in (ii). Set $M = N_G(E_1)$ and $\overline{M} = M/E_1$. We intend to prove that L_1 is normal in M or $2^e = 4$ and $E(\overline{M}) \cong \mathrm{Alt}(7)$.

Since $N_M(L_1)$ contains a Sylow 2-subgroup of M and $C_G(E_1) = E_1$, we have that $\overline{L_1}$ is Sylow 2-embedded in $\overline{M}$. If $C_{E_1}(L_1)$ is normal in M, then we can apply Proposition 5.3 with $V = E_1/C_{E_1}(L_1)$ to see that either L_1 is normal in M or $p^e = 4$ and $E(\overline{M}) \cong \mathrm{Alt}(7)$.

Suppose that $F^*(H) \cong \mathrm{PSp}_4(2^e)$ and $C_{E_1}(L_1)$ is not normal in M.

Since E_1 and E_2 are normal subgroups of S, E_2 acts quadratically on E_1 and, as $E_1/C_{E_1}(L_1)$ is a natural L_1-module, we also have

$$C_{E_1}(e) = C_{E_1}(E_2) = E_1 \cap E_2 = Z(S)$$

for all $e \in E_2 \setminus E_1$. Since $|E_2E_1/E_1| = 2^e = |E_1/C_{E_1}(E_2)|$ we have that E_1 is an F-module for $E(L_1)S_0$. Furthermore, as $2^e > 2$, Lemma 2.14 shows that $\overline{E_2}$ centralizes every odd order subgroup of $\overline{M}$ which is normalized by $\overline{E}_2$.

Set $\overline{K} = E(\overline{M})$ and assume that $\overline{L}_1 \neq \overline{K}$. Then we have that $[\overline{K}, F(\overline{M})] = 1$. As $N_H(E_1)$ contains a Sylow 2-subgroup of M, we see that $\overline{L}_1 \leq E(\overline{M})$, and, in particular, is contained in some component X of $\overline{M}$. Since $E_1/C_{E_1}(X)$ is an F-module for X, Lemma C.21 implies that either X is a group of Lie type in characteristic 2, an alternating group or $3{\cdot}\mathrm{Alt}(6)$. If X is a group of Lie type in characteristic 2, Lemma A.17 implies $\overline{L}_1 = X$, a contradiction. So $X/Z(X)$ is an alternating group. As a Sylow 2-subgroup of X is an extension of an elementary abelian group by a cyclic group, we obtain $X/Z(X) \cong \mathrm{Alt}(7)$ or $\mathrm{Alt}(6)$. Hence $\overline{L}_1 \cong \mathrm{SL}_2(4)$ and $\overline{S}_0 \cong \mathrm{Dih}(8)$. Since $\mathrm{Alt}(6)$ does not contain $\mathrm{Sym}(5) \cong \overline{S_0L_1}$, we must have $X \cong \mathrm{Alt}(7)$. Now referring again to Lemma C.21 and using the fact that $C_{E_1}(L_1)$ is not normalized by $N_G(E_1)$ yields that E_1 is the permutation module for X. Let $v \in C_{E_1}(L_1S_0)^\#$. Then $|v^X| = 1, 7$ or 21. But L_1 has three orbits of length 1 and four of length 15 on E_1. Therefore $C_{E_1}(X) \neq 1$, a contradiction. Thus in this case $X = \overline{L}_1$ and it follows that L_1 is normal in $N_G(E_1)$.

We have shown that one of the following holds:

- $\overline{L}_1$ is normal in $\overline{M}$;or
- $F^*(H) \cong \mathrm{PSL}_3(4)$ or $\mathrm{PSp}_4(4)$, $C_{E_1}(L_1)$ is normalized by M, $\overline{L}_1 \leq X$, $X \cong \mathrm{Alt}(7)$, X normal in $\overline{M}$

Assume first that we are in the second case.

Suppose that $F^*(H) \cong \mathrm{Sp}_4(4)$. Then $N_{F^*(H)}(E_1)/E_1 \cong \mathrm{SL}_2(4) \times 3$ and $N_H(S)$ normalizes E_1. Let $J \in \mathrm{Syl}_3(N_H(E_1))$. Then J induces automorphisms on X and some non-trivial element of $j \in J$ centralizes $\overline{L}_1$. Therefore

$$X\langle j\rangle \cong 3 \times \mathrm{Alt}(7).$$

However this group has to act on $E_1/C_{E_1}(L_1)$ which has order 2^4. Since $\mathrm{PSL}_4(2)$ has Sylow 3-subgroups of order 9, we conclude that some element τ of order 3 centralizes $E_1/C_{E_1}(L_1)$ and so acts faithfully on $C_{E_1}(L_1)$. But then $E_1 = [E_1, \tau] \times C_{E_1}(\tau)$ and, by Lemma D.3, we obtain

$$16 = |S'| = |[E_1, \overline{E_2}]| = |[E_1, \tau, E_2]||[C_{E_1}(\tau), E_2]| \leq 2^3,$$

which is impossible. Therefore if $F^*(H) \cong \mathrm{PSp}_4(4)$, we have $\overline{L}_1$ is normal in $\overline{M}$.

We now make more precise the configuration of normalizers in the exceptional case as detailed in (ii). Suppose that $F^*(H) \cong \mathrm{PSL}_3(4)$. Then $\overline{M}$ is isomorphic to a subgroup of $\mathrm{SL}_4(2)$, which shows

$$\overline{M} \cong \mathrm{Alt}(7).$$

Assume now

$$N_G(E_1)/E_1 \cong \mathrm{Alt}(7) \cong N_G(E_2)/E_2.$$

Then, for non-trivial $z \in Z(S)$, $C_{N_G(E_i)}(z)/E_i \cong \mathrm{SL}_3(2)$ and so $E_1 = O_2(C_{N_G(E_1)}(z))$ and $E_2 = O_2(C_{N_G(E_2)}(z))$, which implies $O_2(C_G(z)) \leq E_1 \cap E_2$, which contradicts $C_G(O_2(C_G(z))) \leq O_2(C_G(z))$. So we have that L_2/E_2 is normal in $N_G(E_2)/E_2$. As E_2 acts quadratically on E_1 we obtain from Lemma C.13 that $\overline{E}_2$ corresponds to $\langle(12)(34), (13)(24)\rangle$ in $\mathrm{Alt}(7)$. This shows that

$$N_{\overline{M}}(\overline{E}_2) \sim 3^2{:}2$$

with the element of order 2 inverting the normal subgroup of order 9. As we have that L_2/E_2 is normal in $N_G(E_2)/E_2$, we now see that

$$N_G(E_2)/E_2 \sim (\mathrm{Alt}(5) \times 3){:}2$$

and this is the configuration described in (ii).

Assume now that L_i is normal in $N_G(E_i)$ for $i = 1, 2$. We have, by Lemma 10.4, that

$$O^2(N_G(E_1E_2)) \leq N_G(E_1) \cap N_G(E_2)$$

normalizes $\langle L_1, L_2\rangle$. As $\langle L_1, L_2\rangle = F^*(H)$, we get $N_G(E_1E_2) \leq H$. By the Frattini Argument we have that

$$N_G(E_i) = N_H(E_i)N_G(E_1E_2) \leq H,$$

for both $i = 1, 2$, which is (i). $\square$

LEMMA 10.6. *Suppose that $N_G(E_1) \not\leq H$ or $N_G(E_2) \not\leq H$. Then $F^*(H) \cong \mathrm{PSL}_3(4)$ and $G \cong \mathrm{Mat}(23)$.*

PROOF. Suppose $N_G(E_1) \not\leq H$. Then Lemma 10.5 yields $F^*(H) \cong \mathrm{PSL}_3(4)$ and

$$N_G(E_1)/E_1 \cong \mathrm{Alt}(7) \text{ and } N_G(E_2)/E_2 \cong (\mathrm{Alt}(5) \times 3){:}2.$$

Let $B \leq N_G(E_1)$ be such that $B/E_1 \cong \mathrm{Alt}(6)$ and $B \cap F^*(H) = L_1$. Set

$$W = N_B(E_2)L_2 \leq N_G(E_2)$$

Then $W/E_2 \cong \mathrm{Sym}(5)$. We now have $B \cap F^*(H) = L_1$, $W \cap F^*(H) = L_2$ and $|W : W \cap B| = 5$. Let $P = \langle F^*(H), B, W\rangle$. Applying Lemma 3.1 yields

$$P \cong \mathrm{Mat}(22).$$

Now we consider the triangle of groups consisting of P, $N_G(E_1)$ and $N_G(E_2)$. We have $N_G(E_1) \sim 2^4{:}\mathrm{Alt}(7)$, $N_G(E_2) \sim 2^4{:}((\mathrm{Alt}(5) \times 3){:}2)$ and $P \cong \mathrm{Mat}(22)$. Furthermore,

$$N_G(E_1) \cap P \sim 2^4 : \mathrm{Alt}(6),$$

$$N_G(E_2) \cap P \sim 2^4{:}\mathrm{Sym}(5)$$

and $N_G(E_1) \cap N_G(E_2) = N_G(E_1E_2)$ with

$$N_G(E_1E_2)/E_1 \cong (3 \times \mathrm{Alt}(4)){:}2.$$

Now application of Lemma 3.2 with $B = N_G(E_1)$, $W = N_G(E_2)$ and P yields

$$M = \langle P, N_G(E_1), N_G(E_2)\rangle \cong \mathrm{Mat}(23).$$

In particular, using [**27**, Table 5.3 d] we now know that P and hence G has exactly one conjugacy class of involutions. Thus, if $r \in E_1^{\#}$, $r^G \cap M = r^M$. In $N_G(E_1)$, we have $C_{N_G(E_1)}(z)/E_1 \cong \mathrm{PSL}_3(2)$. Hence $O_2(C_G(r)) = E_1$ and so $C_G(r) \leq N_G(E_1) = B \leq M$. So $C_G(x) \leq M$ for all involutions x in M. Thus, if $M < G$, then M is strongly 2-embedded in G ([**26**, Proposition 17.11]). Since, by [**9**], G does not have a strongly 2-embedded subgroup we infer that $G = M$ and this completes the proof of the lemma. □

LEMMA 10.7. *Assume that $N_G(E_i) = N_H(E_i)$, $i = 1, 2$. Then*

$$N_G(Z(S)) \leq H.$$

PROOF. Set $M = N_G(Z(S))$ and $U = O_2(M)$. Since $M \geq S_0$, M is of characteristic 2. Assume that $M \not\leq H$. Then $U \neq E_1, E_2$. Let C be a complement to S in $N_G(S)$. Then

$$|C| = (2^e - 1) \times (2^e - 1)/u,$$

where $u = 1$ unless $F^*(H) \cong \mathrm{PSL}_3(2^e)$ with e even in which cases $u = 3$. Thus, if $F^*(H) \ncong \mathrm{PSL}_3(4)$, $(2^e - 1)/u \neq 1$ and there is a subgroup D of C of order $(2^e - 1)/u$, which acts non-trivially on $E_1/Z(S)$ and centralizes $E_2/Z(S)$. Hence E_1 and E_2 are the only proper subgroups of S containing $Z(S)$, which are invariant under C.

Assume $D \neq 1$. Then CS/S admits S_0/S faithfully. Since $C \leq M$, C normalizes U. Since S_0/S acts faithfully on C and $[U, C] \leq U$, we have $Z(S) < U \leq S$. Since C normalizes U, we now have $U = S$. As, by Lemma 10.4, E_1 and E_2 are the only elementary abelian subgroups of maximal order in S, any element of odd order in $N_G(S)$ has to normalize both E_1 and E_2 and so by assumption is in H. Hence we have $O^2(M) \leq H$ and so therefore is $M \leq H$ in this case.

So it remains to consider the case when $D = 1$. So we have

$$F^*(H) \cong \mathrm{PSL}_3(4).$$

Furthermore, if $S \leq U$, then $J(U) = J(S)$ by Lemma D.4(ii). Now we may argue as before that every odd order element in M normalizes E_1 and E_2 and obtain $M \leq H$. If $[U, S] \leq Z(S)$, then [**22**, Chap. 5, Theorem 3.2] implies that $S \leq U$, a contradiction. In particular, $|U/Z(S)| > 2$ and $U \nleq S$.

Since C acts transitively on $Z(S)^{\#}$, we have $U \leq C_G(Z(S))$. Hence $|US/S| \leq 2$. Because C acts fixed-point-freely on S, we now have $|S \cap U| = 2^4$ and $U/Z(S)$ has order 8. Now SC must induce $\mathrm{Alt}(4)$ on $U/Z(S)$. The subgroup structure of $\mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$ can be read from [**33**, Satz 8.27]. Thus we see that $M/U \cong \mathrm{Alt}(4)$, $\mathrm{Sym}(4)$ or $\mathrm{SL}_3(2)$. Hence, either $M \leq H$ or $M/U \cong \mathrm{SL}_3(2)$. But in the latter case, $Z(S) \leq Z(M)$, a contradiction as this is not true in $F^*(H)$, since $C \leq M$. Hence we have the assertion $M \leq H$. $\square$

LEMMA 10.8. *Assume $N_G(E_i) = N_H(E_i)$ for $i = 1, 2$. If $r \in Z(S_0)^{\#}$, then $C_G(r) \leq H$.*

PROOF. Assume that $C_G(r) \nleq H$ and set $U = O_2(C_G(r))$. If $F^*(H) \cong \mathrm{PSp}_4(2^e)$ and r in $C_{E_i}(L_i)$ for some $i \in \{1, 2\}$, then $U \leq O_2(C_H(r)) = E_i$ and we conclude that $U = E_i$. But then

$$C_G(r) \leq N_G(E_i) \leq H,$$

which is a contradiction. Hence, if $F^*(H) \cong \mathrm{PSp}_4(2^e)$, then

$$r \in Z(S) \setminus (C_{E_1}(L_1) \cup C_{E_2}(L_2)).$$

Furthermore, we note that $N_{F^*(H)}(S)$ permutes the members of $Z(S) \setminus (C_{E_1}(L_1) \cup C_{E_2}(L_1))$ transitively.

In particular, if $F^*(H) \cong \mathrm{PSL}_3(2^e)$ or $\mathrm{PSp}_4(2^e)$, then the H-conjugates of r in $Z(S)$ generate $Z(S)$.

We first show

$$Z(S) \leq U.$$

This is of course true if $U \leq S$, as G is of parabolic characteristic 2. So assume that $U \not\leq S$. Choose $t \in U \setminus S$. The either $E_1^t = E_2$ or $[E_i, t] \not\leq Z(S)$ for both i. So we have that $[S, t]$ contains some $u \in S \setminus (E_1 \cup E_2)$. Further $u \in U$.

If $F^*(H) \cong \mathrm{PSL}_3(2^e)$, then $|[E_i, u]| = 2^e$ and is contained in $Z(S)$, so it is equal to $Z(S)$, in particular $Z(S) \leq U$.

So assume that $F^*(H) \cong \mathrm{PSp}_4(2^e)$. By Lemma A.12, $\mathrm{Out}(\mathrm{Sp}_4(2^e))$ is cyclic and so $|\Omega_1(U)S/S| \leq 2$. In particular, $[Z(S), \Omega_1(U), \Omega_1(U)] = 1$ and so

$$[\Omega_1(U), Z(S)] \leq \Omega_1(Z(\Omega_1(U))).$$

Since $\Omega_1(Z(\Omega_1(U)))$ is an elementary abelian normal subgroup of S_0 Lemma D.4, implies that $\Omega_1(Z(\Omega_1(U))) \leq S$. Hence

$$[\Omega_1(Z(\Omega_1(U))), Z(S)] = 1.$$

As $[U, Z(S)] \leq \Omega_1(U)$, we now see that $Z(S)$ stabilizes the chain

$$U \geq \Omega_1(U) \geq \Omega_1(Z(\Omega_1(U))) \geq 1.$$

Application of [**22**, Chap. 5, Theorem 3.2] shows $Z(S) \leq U$.

Let $x \in C_G(r) \setminus H$. Then Lemma 10.7 implies $Z(S^x) = Z(S)^x \neq Z(S)$ and, of course, $Z(S)^x \leq U$. Assume that $Z(S)^x \leq S$. Then $Z(S)Z(S)^x$ is elementary abelian and so we may assume $Z(S)Z(S)^x \leq E_1$. Then

$$E_1 \leq C_G(Z(S)^x) \leq H^x$$

by Lemma 10.7. Now $E_1 \leq S^x$ by Lemma D.4. Hence E_1 is normal in S^x. But then $S^x \leq N_G(E_1) \leq H$, and this means that $S^{xh} = S$ for some $h \in H$. Since $N_G(S) \leq N_G(Z(S)) \leq H$, we have $x \in H$, a contradiction. Hence $Z(S)^x \not\leq S$. Since the G-conjugates of r in $Z(S)^x$ generates $Z(S)^x$, there exists $r^g \in U$ such that r^g induces an outer automorphism of $F^*(H)$. Now $[S, r^g] \leq U$ so, as r^g either swaps E_1 and E_2 or induces a field automorphism on $E_i/Z(S)$ for $i = 1, 2$, we have

$$|U \cap S| \geq |[S, r^g]Z(S)| \geq \begin{cases} 2^{2e} & F^*(H) \cong \mathrm{PSL}_3(2^e) \\ 2^{3e} & F^*(H) \cong \mathrm{PSp}_4(2^e). \end{cases}$$

In addition, as $Z(S)^x \not\leq S$, $|U| \geq 2|U \cap S|$. Assume that $|S_0/S| = t$. Then we have demonstrated that

$$|S_0/U| \leq 2^{e-1}t.$$

Now using Lemma A.16 and the Frattini Argument we get

$$C_{F^*(H)S_0}(r^g) \text{ involves } \begin{cases} \mathrm{SL}_2(2^e).t/2 \\ \mathrm{SL}_3(2^{e/2}).t/2 \\ \mathrm{SU}_3(2^{e/2}).t/2 \end{cases}$$

if $F^*(H) \cong \mathrm{PSL}_3(2^e)$ and

$$C_{F^*(H)S_0}(r^g) \text{ involves } \begin{cases} \mathrm{Sp}_4(2^{e/2}).t/2 \\ {}^2\mathrm{B}_2(2^e).t/2 \end{cases}$$

when $F^*(H) \cong \mathrm{Sp}_4(2^e)$. Since one of these groups has to be involved in $C_G(r^g)/U$ and $|S_0/U| \leq 2^{e-1}t$, we conclude by comparing the size of the Sylow 2-subgroups that only $F^*(H) \cong \mathrm{PSL}_3(2^e)$, $C_H(r^g)$ involves $\mathrm{SL}_2(2^e).t/2$ and r^g induces a graph automorphism of $F^*(H)$ remains. Furthermore, $|U| = 2^{2e+1}$. Since r^g induces the graph automorphism of $F^*(H)$, we see that $[Z(S), r^g] = 1$ and so $[Z(S), Z(S^g)] = 1$. Set $W = \langle Z(S)^{C_G(r)} \rangle$. Then W is elementary abelian. Since $\mathrm{SL}_2(2^e)$ acts on W we have that W admits an element of order $2^e + 1$ faithfully, we have $|W| \geq 2^{2e}$. Hence W is a maximal order elementary abelian subgroup of H and so $Z(S)^x \leq W \leq S$ by Lemma D.4, but we have already seen that this is impossible. This concludes the proof. □

Proof of Proposition 10.1. Suppose that $G \neq H$. Then Lemmas 10.3 and 10.2 show that if $e = 1$, then (ii) holds. Similarly, Lemma 10.6 shows that if $N_G(E_1)$ or $N_G(E_2) \not\leq H$, then (iii) holds.

Thus we may suppose that $N_G(E_i) = N_H(E_i)$ for $i = 1, 2$. Then Lemma 10.8 implies that the centralizer of any 2-central involution r of H is contained in H. We may choose r such that $C_H(r)$ is soluble. Let r be conjugate to some involution $u \in H \setminus F^*(H)$. Then $C_H(u)$ must be soluble. By Lemma A.16 u induces a field, graph or graph-field automorphisms on $F^*(H)$. The only possibility for a soluble centralizer occurs with $F^*(H) \cong \mathrm{PSL}_3(4)$ and u a graph-field automorphism. But then u centralizes a group of order 9, while r does not. So we have that $r^G \cap H = r^H$ and then H controls fusion of r. Recall that in case of $\mathrm{PSL}_3(2^e)$ we just have one conjugacy class of involutions in $F^*(H)$, in case of $\mathrm{Sp}_4(2^e)$ we have three $F^*(H)$-classes and only one has a solvable centralizer. Hence in both cases $r^G \cap H = r^{F^*(H)}$. Together with Lemma 2.5 application of Lemma 4.4 gives a contradiction. Hence $H = G$. □

11. The groups with $\mathbf{F^*(H) \cong Sp_{2n}(2^e)}$, $\mathbf{n \geq 3}$

In this section we will treat those cases with $F^*(H) \cong \mathrm{Sp}_{2n}(2^e)$ with $n \geq 3$. Our aim is to prove the following statement.

Proposition 11.1. *Suppose Hypothesis* 6.1 *holds with* $F^*(H) \cong \mathrm{Sp}_{2n}(2^e)$, $n \geq 3$. *Then* $G = H$.

Assume that
$$F^*(H) \cong \mathrm{Sp}_{2n}(2^e).$$
Then S_0/S is cyclic and is generated by field automorphisms of $F^*(H)$. Taking V to be the natural symplectic space for $F^*(H)$, we focus our attention on the parabolic subgroups K and M of $F^*(H)$ which contain S and leave an isotropic one space and a maximal totally isotropic subspace of V invariant respectively. As usual let R be a long root subgroup contained in $Z(S)$.

Lemma 11.2. *The following hold:*

(i) $O^{2'}(K/O_2(K)) \cong \mathrm{Sp}_{2n-2}(2^e)$, $O_2(K)$ *is elementary abelian with* $|O_2(K)| = 2^{e(2n-1)}$ *and* $O_2(K)/R$ *is a natural module for* $O^{2'}(K/O_2(K))$*;*

(ii) $O^{2'}(M/O_2(M)) \cong \mathrm{SL}_n(2^e)$ *and* $O_2(M) = J(S)$ *is elementary abelian of order* $2^{en(n+1)/2}$*; and*

(iii) *every involution of* $F^*(H)$ *is conjugate to an element of* $J(S)$.

Proof. Part (i) is Lemma D.5 (ii)(a) and part (ii) comes from Lemma D.6.

To prove part (iii), we note that $[V,t] \leq C_V(t) = [V,t]^\perp$ for all involutions t in $F^*(H)$. In particular, if U is a maximal isotropic subspace of V containing $[V,t]$, then $U \leq [V,t]^\perp = C_V(t)$ and so $[U,t] = 0$. Since the centralizer in $F^*(H)$ of U is conjugate to $O_2(M)$, we conclude that t is contained in a conjugate of $O_2(M)$ as claimed. □

Our plan is to apply Holt's result Lemma 4.4 to deduce that $H = G$ and so we intend to show that $r^G \cap H = r^{F^*(H)}$ and $C_G(r) \leq H$ for $r \in R^\#$. Recall that by Lemma A.4 all involutions in $R^\#$ are conjugate in $F^*(H)$.

Lemma 11.3. *If* $F^*(H) \cong \mathrm{Sp}_6(2)$, *then* $G = H$.

Proof. By Hypothesis 6.1, G is of parabolic characteristic 2 and S is a Sylow subgroup of G. Hence as $O_2(K)$ is abelian, we have $O_2(K) = O_2(C_G(r))$. Now $O_2(K)/\langle r\rangle$ has order 16. As $\mathrm{Sym}(6)$ is maximal in $\mathrm{Alt}(8) \cong \mathrm{GL}_4(2)$ and $|\mathrm{Sym}(6)|_2 < |\mathrm{Alt}(8)|_2$, we see $C_G(r) = K \leq H$.

Let r_1 be the root element in $Z(S) \setminus \{r\}$. Then $|C_H(r_1)| = 2^9 \cdot 3^2$. If $r_1^g = r$ for some $g \in G$, then K has a subgroup $C_H(r_1)^g$ of index 5, a contradiction as $K/O_2(K) \cong \mathrm{Sym}(6)$. Thus r and r_1 are not G-conjugate and therefore $N_G(S) \leq H$ as $S \in \mathrm{Syl}_2(G)$ and $N_G(S)$ is not transitive on $Z(S)^\sharp$. Thus G has three conjugacy classes of 2-central involutions.

Let $J = J(S)$. As H has four conjugacy classes of involutions, so M has four orbits on $J^\#$ with representatives r, r_1, rr_1, j and orbit lengths 7, 7, 21 and 28. We claim that $r^H = r^G \cap H$. If not, $j \in r^G$ and $N_G(J)$ is transitive on $r^G \cap J$ of length 35, so $|N_G(J) : M| = 5$. As $N_G(J)/J$ has dihedral Sylow 2-subgroups S/J we conclude $N_G(J)/J = M/J \times U/J$ with $|U/J| = 5$. But then $U \leq N_G(S) \leq H$, a contradiction that establishes the claim. Now Theorem 4.3 implies that $G = H$. $\square$

From now on we may assume that G is a $\mathcal{K}_2$-group and that $e > 1$.

LEMMA 11.4. *Suppose that* $r \in (R \cap Z(S_0))^\#$. *Then* $O^{2'}(K)$ *is normal in* $C_G(r)$ *and* $O^{2'}(M)$ *is normal in* $N_G(J(S))$.

PROOF. Since $r \in Z(S_0)$ and G is of parabolic characteristic 2, we have $C_G(r)$ has characteristic 2. Furthermore, $O^{2'}(K) \leq C_G(r)$ and consequently $O_2(C_G(r)) \leq O_2(C_H(r)) = O_2(K)$. Since $O_2(K)$ is abelian by Lemma 11.2 (i) and $C_G(r)$ has characteristic 2, we obtain $O_2(C_G(r)) = O_2(K)$. Because $J(S)$ is normal in S_0, $N_G(J(S))$ also is of characteristic 2 and, as $J(S) = O_2(M)$ is abelian by Lemma 11.2 (ii), we have $O_2(M) = O_2(N_G(J(S)))$.

We make the following observation $O_2(K \cap M) = O_2(K)O_2(M)$, $|O_2(K) \cap O_2(M)| = 2^{en}$,

$$|O_2(K) : O_2(K) \cap O_2(M)| = 2^{e(n-1)}$$

and

$$|O_2(M) : O_2(M) \cap O_2(K)| = 2^{en(n-1)/2}.$$

Set $X = C_G(r)$. If $e = 1$, then $K/O_2(K) \cong \mathrm{Sp}_{2n}(2)$, $O_2(K)/\langle r\rangle$ is the natural $K/O_2(K)$-module and $O^{2'}(K)$ is Sylow embedded in $X/O_2(K)$. Furthermore $K/O_2(K)$ satisfies Hypothesis 5.2 (a) when acting on $O_2(K)/\langle r\rangle$. Hence Proposition 5.3 implies that $O^{2'}(K)'$ is normal in M. This proves the result for $e = 1$ and so $e > 1$. Let R_* be a root subgroup contained in $J(S)$ with $R_* \cap O_2(K) = 1$. Then by Lemma A.8 $C_{F^*(H)}(x) = C_{F^*(H)}(R_*)$ for all non-trivial $x \in R_*$. Thus Lemma 2.14 implies that R_* centralizes $O(X/O_2(K))$ and therefore $O(X/O_2(K))$ is centralized by $O^{2'}(K/O_2(K))$. Therefore $E(X/O_2(K)) \neq 1$ and it follows that

$$O^{2'}(K/O_2(K)) \leq E(X/O_2(K)).$$

Furthermore, $E(X/O_2(K))$ is quasisimple. Since

$$2^{en(n-1)/2} = |O_2(M)O_2(K)/O_2(K)| \geq |O_2(K)/C_{O_2(K)}(O_2(M))| = 2^{e(n-1)},$$

$O_2(K)/\langle r\rangle$ is an F-module for $E(X)/O_2(K)$. Thus Lemma C.21 applies. If $E(X/O_2(K))$ is a group of Lie type in characteristic 2, then Lemma A.17 implies that $E(X/O_2(K)) = O^{2'}(K/O_2(K))$ and we are done. Hence $E(X/O_2(K))$ modulo its centre is an alternating group. Applying Lemma B.1 shows that $F^*(K/O_2(K))$ is an alternating group. Hence [**37**, Proposition 2.9.1] yields that $O^{2'}(K/O_2(K)) \cong \mathrm{Sp}_4(2)$, contrary to $e > 1$. We conclude that $O^{2'}(K)$ is normal in X as required.

Set $X = N_G(J(S))$. We have $O_2(X) = O_2(M)$ is elementary abelian of order $2^{en(n+1)/2}$ by Lemma 11.2 (ii). Set $\overline{X} = X/O_2(X)$,

$$\overline{M}^* = O^{2'}(\overline{M}) \cong \mathrm{SL}_n(2^e)$$

and $W_0 = O(\overline{X})$. Suppose that $\overline{M}^*$ does not centralize W_0. Choose a root subgroup R_* in $O_2(K)$ such that $R_* \cap O_2(M) = 1$. Then by Lemma A.8, for all $x \in R_*^{\#}$, $C_{O_2(M)}(x) = C_{O_2(M)}(R_*)$ and so, if $e > 1$, Lemma 2.14 shows that R^* and hence also $\overline{M}^*$ centralizes W_0, a contradiction. Therefore $e = 1$ and, in particular, $Z(\overline{M}) = 1$, $\overline{M} = \overline{M}^*$ operates faithfully on W_0 and also in $F(W_0)$. By the Critical Subgroup Theorem [**27**, Proposition 11.11], there exists an odd prime ℓ and an ℓ-group $W^* \leq F(W_0)$ such that $\overline{M}$ acts faithfully on $W^*/\Phi(W^*)$. Furthermore, W^* has exponent ℓ and is nilpotent of class at most 2. From among all $\overline{M}$-invariant subgroups of W^* we choose W of smallest order such that W not centralized by $\overline{M}$. Let U be a proper $\overline{M}$-invariant subgroup of W. Then $[U, \overline{M}] = 1$ by the definition of W. Hence $[U, \overline{M}, W] = 1$, and as $[U, W] < W$, $[[U, W], \overline{M}] = 1$. Hence the Three Subgroup Lemma implies that $[U, [W, \overline{M}]] = 1$. As $W = [W, \overline{M}]$, we have $U \leq Z(W)$. In particular, if W is non-abelian, then $Z(W)$ is the unique maximal $\overline{M}$-invariant proper subgroup of W. Since $\overline{M}$ acts faithfully on $W/\Phi(W)$ and, as $(2^n - 1)/(2 - 1) - n \geq 2^{n-1}$ for $n \geq 3$, Lemma C.5 implies that

$$|W/\Phi(W)| \geq \ell^{2^{n-1}-1}$$

using $\mathrm{PSL}_3(2)$ is not a subgroup of $\mathrm{SL}_2(\ell^a)$ for all $a \geq 1$. We claim that some subgroup of $O_2(M)$ admits a faithful action of an elementary abelian ℓ-subgroup of rank $2^{n-2} + 1$. If $\Phi(W) = 1$, then we have nothing further to do. Suppose that $\Phi(W) \neq 1$. Set $Y = [O_2(M), Z(W)]$. Then there is a hyperplane W_1 of $Z(W)$ such that $C_Y(W_1) \neq 1$. Hence $W\overline{M}/W_1$ acts faithfully on $C_Y(W_1)$. If $\Phi(W) \leq W_1$, then W/W_1 is elementary abelian and we are done again. If $\Phi(W) \not\leq W_1$, then W/W_1 is extraspecial. In particular, Y admits W/W_1 faithfully. Since W has exponent ℓ and W/W_1 is even dimensional as a $\mathrm{GF}(\ell)$-space, W/W_1 has an elementary abelian ℓ-subgroup of order $2^{n-2} + 1$ and this proves our claim.

Now the ℓ-rank of $\mathrm{GL}_{n(n+1)/2}(2)$ is bounded above by $n(n+1)/4$ (this is attained by an elementary abelian 3-group). Thus we have

$$n(n+1)/4 \geq 2^{n-2} + 1$$

and this yields $n = 4$ using Lemma 11.3. Then $|O_2(M)| = 2^{10}$ and $\overline{M} \cong \mathrm{SL}_4(2)$ with $|W/\Phi(W)| \geq \ell^7$. Thus the fact that $|W|$ does not divide $|\mathrm{GL}_{10}(2)|$ provides a contradiction. This proves that $\overline{M}$ centralizes W_0 as desired.

Since $\overline{M}$ centralizes $O(\overline{X})$, we have $E(\overline{X}) \neq 1$. Then $\overline{M}^*$ is quasisimple so, as $p = 2$, $\overline{M}^*$ is contained in a component $\overline{X}^*$ of $\overline{X}$.

We intend to show that $\overline{X}^* = \overline{M}^*$. Notice first that $O_2(K)$ acts quadratically on $O_2(M)$. If X^* is a group of Lie type in characteristic 2 then Lemma A.17 implies that X^* and M^* are equal. Thus suppose that X^* is not such a group. We exploit the quadratic action of $O_2(K)$ on $O_2(M)$.

By Lemma 11.3, $|O_2(M)O_2(K) : O_2(M)| \geq 8$. Suppose that $\overline{M}^* \neq \overline{X}^*$. We have that $O_2(K)O_2(M)/O_2(M)$ and acts quadratically on $O_2(M)$. Therefore Lemma C.13 shows that $\overline{X}^* \cong 3{\cdot}\mathrm{Mat}(22)$, $3{\cdot}\mathrm{PSU}_4(3)$ or $\mathrm{Alt}(m)$ for some $m \geq 8$. In the first two cases, $|\overline{X}^*|_2 = 2^7$. Since $\overline{X}^* \geq \overline{M}^* \cong \mathrm{SL}_n(2^e)$, the only possibility is that $\overline{M}^* \cong \mathrm{SL}_3(4)$. But then we may cite [**14**, Table 8.11] to see that $\mathrm{SL}_3(4).2$ is not a subgroup of $3{\cdot}\mathrm{PSU}_4(3)$ and [**27**, Table 5.3c] to see the same is true for $3{\cdot}\mathrm{Mat}(22)$. So we have that

$$\overline{X}^* \cong \mathrm{Alt}(m)$$

for some $m \geq 8$. Since $\overline{M}S_0 \cap \overline{X}^*$ contains a Sylow 2-subgroup of $\overline{X}^*$ we must have $\overline{M}^* \cong \mathrm{Alt}(t)$ for some $t \in \{m-3, m-2, m-1, m\}$ by Lemma B.1. Using [**37**, Proposition 2.9.1] gives $n = 4$, $e = 1$, $m > 2n = 8$, $\overline{M}^* \cong \mathrm{SL}_4(2)$ and $|O_2(M)| = 2^{10}$. Furthermore, $O_2(K)O_2(M)/O_2(M)$ is a quadratic subgroup of order 8. It follows from Lemma C.13 that $\overline{X}^*$ has one non-central chief factor U in $O_2(M)$ and either U is the natural permutation module or $m = 9$ and U is the spin module. If U is the permutation module, then $\overline{M}^*$ centralizes a non-trivial subspace of U and hence M^* centralizes a 2-central involution in $O_2(M)$. Therefore on the natural 8-dimensional module V for H, M^* fixes a 1-space or a 2-space and this is impossible. Hence U is the spin module and $\overline{X}^* \cong \mathrm{Alt}(9)$. Since $|O_2(M)| = 2^{10}$, Lemma C.30 shows that $C_{O_2(M)}(X^*) \neq 1$. But then again M^* centralizes a 2-central involution, a contradiction. $\square$

LEMMA 11.5. *Suppose that $r \in (R \cap Z(S_0))^\#$. Then $C_G(r) \leq H$ and $N_G(J(S)) \leq H$.*

PROOF. Let $X = C_G(r)$ or $N_G(J(S))$. Then, by Lemma 11.4, we have that $X = O^{2'}(X \cap H)N_X(S)$. Thus, to prove the lemma all we have to do is show that $N_X(S) \leq H$.

We have that $N_X(S) \leq N_G(J(S))$. If $X = C_G(r)$, then by Lemma 11.4 we have that $N_X(S)$ normalizes

$$\langle O^{2'}(C_{F^*(H)}(r)), O^{2'}(N_{F^*(H)}(J(S)))\rangle = F^*(H).$$

As $H = N_G(F^*(H))$, $N_X(S) \leq H$ and we have

$$C_G(r) = C_H(r).$$

Assume $X = N_G(J(S))$. Then $N_X(S)$ normalizes $Z(S)$ and, since $C_G(Z(S)) \leq C_G(r) = C_H(r)$, we have $C_G(Z(S)) = C_H(Z(S))$. Thus $N_X(S)$ normalizes

$$\langle C_{F^*(H)}(Z(S)), O^{2'}(N_{F^*(H)}(J(S)))\rangle = F^*(H).$$

This shows $N_X(S) \leq H$ and so $N_G(J(S)) \leq H$. $\square$

Lemma 11.6. *Suppose that $r \in (R \cap Z(S_0))^{\#}$. Then $r^G \cap H = r^{F^*(H)}$.*

Proof. By Lemma 11.5, if $r^G \cap H \subset F^*(H)$, then the result is valid by Lemma 11.2 as every involution of H is conjugate to an element of $J(S)$ and $N_G(J(S)) \leq H$ controls fusion in $J(S)$. Thus we may assume that $x = r^g \in H \setminus F^*(H)$. Then by Theorem A.10 x acts as a field automorphism on $F^*(H)$. Therefore by Lemma A.16 $C_{F^*(H)}(x) \cong \mathrm{Sp}_{2n}(2^{e/2})$. As $C_G(r) \leq H$, we have $E(C_G(r)/O_2(C_G(r))) \cong \mathrm{Sp}_{2n-2}(2^e)'$ and therefore, by Lemma C.5, $C_G(r)$ has no subgroup isomorphic to $C_{F^*(H)}(x)$, and this proves our claim. $\square$

Proof of Proposition 11.1. The result holds when $H \cong \mathrm{Sp}_6(2)$ by Lemma 11.3. Lemmas 11.5, 11.6 and 2.5 provide the hypothesis of Holt's Lemma 4.4. As $F^*(H)$ is not an alternating group we obtain $G = H$. $\square$

12. The groups with $\mathbf{F^*(H) \cong {}^2F_4(2^{2e+1})'}$

This section is devoted to possible configurations which satisfy Hypothesis 6.1 with $F^*(H) \cong {}^2\mathrm{F}_4(2^{2e+1})'$. We note, however, that we do not require the $\mathcal{K}_2$-hypothesis. In this section we will prove the following proposition.

Proposition 12.1. *Suppose Hypothesis 6.1 holds with $F^*(H) \cong {}^2\mathrm{F}_4(2^{2e+1})'$. Then $G = H$.*

We continue with our standard notation. So

$$S_0 \in \mathrm{Syl}_2(H) \subseteq \mathrm{Syl}_2(G),$$

$R \leq Z(S)$ is a long root subgroup and $Q = O_2(C_{F^*(H)}(R))$. Set

$$Q_0 = O_2(C_H(R)).$$

The structure of $N_{F^*(H)}(R)$ is described in Lemma D.13.

By Lemma A.13, for $2^{2e+1} > 2$, $\mathrm{Out}(F^*(H))$ has odd order, so we have $S = S_0$ if $2^{2e+1} > 2$. In particular, we note that $Q = Q_0$ unless $H \cong {}^2\mathrm{F}_4(2)$ in which case $|Q_0/Q| = |S_0/S| = 2$.

LEMMA 12.2. *$Q_0 = O_2(C_H(r)) = O_2(C_G(r))$ for all $r \in R^\#$.*

PROOF. Set $U = O_2(C_G(r))$. Since $U \le S_0$, U is normal in $C_H(r)$ and, as $R = Z(S_0)$ and G is of parabolic characteristic 2, we have $R \le U$ and $U > R$. This implies that

$$U \cap Z_2(S_0) > R.$$

As $C_H(r)$ acts irreducibly on $Z_2(Q_0)/R$ by Lemma D.13 (ii) and (v), we obtain

$$Z_2(Q_0) \le U.$$

Assume that either $2^{2e+1} > 2$ or $U \not\le F^*(H)$. Suppose $U \le C_S(Z_2(Q_0))$. Then, by Lemma D.13 (iii), $Z_2(Q_0) = \Omega_1(U)$. Hence also $Z_2(Q_0)$ is normal in $N_G(U)$. In particular $[C_S(Z_2(Q_0)), U] \le Z_2(Q_0)$ and so $C_S(Z_2(Q_0))$ centralizes a series, which is normalized by $N_G(U)$ and so by [**22**, Chap. 5, Theorem 3.2] $U = C_S(Z_2(Q_0))$. Now by Lemma D.13(iii) $\Phi(U) = R$. Hence R is normal in $N_G(U)$. In particular as

$$\begin{aligned} [Q_0, R] &= 1 \\ [Q_0, Z_2(Q_0)/R] &= 1 \\ [Q_0, U/Z_2(Q_0)] &= 1, \end{aligned}$$

we see that Q_0 centralizes a chain of subgroups in U which is normalized by $N_G(U)$. Again, by [**22**, Chap. 5, Theorem 3.2], we have

$$Q_0 \le O_2(N_G(U)) = U = Z_2(Q_0),$$

a contradiction. This now shows that

$$U \not\le C_S(Z_2(Q_0)).$$

Now $UZ_2(Q_0)/Z_2(Q_0)$ is normalized by $C_H(r)$ and so, as $U \not\le C_S(Z_2(Q_0))$, and $U \not\le F^*(H)$ when $2^{2e+1} = 2$, we have $U = Q_0$ by Lemma D.13 (iv) and (v).

Assume now that the remaining case holds. Thus $2^{2e+1} = 2$ and $U \le F^*(H)$. If $U = C_S(Z_2(Q)) = Z_2(Q)$, we have $[Q, U] = \langle r \rangle$ and as before $Q \le U$, a contradiction. As $C_H(r)$ acts irreducibly on $Q/Z_2(Q_0)$ we then get $Q = U$. So we may assume that $S_0 > S$ and so that $H > F^*(H)$. Then Q_0 centralizes the Frattini factor group of $U = Q$, a contradiction to the fact that $U = O_2(C_G(r))$.

Hence in any case we proved $U = Q_0$. $\square$

LEMMA 12.3. *If $F^*(H) \cong {}^2\mathrm{F}_4(2)'$, then $C_G(r) \le H$, for $r \in Z(S_0)$.*

PROOF. By Lemma 12.2 we have

$$Q_0 = O_2(C_G(r)).$$

In particular, by Lemma D.13 (i), $N_H(Q_0)/Q_0$, and hence also $N_G(Q_0)/Q_0$ has cyclic Sylow 2-subgroups of order 4 and consequently $N_G(Q_0)/Q_0$ has a normal 2-complement. Assume $C_G(r) \not\leq H$. Then $N_G(Q_0) \not\leq H$. Since $Z(Q_0) = R = \langle r \rangle$ and $Z_2(Q_0)$ is elementary abelian of order 2^5, the quotient $N_G(Q_0)/C_G(Z_2(Q_0))$ embeds into the parabolic subgroup of $\mathrm{SL}_5(2)$ stabilising R of shape $2^4{:}\mathrm{SL}_4(2)$. As $Q_0/C_{Q_0}(Z_2(Q_0))$ is elementary abelian of order 2^4, we now have $N_G(Q)/Q$ is isomorphic to a subgroup X of $\mathrm{SL}_4(2) \cong \mathrm{Alt}(8)$.

We have that 5 divides $|X|$. Assume 5 does not divide $|F(X)|$, then $F(X)$ has order dividing $3^2 \cdot 7$ and hence no automorphism of order 5. So $F(G)$ has order divisible by 5. In particular $O_5(X) \neq 1$. As the centralizer of an element of order 5 in $\mathrm{Alt}(8)$ has order 15, we now get that $O^2(N_G(Q_0)/Q_0)$ must be a cyclic group of order 15, as otherwise $N_G(Q) = N_H(Q)$. Hence

$$N_G(Q)/Q \sim (3 \times 5) : 4,$$

is the normalizer in $\mathrm{SL}_4(2)$ of the cyclic group of order 15. It follows that $N_G(Q)$ acts transitively on $Q/Z_2(Q_0) = [Q_0, N_H(Q_0)]$. Now Q/R has centre $Z_2(Q)/R$ and $Q/R \setminus Z_2(Q/R)$ contains involutions by Lemma D.14 (iii). Since $N_G(Q)$ acts transitively on $Q/Z_2(Q_0)$, we conclude that Q/R has exponent 2. Therefore $R = \Phi(Q)$ and $Q = Z_2(Q) < Q$, a contradiction. Hence $N_G(Q) \leq H$. □

LEMMA 12.4. *Suppose that* $F^*(H) \cong {}^2\mathrm{F}_4(2^{2e+1})$ *with* $2^{2e+1} > 2$. *Then* $C_G(r) = C_H(r)$ *for all* $r \in R^{\#}$.

PROOF. By Lemma 12.2 we have that $Q = Q_0 = O_2(C_H(r)) = O_2(C_G(r))$. Set $M = C_G(r)$ and $\overline{M} = M/Q$. Then, by Lemma D.13 (i),

$$\overline{C_{F^*(H)}(r)} \cong {}^2\mathrm{B}_2(2^{2e+1}).$$

In addition, we recall that $C_{F^*(H)}(r)$ contains a Sylow 2-subgroup S of G. In particular $\Omega_1(\overline{S})$ is a strongly closed elementary abelian subgroup in $\overline{S}$. Hence application of [**21**] yields

$$O^{2'}(\overline{M}/O(\overline{M})) \cong {}^2\mathrm{B}_2(2^{2e+1}).$$

We first show that $\overline{C_{F^*(H)}(r)}$ centralizes $O(\overline{M})$. Otherwise there is an odd prime s and a non-trivial s-group $\overline{P} \leq O(\overline{M})$ which is normalized but not centralized by $\overline{C_{F^*(H)}(r)}$. As $N_{\overline{C_{F^*(H)}(r)}}(\overline{S})$ acts transitively on $\Omega_1(S/Q)$ by Lemma A.19 (iii), we get by Lemma 2.22 that $|\overline{P}/\Phi(\overline{P})| \geq p^{2^{2e+1}-1}$. On the other hand, $\overline{P}$ acts faithfully on $Q/C_Q(Z_2(Q))$ which is of GF(2)-dimension $4(2e+1)$. Hence $2(2e+1) \geq 2^{2e+1}-1$, which is impossible. Now we have that

$$C_{F^*(H)}(r) \text{ is normal in } C_G(r).$$

By the Frattini Argument we have that $C_G(r) = C_{F^*(H)}(r)N_{C_G(r)}(S)$. Hence to complete the proof of the lemma we just have to prove that $N_{C_G(r)}(S) \leq H$.

We have that $N_G(S)$ acts on $Z_2(S)$. By Lemma D.13(vi), we have that all elements in $Z_2(S)^{\#}$ are conjugate to r in H. Furthermore

$$U = \langle O_2(C_H(t)) \mid 1 \neq t \in Z_2(S)\rangle S$$

is a parabolic subgroup of $F^*(H)$ with $\langle U, C_{F^*(H)}(r)\rangle = F^*(H)$. By Lemma 12.2 we have that

$$U = \langle O_2(C_G(t)) \mid 1 \neq t \in Z_2(S)\rangle S$$

and so U is normalized by $N_G(S)$. Hence $F^*(H)$ is normalized by $N_{C_G(r)}(S)$. As $H = N_G(F^*(H))$, we get $N_{C_G(r)}(S) \leq H$ and then $C_G(r) \leq H$. This completes the proof. $\square$

Proof of Proposition 12.1. By Lemmas 12.3 and 12.4, we have that $C_G(r) \leq H$ for any 2-central involution r of H. By [**69**, Corollary 2] we know that H has exactly two classes of involutions which both by Lemma A.13 are contained in $F^*(H)$. Lemma D.13(vi) yields an involution t in $Z_3(S)$ such that $|C_{F^*(H)}(t)|$ is divisible by $2^{2e+1}+1$. Suppose that $t = r^g$ for some $g \in G$. Then $C_{F^*(H)}(t) \leq H^g$ and

$$C_{F^*(H)}(t) \cap F^*(H^g) \leq C_{F^*(H^g)}(t)$$

where the latter group has order coprime to $2^{2e+1}+1$. As $\mathrm{Out}(F^*(H))$ has order dividing $2e$, it follows that $2^{2e+1}+1$ divides e which is nonsense. Hence t and r are not G-conjugate. Hence $r^G \cap H = r^H$. As all involutions are in $F^*(H)$ we even have $r^G \cap H = r^{F^*(H)}$. In addition, by Lemma 2.5 we have $O(G) = 1$. Therefore application of Lemma 4.4 yields $G = H$. $\square$

13. The groups with $\mathbf{F^*(H) \cong F_4(2^e)}$

We continue the investigation of groups which satisfy Hypothesis 6.1 by studying the case in which $F^*(H) \cong \mathrm{F}_4(2^e)$. We shall prove the following result.

Proposition 13.1. *If Hypothesis 6.1 holds with $F^*(H) \cong \mathrm{F}_4(2^e)$, then $G = H$.*

By Lemma A.3, $Z(S) = R_1R_2$ with R_1 a long root subgroup and R_2 a short root subgroup of $F^*(H)$. Furthermore Lemma D.7 gives

$$C_{F^*(H)}(R_1) \cong C_{F^*(H)}(R_2) \sim 2^e.2^{6e}.2^{8e}.\mathrm{Sp}_6(2^e).$$

The fact that S_0 may contain elements which conjugate R_1 to R_2 leads to the main complication of the section. That is that $Z(S_0)$ may not contain a root element. Thus the hypothesis that G has parabolic characteristic 2 does not necessarily lead to the statement that $C_G(R_1)$ or $C_G(R_2)$ has characteristic 2. This forces us to consider elements in $Z(S_0)$ which are not contained in either R_1 or R_2. Such elements are products of elements from R_1 and R_2. For $r_1 \in R_1^\#$ and $r_2 \in R_2^\#$ with $r_1r_2 \in Z(S_0)$ we use the abbreviation $r_{12} = r_1r_2$ and note that

$$C_{F^*(H)}(r_{12}) = C_{F^*(H)}(R_1) \cap C_{F^*(H)}(R_2)$$

by Lemma D.8. Furthermore, we know that in $F^*(H)$ all the elements of $R_1^\#$, $R_2^\#$ and the elements of $Z(S) \setminus (R_1 \cup R_2)$ are all conjugate by elements of $N_{F^*(H)}(S)$. Finally, for $i = 1, 2$, we set

$$Q_i = O_2(C_{F^*(H)}(R_i)) \text{ and } Q_{12} = O_2(C_{F^*(H)}(R_1R_2)).$$

We continue with this notation for the remainder of the section.

LEMMA 13.2. *The group $F^*(H)$ has exactly 4 conjugacy classes of involutions, $r_1^{F^*(H)}$, $r_2^{F^*(H)}$, $r_{12}^{F^*(H)}$ and $j^{F^*(H)}$ where*

$$C_{F^*(H)}(j)/O_2(C_{F^*(H)}(j)) \cong \mathrm{SL}_2(2^e) \times \mathrm{SL}_2(2^e)$$

and $|O_2(C_{F^*(H)}(j))| = 2^{18e}$. Furthermore, r_1, r_2 and r_{12} are 2-central and j is not.

PROOF. This follows from [**69**, Corollary 1] or [**31**, (5.1)]. □

LEMMA 13.3. *Suppose that $r_i \in R_i^\# \cap Z(S_0)$ for $i \in \{1, 2\}$. Then $C_{F^*(H)}(r_i)$ is normal in $C_G(r_i)$.*

PROOF. We prove the result for $i = 1$. We have that $O_2(C_G(r_1)) \leq S_0$ normalizes $C_{F^*(H)}(r_1)$ and, as S_0 normalizes R_1, we know that S_0/S is cyclic and induces field automorphisms on $F^*(H)$ by Theorem A.11 (v). In particular, $C_{S_0/Q_1}(C_{F^*(H)}(r_1)/Q_1) = 1$. Thus $O_2(C_G(r_1)) \leq Q_1$ and, since $C_G(r_1)$ is of characteristic 2, $Z(Q_1) < O_2(C_G(r_1))$. Hence, as $C_{F^*(H)}(r_1)$ acts irreducibly on $Q_1/Z(Q_1)$ by Lemma D.7, we have

$$Q_1 = O_2(C_G(r_1)).$$

Set

$$V = Z(Q_1)/Q_1' = Z(Q_1)/R_1.$$

Then $Q_1 \in \mathrm{Syl}_2(C_{C_G(r_1)}(V))$ and so $C_{C_G(r_1)}(V)/Q_1$ has odd order. Since Q_1/R_1 is an indecomposable module by Lemma D.7, we have

$C_{C_G(r_1)}(V) = Q_1$. Hence $C_{F^*(H)}(r_1)/Q_1$ is Sylow embedded in $C_G(r_1)/Q_1$ when acting on V. As V is the natural $C_{F^*(H)}(r_1)/Q_1$-module Hypothesis 5.2 (a) holds and Proposition 5.3 implies that $C_{F^*(H)}(r_1)$ is normalized by $C_G(r_1)$. This proves the result. $\square$

LEMMA 13.4. *Suppose that $R_i \cap Z(S_0) \neq 1$ for $i \in \{1, 2\}$. Then, for $S \leq T \leq S_0$, $N_G(T) \leq H$. In particular, $C_G(r_i) = C_H(r_i)$ for $r_i \in R_i^\#$.*

PROOF. We have that $N_G(T)$ normalizes $Z(T) = C_{R_1}(T)C_{R_2}(T)$ and hence normalizes $C_G(Z(T)) \leq C_G(r_1) \cap C_G(r_2)$. Now $C_G(r_1) \cap C_G(r_2)$ normalizes $Q_{12} = Q_1Q_2$ and $O_2(C_{F^*(H)}(R_1R_2)) = Q_{12}$. Therefore

$$O_2(C_G(Z(T))) = Q_{12}$$

and this means that $N_G(T)$ normalizes Q_{12}. Since S_0 normalizes Q_1 and Q_2 and $S_0 \in \mathrm{Syl}_2(N_G(T))$, Lemma D.8(vi) implies that Q_1 and Q_2 and hence R_1 and R_2 are normalized by $N_G(T)$. Since $C_{F^*(H)}(r_i) = C_{F^*(H)}(R_i)$ is normalized by $C_G(R_i)$, we conclude that

$$F^*(H) = \langle C_{F^*(H)}(R_1), C_{F^*(H)}(R_2)\rangle$$

is normalized by $N_G(T)$. Hence $N_G(T) \leq N_G(F^*(H)) = H$, as claimed.

Finally, for $i = 1, 2$, $S \in \mathrm{Syl}_2(C_{F^*(H)}(r_i))$ and $C_G(r_i)$ normalizes $C_{F^*(H)}(r_i)$. Thus the Frattini Argument implies that $C_G(r_i) \leq H$. $\square$

LEMMA 13.5. *Either $G = H$ or $R_i \cap Z(S_0) = 1$ for $i \in \{1, 2\}$.*

PROOF. Suppose that $R_1 \cap Z(S_0) \neq 1$. Then, by Lemma 13.4 $C_G(r_1) = C_H(r_1)$ for all $r_1 \in R_1^\#$ and $C_G(r_2) = C_H(r_2)$ for all $r_2 \in R_2^\#$. By Lemma 13.4, $N_G(S_0) \leq H$ and $N_G(S_0) = N_H(S_0)$ normalizes both R_1 and R_2. Hence r_1, r_2 and r_{12} are in distinct $N_G(S_0)$-conjugacy and therefore also in distinct G-conjugacy classes. By Lemma 13.2 $F^*(H)$ has one further conjugacy class of involutions with representative j. Since r_1 and r_2 are not G-conjugate, j cannot be G-conjugate to both r_1 and r_2. Hence we may, without loss of generality, suppose that $r_1^G \cap F^*(H) = r_1^H = r_1^{F^*(H)}$. If r_1 is G-conjugate to some involution $i \in H \setminus F^*(H)$, then Lemmas A.12 and A.16 (i) and (ii)(c) imply that $O^{2'}(C_{F^*(H)}(i))$ is isomorphic to $\mathrm{F}_4(2^{e/2})$. Since this group is not isomorphic to subgroups of $\mathrm{Sp}_6(2^e)$, we have a contradiction. Thus $r_1^G \cap H = r_1^{F^*(H)}$ and $C_G(r_1) \leq H$. Application of Lemma 2.5 and Lemma 4.4 now yields $G = H$ as claimed. $\square$

From now on we may assume that S_0 contains an element which conjugates R_1 to R_2. We fix an element $r_{12} = r_1r_2 \in Z(S_0)^\#$ where $r_i \in R_i^\#$ for $i = 1, 2$.

LEMMA 13.6. *If $R_i \cap Z(S_0) = 1$ for $i \in \{1,2\}$, then $C_G(r_{12}) \leq H$. Furthermore r_1 and r_2 are not G-conjugate to r_{12}.*

PROOF. Set $I_{12} = O^{2'}(C_H(r_{12}))$ and $L_{12} = I_{12}/Q_{12}$. Then $L_{12} \cong \mathrm{Sp}_4(2^e)$ by Lemma D.8 and, in addition, Lemma D.8 gives the following L_{12}-invariant series of normal subgroups of Q_{12}:

$$1 < R_1R_2 < V_{12} < W_{12} < Q_{12},$$

where $V_{12} = Q_1 \cap Q_2$, V_{12}/R_1R_2 is a direct sum of two L_{12}-modules which are not isomorphic as GF(2)-modules and the same applies for Q_{12}/W_{12}. The subgroup W_{12} is described in Lemma D.8 as

$$W_{12} = Z(Q_1)Z(Q_2),$$

and we have

$$W_{12}' = R_1R_2, \text{ or } q = 2 \text{ and } W_{12}' = \langle r_{12}\rangle.$$

Furthermore, as R_1 and R_2 are conjugate in $C_H(r_{12})$, we see that Q_{12}/W_{12} and V_{12}/R_1R_2 are irreducible $I_{12}S_0/Q_{12}$-modules. We also note that W_{12}/V_{12} is centralized by L_{12} and has order 2^{2e}. Set $J_{12} = O_2(C_G(r_{12}))$. We intend to demonstrate that $J_{12} = Q_{12}$. As $J_{12} \leq S_0$, $C_{F^*(H)}(r_{12})$ normalizes J_{12} and $C_{S_0/Q_{12}}(L_{12}) = 1$, we obtain

$$J_{12} \leq Q_{12}.$$

Therefore, as G is of parabolic characteristic 2, we have that $C_G(r_{12})$ is of characteristic 2 and with the help of Lemma D.8 this implies

$$R_1R_2 = Z(Q_{12}) \leq Z(J_{12}).$$

Assume that $J_{12} \leq W_{12}$. As $V_{12} = Z(W_{12})$ by Lemma D.8 (iii), we know that $J_{12} > V_{12}$. Because $Z(Q_1)V_{12} \cap Z(Q_1)V_{12} = V_{12}$ and Q_1 and Q_2 are conjugate in S_0, we cannot have $J_{12} \leq Z(Q_i)V_{12}$ for $i = 1,2$. Now we exploit that fact that, for $i = 1,2$, $Z(Q_i)/R_i$ are GF(2^e)-modules to obtain $Z(J_{12}) \cap Z(Q_i) \leq V_{12}$ and so $Z(J_{12})/V_{12}$ has order at most 2^e. Suppose that $x \in W_{12}$ then $x = ab$ where $a \in Z(Q_1)$ and $b \in Z(Q_2)$. If x has order 2, then, as $Z(Q_1)$ and $Z(Q_2)$ are abelian $x^2 = abab = [a,b] = 1$ and so $x \in Z(Q_1)$ or $x \in Z(Q_2)$ again as $Z(Q_i)/R_i$ are GF(2^e)-modules. It now follows that $\Omega_1(Z(J_{12})) = V_{12}$. In particular W_{12}-centralizes the chain $J_{12} > V_{12} > 1$ of normal subgroup of $C_G(r_{12})$ and so $W_{12} \leq J_{12}$ by [**22**, Chap. 5, Theorem 3.2]. Hence $J_{12} = W_{12}$. Now assume first $W_{12}' = R_1R_2$. This happens precisely when $e > 1$. Then Q_{12} centralizes the normal series of subgroups $J_{12} > V_{12} > R_1R_2$ and this means that $J_{12} < Q_{12} \leq J_{12}$ which is absurd. Hence $J_{12} \not\leq W_{12}$. As $Q_{12}W_{12}/W_{12}$ is an irreducible $I_{12}S_0/Q_{12}$-module, we have $W_{12}J_{12} = Q_{12}$. Using the fact that Q_{12}/V_{12} is a direct sum of indecomposable L_{12}-modules by Lemma D.8 (v) this yields $Q_{12} =$

$J_{12}V_{12}$ and, as $V_{12} = \Phi(Q_{12})$ by the construction of V_{12} in Lemma D.8, we finally obtain

$$Q_{12} = J_{12},$$

as claimed.

Assume now $e = 1$ and so $H \cong \mathrm{Aut}(\mathrm{F}_4(2))$. Then $W'_{12} = \langle r_{12}\rangle$. Furthermore, for $i \in \{1, 2\}$, $[Q_i, V_{12}] = R_i$, which implies that $[Q_{12}, V_{12}] = R_1R_2$. As $C_{Q_1Q_2}(V_{12}/\langle r_{12}\rangle) = W_{12}$, we get that Q_{12}/W_{12} is the full group of transvections on $V_{12}/\langle r_{12}\rangle$ to $R_1R_2/\langle r_{12}\rangle$. Now choose $g \in C_G(r_{12})$ and assume that $Q^g_{12} \neq Q_{12}$. Then first of all $(R_1R_2)^g \neq R_1R_2$ and Q^g_{12} induces the full transvection group to $(R_1R_2)^g/\langle r_{12}\rangle$. This implies $Q_{12} \cap Q^g_{12} = W_{12}$. Set $X = \langle Q_{12}, Q^g_{12}\rangle$. Then X acts on $\langle r_1, r^g_1, r_{12}\rangle/\langle r_{12}\rangle$ and induces $\mathrm{SL}_2(2)$ on this group. Furthermore $[X, V_{12}] \leq \langle r_1, r_2, r_{12}\rangle$. This shows that $C_X(\langle r_1, r^g_1\rangle)$ stabilizes a chain and so $X/O_2(X) \cong \mathrm{SL}_2(2)$. As $Q_{12} \cap Q^g_{12} = W_{12}$, we now get that $|O_2(X)| \geq |(O_2(X) \cap Q_{12})(O_2(X) \cap Q^g_{12})| \geq 2^7 \cdot 2^7|W_{12}| = 2^{26}$, while $|S_0| = 2^{25}$. This contradiction shows $Q^g_{12} = Q_{12}$ and so again

$$Q_{12} = J_{12}.$$

Now $C_G(r_{12})$ normalizes Q_{12} and hence, using Lemma D.8 (v), $C_G(r_{12})$ permutes $\{Q_1, Q_2\}$. Let $K = N_{C_G(r_{12})}(Q_1)$. Then K is a normal subgroup of index 2 in $C_G(r_{12})$ and acts on $V = X_1 \times X_2$ where $X_i = Q_iW_{12}/W_{12}$ preserving both summands. As before, using the indecomposable property of Q_{12}/V_{12} we obtain $C_K(V) = Q_{12}$.

Let $K_1 = C_{C_G(r_{12})}(X_1)$, then I_{12}/K_1 is Sylow maximal in K/K_1 acting on K_1 and hence by Proposition 5.3 $I_{12}K_1$ is a normal subgroup of $C_G(r_{12})$ or $L'_{12} \cong \mathrm{Alt}(6)$ and $C_G(r_{12})/K_1 \cong \mathrm{Alt}(7)$. Since $I_{12}K_1/K_1 \cong \mathrm{Sp}_4(2)$, this latter possibility does not occur. Hence $I_{12}K_1$ is normal in $C_G(r_{12})$. By considering the action of K_1I_{12} on X_2 and applying Proposition 5.3 again, we find that L_{12} is normal in $I_{12}K_1/Q_1$ Hence L_{12} is normal in $C_G(r_{12})/Q_1$ and $E(C_G(r_{12})/Q_{12}) = L'_{12}$. In particular, by the Frattini Argument

$$C_G(r_{12}) = I_{12}N_{C_G(r_{12})}(S).$$

Since $C_{F^*(H)}(r_1)/Q_1 \cong \mathrm{Sp}_6(2^e)$ by Lemma D.7, and $\mathrm{Sp}_6(2^e)$ is not isomorphic to a subgroup of $\mathrm{Sp}_4(2^e)$, we have r_{12} is not G-conjugate to r_1 (which is H-conjugate to r_2).

Now we consider the normalizer of $Z_2(S)$. By Lemma D.9 we have that $|Z_2(S)| = 2^{4e}$ and that

$$O^{2'}(N_{F^*(H)}(Z_2(S))/O_2(N_{F^*(H)}(Z_2(S))) \cong \mathrm{SL}_2(2^e) \times \mathrm{SL}_2(2^e),$$

where $Z_2(S) = U_1 \oplus U_2$, with $U_i = \langle R_i^{N_{F^*(H)}(Z_2(S))}\rangle$ for $i \in \{1,2\}$. As $r_1 \notin r_{12}^G$, we get that

$$\begin{aligned}\langle O_2(C_G(x)) \mid x \in r_{12}^G \cap Z_2(S)\rangle &= \langle O_2(C_H(x)) \mid x \in r_{12}^G \cap Z_2(S)\rangle \\ &= O^{2'}(N_{F^*(H)}(Z_2(S))).\end{aligned}$$

Using the fact that $N_{C_G(r_{12})}(S)$ normalizes $Z_2(S)$ and I_{12}, we get that $N_{C_G(r_{12})}(S)$ normalizes

$$\langle O^{2'}(I_{12}), O^{2'}(N_{F^*(H)}(Z_2(S)))\rangle = F^*(H).$$

As $H = N_G(F^*(H))$, we have $N_{C_G(r_{12})}(S) \le H$ and so finally we obtain $C_G(r_{12}) = I_{12}N_{C_G(r_{12})}(S) \le H$, the assertion. $\square$

LEMMA 13.7. *If $R_i \cap Z(S_0) = 1$ for $i = 1,2$, then $(r_{12})^G \cap H = (r_{12})^{F^*(H)}$.*

PROOF. By Lemma 13.2, $F^*(H)$ has three H-conjugacy classes of involutions. They are r_1^H, r_{12}^H and j^H. Furthermore $r_{12}^H = r_{12}^{F^*(H)}$. Set $Y = C_{F^*(H)}(j)$, $X = O_2(Y)$. Then $Y/X \cong \mathrm{SL}_2(2^e) \times \mathrm{SL}_2(2^e)$ and as j is not H-conjugate to r_1, or r_{12} and Y has characteristic 2, we have $|Z(X)| \ge 2^{2e+1}$.

Suppose that r_{12} is G-conjugate to $j \in H$. Then $j^g = r_{12}$ and $Y^g \le C_G(r_{12}) = C_H(r_{12})$ by Lemma 13.6. As there is no non-trivial 2-subgroup in $\mathrm{Sp}_4(2^e)$ which is normalized by $O^2(Y^g/X^g) \cong O^2(\mathrm{SL}_2(2^e)\times \mathrm{SL}_2(2^e))$ (see Lemma D.5 and [**27**, Theorem 2.6.7]), we get that X^g is a subgroup of index 2^{2e} in Q_{12} and $O^2(Y^gQ_{12}/Q_{12}) \cong O^2(\mathrm{SL}_2(e) \times \mathrm{SL}_2(2^e))$ normalizes X^g. Notice that $O^2(Y^gQ_{12}/Q_{12})$ either acts irreducibly on the natural $\mathrm{Sp}_4(2^e)$-module or acts as a direct sum of two 2-dimensional submodules. In any case, it does not fix 1-dimensional subspaces. Since $O^2(Y^g)$ normalizes Q_{12}, it also normalizes Q_1 and Q_2 by Lemma D.8 (vi). Now using the action of $O^2(Y^g)$ and the fact that Q_{12}/Q_i is an indecomposable 5-dimensional $\mathrm{GF}(2^e)$-module for $\mathrm{Sp}_4(2^e)$ shows that X^gQ_i has index at most 2^e in Q_{12}. Thus, for $i \in \{1,2\}$,

$$|Q_i : Q_i \cap X^g| \le 2^e.$$

The $C_{F^*(H)}(R_1R_2)$ chief-factors of Y^g on $Q_i/Z(Q_i)$ are both 4-dimensional. Hence $Q_i = (X^g \cap Q_i)Z(Q_i)$ for $i \in \{1,2\}$ and so $Z(X^g)$ centralizes $Q_i/Z(Q_i)$ which means that $Z(X^g) \le Z(Q_i)$. Thus we have shown that

$$Z(X^g) \le Z(Q_1) \cap Z(Q_2) = R_1R_2.$$

Since $|R_1R_2| = 2^{2e}$ and $|Z(X^g)| \ge 2^{2e+1}$, we have a contradiction. Hence

$$r_{12}^G \cap F^*(H) = r_{12}^H = r_{12}^{F^*(H)}$$

by Lemmas 13.2 and 13.6.

Assume now that r_{12} is G-conjugate to some involution $i \in H \setminus F^*(H)$. Then Lemmas A.12 and A.16 (i) and (ii)(c) imply $O^{2'}(C_{F^*(H)}(i))$ is isomorphic to either $\mathrm{F}_4(2^{e/2})$ or ${}^2\mathrm{F}_4(2^e)$ depending on whether or not e is even. Since these groups are not isomorphic to subgroups of $\mathrm{Sp}_4(2^e)$, we have a contradiction. This proves the lemma. □

Proof of Proposition 13.1. If $Z(S_0) \cap R_i = 1$ for $i \in \{1, 2\}$, then Lemmas 13.7, 13.6 and 2.5 provide the hypothesis of Holt's Lemma 4.4 and this implies that $G = H$. Hence using Lemma 13.5, we have $G = H$ and this proves the proposition. □

We collect the results of Sections 10, 11, 12, and 13 in the following proposition which was cited in the introduction.

Proposition 13.8. *Let G be a $\mathcal{K}_2$-group of parabolic characteristic 2. If $H \le G$, $F^*(H) \cong {}^2\mathrm{F}_4(2^{2e+1})'$, $\mathrm{F}_4(2^e)$, $\mathrm{Sp}_{2n}(2^e)$, $n \ge 3$, $\mathrm{Sp}_4(2^e)$, $e > 1$ or $\mathrm{PSL}_3(2^e)$, $e \ne 2$, $H = N_G(F^*(H))$, $|G : H|$ odd, then $G = H$.*

Proof. We have that Hypothesis 6.1 holds. Thus the statements follow from Propositions 10.1, 11.1, 12.1 and 13.1. □

14. The case when p = 2 and centralizer of some 2-central element of H is soluble

For this section we work under the following hypothesis:

Hypothesis 14.1. *Hypothesis 6.1 holds with $p = 2$, $F^*(H)$ is a group of Lie type in characteristic 2 and $C_H(z)$ is soluble for some 2-central involution z in H.*

The main result of this section is

Proposition 14.2. *Suppose that Hypothesis 14.1 holds. Then either $G = H$ or the pair $(F^*(G), F^*(H))$ is one of $(\mathrm{Mat}(11), \mathrm{Sp}_4(2)')$, $(\mathrm{Mat}(23), \mathrm{PSL}_3(4))$, $(\mathrm{Alt}(9), \mathrm{PSL}_4(2))$, $(\mathrm{Alt}(10), \mathrm{PSL}_4(2))$, $(\mathrm{PSL}_4(3), \mathrm{PSU}_4(2))$, $(\mathrm{G}_2(3), \mathrm{G}_2(2)')$ or $(\mathrm{P}\Omega_8^+(3), \Omega_8^+(2))$.*

Suppose that Hypothesis 14.1 holds. Then, by Lemma D.15 we have that $F^*(H) \cong \mathrm{PSL}_3(2^e)$, $\mathrm{Sp}_6(2)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_5(2)$, $\mathrm{G}_2(2)'$, ${}^2\mathrm{F}_4(2)'$, $\mathrm{PSL}_4(2)$, $\mathrm{P}\Omega_8^+(2)$ or $\mathrm{Sp}_4(2^e)'$. Because of Lemma 3.15 and Proposition 13.8, the cases that remain to be studied are those with

$$F^*(H) \cong \mathrm{PSL}_4(2), \mathrm{PSU}_4(2), \mathrm{G}_2(2)' \text{ and } \mathrm{PSU}_5(2).$$

This section investigates these cases.

Recall that by Lemma 2.5 we have

$$O(G) = 1$$

in all cases.

We start with the cases $F^*(H) \cong \mathrm{PSL}_4(2)$ and $F^*(H) \cong \mathrm{PSU}_4(2)$ and prove

Proposition 14.3. *Suppose that $F^*(H) \cong \mathrm{PSL}_4(2)$ or $\mathrm{PSU}_4(2)$. If $G \neq H$, then $F^*(G) \cong \mathrm{Alt}(9)$, $\mathrm{Alt}(10)$ or $\mathrm{PSL}_4(3)$, where in the first two cases $F^*(H) \cong \mathrm{PSL}_4(2)$ and in the third case $F^*(H) \cong \mathrm{PSU}_4(2)$. In all these groups we have that $C_G(z) = C_H(z)$ for z a 2-central involution in S_0.*

Proof. By [**37**, Proposition 2.9.1], $\mathrm{PSL}_4(2) \cong \Omega_6^+(2) \cong \mathrm{Alt}(8)$ and $\mathrm{PSU}_4(2) \cong \Omega_6^-(2)$. Also by Lemma E.9 the Sylow 2-subgroups of $\mathrm{Aut}(\mathrm{PSL}_4(2))$ and $\mathrm{Aut}(\mathrm{PSU}_4(2))$ are isomorphic as are those of $\mathrm{PSL}_4(2)$ and $\mathrm{PSU}_4(2)$. In particular, $Z(S) = Z(S_0) = R$ has order 2. Let $z \in Z(S)^\#$.

If $H = F^*(H)$, then S is isomorphic to a Sylow 2-subgroup of $\mathrm{Alt}(8)$. Furthermore, by Lemmas D.1 and D.16, $O_2(C_H(z))$ is extraspecial of order 32 of $+$-type. Since G has parabolic characteristic 2, since $O_2(C_G(z)) \leq O_2(C_H(z))$, and since $[O_2(C_G(z)), O_2(C_H(z))] \leq \langle z \rangle$, we have $O_2(C_G(z)) = O_2(C_H(z))$. In particular, the quotient $C_G(z)/O_2(C_G(z))$ embeds into $\mathrm{O}_4^+(2)$ by [**79**, Theorem 1] and consequently $C_G(z)$ is soluble. Using Lemma 3.13, we obtain $G \cong \mathrm{Alt}(8)$, $\mathrm{Alt}(9)$ or $\mathrm{PSU}_4(2)$. Thus if $G \neq H$, then $G \cong \mathrm{Alt}(9)$ and $H \cong \mathrm{Alt}(8)$. Finally we note that in this case $C_H(z) = C_G(z)$.

Suppose that $H \neq F^*(H)$. Then S is isomorphic to a Sylow 2-subgroup of $\mathrm{Sym}(8)$ and so is isomorphic to $\mathrm{Dih}(8) \wr 2$. If G possesses a subgroup G_1 of index two, then by Lemma 2.4 G_1 is of parabolic characteristic 2 and $F^*(H) < G_1$ with $N_{G_1}(F^*(H)) = F^*(H)$. Hence G_1 is recognized by the previous case and we are done. So we may assume G is simple and so by Lemma 3.14 we obtain $G \cong \mathrm{Alt}(10)$, $\mathrm{Alt}(11)$ or $\mathrm{PSL}_4(q)$ with $q \equiv 3 \pmod 4$ or $\mathrm{PSU}_4(q)$ with $q \equiv 1 \pmod 4$. We have $G \not\cong \mathrm{Alt}(11)$ as in $\mathrm{Alt}(11)$ the centralizer of $(12)(34)(56)(78)$ is not of characteristic 2. Similarly, in $\mathrm{PSL}_4(q)$ and $\mathrm{PSU}_4(q)$ the centralizer of z contains a normal subgroup isomorphic to $\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$, and this is of characteristic 2 if and only if $q = 3$. Thus $G \cong \mathrm{PSL}_4(3)$ in this case. As the order of $\mathrm{PSL}_4(3)$ is not divisible by 7, we get $F^*(H) \cong \mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$ with $H \cong \mathrm{Aut}(\mathrm{PSU}_4(2))$ in this case. Finally we observe that $C_H(z) = C_G(z)$ to conclude the proof. $\square$

Next we consider $F^*(H) \cong \mathrm{G}_2(2)'$.

Proposition 14.4. *If $F^*(H) \cong \mathrm{G}_2(2)'$, then either $G = H$ or $G \cong \mathrm{G}_2(3)$.*

PROOF. Again let $z \in Z(S)^\#$. By Lemma D.12 (i) we have

$$O_2(C_{F^*(H)}(z)) \cong 4 \circ \mathrm{Q}_8.$$

Assume $H = F^*(H)$. Then, as $C_G(O_2(C_G(z))) \leq O_2(C_G(z))$ and G has parabolic characteristic 2, we have $O_2(C_H(z)) = O_2(C_G(z))$. In particular, $C_G(z) = C_H(z)$. Since H has exactly one conjugacy class of involutions, Lemma 4.4 yields $H = G$.

So we may assume that $H \cong \mathrm{G}_2(2)$. If G has a subgroup G_1 of index 2, then, as $\Omega_1(Z(S)) = \Omega_1(Z(S \cap G_1))$, G_1 has parabolic characteristic 2 by Lemma 2.4 and we obtain $G = H$.

So we may assume that G has no subgroup of index 2. Recall $z \in Z(S_0^\#)$. By the Thompson Transfer Lemma [**26**, Lemma 15.16], as $F^*(H)$ has exactly one conjugacy class of involutions, so does G. Furthermore by Lemma D.12 (iv) we have that $O_2(C_H(z))$ is extraspecial of order 32 and $+$-type. Choose t an involution in $H \setminus F^*(H)$. By Lemma D.12 (iv) we have $C_H(t) \cong 2 \times \mathrm{Sym}(4)$. Because t and z are G-conjugate, a Sylow 2-subgroup T of $C_H(t)$ is not a Sylow 2-subgroup of $C_G(t)$. Let $T_1 \leq C_G(t)$ with $|T_1 : T| = 2$. We may assume that $\langle z \rangle = T'$, so $T_1 \leq C_G(z)$. In particular $C_{C_G(z)}(t) > C_H(t)$ and so $C_G(z) > C_H(z)$. Since $O_2(C_G(t))$ is extraspecial of $+$-type and $C_G(z)$ is of characteristic 2, this means that $|C_G(z) : C_H(z)| = 3$. Hence

(14.4.1) there is a subgroup X of index 2 in $C_G(z)$ such that $X \cong \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)$.

We now consider the parabolic subgroup P of H containing S with $P \neq C_H(z)$. By Lemma D.12 (iii) we know that P has shape $((4 \times 4):2).\mathrm{Sym}(3)$, where the homocyclic subgroup of shape 4×4 is inverted in $O_2(P)$. Let $U = \langle z^P \rangle$. Then as $z \in Z(O_2(P))$, we have $U = Z(O_2(P))$ which is elementary abelian of order 4. Consequently $[U, O_2(C_H(z))] \leq \langle z \rangle$ and so obtain $U \leq O_2(C_H(z))$. Let $x \in P \setminus C_H(z)$ and consider the subgroup $E = O_2(C_H(z)) \cap O_2(C_H(z))^x$. We have $\Phi(E) \leq \langle z \rangle \cap \langle z^x \rangle = 1$ and so E is elementary abelian and contains U. Moreover, as

$$|O_2(C_G(z)) : O_2(C_G(z)) \cap O_2(P)| = 2$$

and

$$|(O_2(C_G(z)) \cap O_2(P))O_2(C_G(z))^x / O_2(C_G(z))^x| \leq 2,$$

we calculate that $|E|$ has order 8. Using that P has two non-central chief factors in $O_2(P)$, yields

$$P/E \cong \mathrm{Sym}(4).$$

As G has just one conjugacy class of involutions, all the involutions in E are G-conjugate. Let $t \in E \setminus F^*(H)$. Then t^P has order 4 and

$C_P(t)E/E \cong \mathrm{Sym}(3)$. Therefore $E = \langle t\rangle[E, C_P(t)]$ with $[E, C_P(t)] = U$. As $t \in C_G(t)'$, it follows that $E \leq C_G(t)'$. Now $UO_2(C_G(t))$ is normalized by $C_P(t)$ and so, as $C_G(t)/O_2(C_G(t))$ has Sylow 2-subgroups of order 2, we have $E = \langle t\rangle U \leq O_2(C_G(t))$. Since $Z(O_2(C_G(t))) = \langle t\rangle \not\leq U$, we get that $O_2(C_G(t)) \not\leq P$. We also know that $C_G(E) = C_{C_G(z)}(E) = E$ and so $N_G(E)/C_G(E)$ is isomorphic to a subgroup of $\mathrm{SL}_3(2)$. Since $P/E \cong \mathrm{Sym}(4)$ is a maximal subgroup of $\mathrm{SL}_3(2)$ and $O_2(C_G(t)) \not\leq P$, we now have

(14.4.2) $N_G(E)/E \cong \mathrm{SL}_3(2)$.

Finally (14.4.1) and (14.4.2) provide the hypotheses of Lemma 3.12. Thus $G \cong \mathrm{G}_2(3)$. □

We finally will consider $F^*(H) \cong \mathrm{PSU}_5(2)$.

LEMMA 14.5. *If $F^*(H) \cong \mathrm{PSU}_5(2)$ and $z \in Z(S_0)^{\#}$, then $C_G(z) \leq H$.*

PROOF. We have that $Q = O_2(C_{F^*(H)}(z))$ is extraspecial of order 2^7 with outer automorphism group $\mathrm{O}_6^-(2)$ by [**79**, Theorem 1]. Assume $N_G(Q) \neq N_H(Q)$. As, by Lemma E.5, $N_H(Q)/Q \cong \mathrm{GU}_3(2)$ is a maximal subgroup of $\mathrm{P\Omega}_6^-(2)$, we have $N_G(Q)/Q$ contains a subgroup isomorphic to $\mathrm{P\Omega}_6^-(2)$.This is ridiculous as $|S_0 : Q| = |N_H(Q)/Q|_2 \leq 2^4$ and therefore

$$N_G(Q) = N_H(Q).$$

As $N_H(Q)$ acts irreducibly on $Q/\langle z\rangle$, and G is of parabolic characteristic 2, we have that $Q = O_2(C_G(z)))$, which implies $C_G(z) = C_H(z)$ and the lemma is true. □

PROPOSITION 14.6. *Suppose that $F^*(H) \cong \mathrm{PSU}_5(2)$. Then $G = H$.*

PROOF. Let $z \in Z(S_0)$. Then $C_G(z) = C_H(z)$ by Lemma 14.5. We will show that $z^G \cap H = z^H$. By Lemma E.1 $\mathrm{Aut}(\mathrm{PSU}_5(2))$ has exactly three conjugacy classes of involutions. If $i \in H \setminus F^*(H)$ is an involution then again by Lemma E.1 we get that 5 divides $|C_H(i)|$ and as $C_G(i)$ is a $\{2, 3\}$-group we conclude that i and z are not G-conjugate.

Assume now $z^G \cap H \neq z^H = z^{F^*(H)}$. Then we have that all the involutions in $F^*(H)$ are G-conjugate. If $H \neq F^*(H)$ then the Thompson Transfer Lemma [**26**, Lemma 15.16] implies G has a normal subgroup G_1 of index 2 and Lemma 2.4 yields that G_1 has characteristic 2. Hence we may assume that $F^*(H) = H$. Set $Q = O_2(C_H(z))$ and let $t \in Q \setminus \langle z\rangle$ be an involution. Let S_1 be a Sylow 2-subgroup of $C_{C_H(z)}(t)$ containing $C_S(t)$. Then we have that $Z(S_1) = \langle z, t\rangle$ as $Z(C_S(t)) \leq C_Q(t) = \langle z, t\rangle$. Now there is $S_2 \leq C_G(t)$ with $|S_2 : S_1| = 2$. This shows that $\langle Q, S_2\rangle$

induces Sym(3) on $\langle z, t\rangle$. In particular $\langle z, t\rangle \le Q^g$ for $g \in G$. Hence $|Q^g : C_{Q^g}(z)| = 2$. We consider $QC_{Q^g}(z)$. As $t \in Q$, we have that $QC_{Q^g}(z)/Q$ is elementary abelian. As S/Q is quaternion of order 8, we get $|QC_{Q^g}(z) : Q| \le 2$ and so $|Q \cap Q^g| \ge 2^5$. But then $Q \cap Q^g$ is not abelian and so, as $Q' = \langle z\rangle \ne \langle t\rangle = (Q^g)'$, we have a contradiction. This proves

$$z^G \cap H = z^H = z^{F^*(H)}.$$

Now as $C_G(z) \le H$ and $O(G) = 1$, Lemma 4.4 implies that $G = H$. □

Proof of Proposition 14.2. The candidates for $F^*(H)$ are given by Lemma D.15. With this information, the proposition follows by combining the statements from Lemma 3.15, Propositions 12.1, 11.1 and 10.1 combined with Propositions 14.3, 14.4 and 14.6. □

15. The groups with $\mathbf{F^*(H) \cong G_2(3^e)}$

In this section we assume Hypothesis 6.1 (i), (ii), (iii) hold with $F^*(H) \cong \mathrm{G}_2(3^e)$ and $e \ge 1$. As usual, $S_0 \in \mathrm{Syl}_3(H) \subseteq \mathrm{Syl}_3(G)$ and $S = S_0 \cap F^*(H)$. We have $Z(S) = R_1R_2$ where R_1 and R_2 are root subgroups of $F^*(H)$ which are not $F^*(H)$-conjugate by Lemma A.3. The structure of the parabolic subgroups of $F^*(H)$ is described in Lemma D.11. Thus the maximal parabolic subgroups in $F^*(H)$ containing S are $H_i = N_H(R_i)$, $i = 1, 2$. Set

$$Q_i = O_3(H_i)$$

and recall that

$$O^{3'}(H_i/O_3(H_i)) \cong \mathrm{SL}_2(3^e).$$

Our objective in this section is to prove:

Proposition 15.1. *Suppose that Hypothesis* 6.1 *(i), (ii) and (iii) hold with $F^*(H) \cong \mathrm{G}_2(3^e)$ and $e \ge 1$. Then $N_G(E) \le H$ for any non-trivial normal subgroup E of S_0. If, furthermore, G is of local characteristic* 3*, then $G = H$ or H is strongly* 3*-embedded in G.*

Our first result investigates the normalizer of S and the centralizers of root elements.

Lemma 15.2. *For $i = 1, 2$ the following hold:*

(i) $N_G(Q_i) = N_H(Q_i)$;
(ii) $N_G(S) = N_H(S)$;
(iii) $N_G(R_i) = N_H(R_i)$; *and*
(iv) *for $r \in R_i^{\#}$, $C_G(r) = C_H(r)$.*

Proof. By symmetry it is enough to prove the lemma for $i = 1$. Let $M = N_G(Q_1)$. Obviously, M/Q_1 acts on $Q_1/Z(Q_1)$ which is a natural $O^{3'}(H_1/Q_1)$-module by Lemma D.11. Suppose $C = C_M(Q_1/Z(Q_1)) \neq Q_1$. Then C/Q_1 has order coprime to 3 and is normalized by H_1/Q_1. As G is of parabolic characteristic 3, C does not centralize Q_1/R_1. Therefore, as $\Phi(Q_1) = R_1$ by Lemma D.11(iii),

$$Q_1/R_1 = [Q_1/R_1, C] \times C_{Q_1/R_1}(C)$$

is a non-trivial decomposition of Q_1/R_1 which is H_1 invariant. This contradicts Lemma D.11 (v). Hence $C = Q_1$ and M/Q_1 acts faithfully on $Q_1/Z(Q_1)$. It follows from Lemma 2.27 that $O^{3'}(H_1)$ is normal in $N_G(Q_1)$. In the same way we see that $O^{3'}(H_2)$ is normal in $N_G(Q_2)$.

We have $N_G(Q_1) = O^{3'}(H_1)N_{N_G(Q_1)}(S)$, and $N_{N_G(Q_1)}(S)$ permutes the set $\{Q_1, Q_2\}$ by Lemma D.11 (ix). Hence $N_{N_G(Q_1)}(S)$ normalizes Q_2 and therefore $N_{N_G(Q_1)}(S)$ normalizes $\langle O^{3'}(H_1), O^{3'}(H_2)\rangle = F^*(H)$. Since $H = N_G(F^*(H))$, we have

$$N_G(Q_1) = O^{3'}(H_1)N_{N_G(Q_1)}(S) \leq H.$$

This proves (i).

Since $N_G(S)$ permutes $\{Q_1, Q_2\}$, part (i) yields

$$\langle N_G(Q_1), N_G(Q_2)\rangle = \langle N_H(Q_1), N_H(Q_2)\rangle \leq H$$

is normalized by $N_G(S)$. Since $F^*(\langle N_H(Q_1), N_H(Q_2)\rangle) = F^*(H)$, we now have $N_G(S) \leq N_G(F^*(H)) \leq H$. Thus (ii) holds.

Now consider $N_G(Y)$ for Y a non-trivial subgroup of R_1. First of all we have that $O_3(N_G(Y)) \leq Q_1$ as $Y \leq Z(N_{F^*(H)}(R_1))$. Since G is of parabolic characteristic 3, we obtain $Z(Q_1) < O_3(N_G(Y))$. Using the fact that H_1 acts irreducibly on $Q_1/Z(Q_1)$ by Lemma D.11 (iv) and (v), we conclude that $O_3(N_G(Y)) = Q_1$. Now (iii) and (iv) follows from (i). □

Lemma 15.3. *For $i = 1, 2$ and $r_i \in R_i^{\#}$, we have $r_i^G \cap H \subseteq F^*(H)$.*

Proof. Suppose that $r_1^g \in H \setminus F^*(H)$ for some element $g \in G$ and $r_1 \in R_1^{\#}$. Then r_1^g acts as a field automorphism on $F^*(H)$ by Theorem A.11 and Lemmas A.12 and A.15 all elements of order three in the coset $F^*(H)r_1^g$ are $F^*(H)$-conjugate. In particular $C_{F^*(H)}(r_1^g) \cong \mathrm{G}_2(3^{e/3})$. By Lemma 15.2, $C_G(r_1)/O_2(C_G(r_1)) = C_H(r_1)/Q_1$ and $O^{3'}(C_H(r_1)/Q_1) \cong \mathrm{SL}_2(3^e)$ by Lemma D.11 (ii). But then $C_{F^*(H)}(r_1^g)$ does not embed in $C_G(r_1)$ (by [**33**, Satz 8.27] for example). This proves the lemma. □

Lemma 15.4. *We have $N_G(S_0) \leq N_G(S) = N_H(S)$.*

Proof. Since S is generated by root elements, for $g \in N_G(S_0)$, $S^g \le F^*(H)$ by Lemma 15.3. Hence $g \in N_G(S)$ and $N_G(S_0) \le N_G(S)$. $\square$

Lemma 15.5. *For $i = 1, 2$ let $r_i \in R_i^{\#}$. Then $r_i^G \cap H = r_i^H$.*

Proof. By Lemma 15.4 we have that H controls fusion in $Z(S_0)^{\#}$. This means that $r_1^G \cap Z(S_0)^H = r_1^H$ for all $r_1 \in Z(S_0) \cap R_1$.

By Lemma 15.3 we have $r_1^G \cap H \subset F^*(H)$, so suppose that $r_1^g \in S \setminus r_1^H$ for some $g \in G$ and $r_1 \in R_1^{\#}$. Then, by Lemma D.11 (vi), (viii) and (ix), we may suppose that $r_1^g \in Q_1 \cap Q_2$ and that

$$C_S(r_1^g) = Q_1 \cap Q_2 \in \mathrm{Syl}_3(C_{F^*(H)}(r_1^g)).$$

We have $Q_1 \cap Q_2 \le C_G(r_1^g) \le H^g$ by Lemma 15.2. Since $Q_1 \cap Q_2$ is generated by root elements, Lemmas D.11(iv) and 15.3 imply that $Q_1 \cap Q_2 \le F^*(H^g)$. Let $T \in \mathrm{Syl}_3(C_{F^*(H^g)}(r_1^g))$ with $Q_1 \cap Q_2 \le T$ and let T_a and T_b be the conjugates of Q_1 and Q_2 in T. By Lemma D.11 (ix) we may suppose that $Q_1 \cap Q_2 \le T_a$. Since $Q_1 \cap Q_2$ is elementary abelian of order 3^{4e}, $Z(T_a) \le Q_1 \cap Q_2$. Now $[Z(Q_1), O^{3'}(H_1)] \le Q_1 \cap Q_2$ has order 3^{2e} and $Z(T)$ has order 3^{3e}. Hence $[Z(Q_1), O^{3'}(H_1)] \cap Z(T_a) \ne 1$. Let $x \in ([Z(Q_1), O^{3'}(H_1)] \cap Z(T_a))^{\#}$. Then x is H-conjugate to an element of R_2 by Lemma D.11 (iv). Therefore $T_a \le C_G(x) \le H$ by Lemma 15.2. Since T_a is generated by conjugates of R_1 and R_2, we have $T_a \le F^*(H)$ by Lemma 15.3. Hence T_a is H-conjugate to Q_1 or Q_2, and so $r^g \in Z(T) \le Z(T_a)$ which is conjugate to either $Z(Q_1)$ or $Z(Q_2)$. But then r^g is H-conjugate to an element of $Z(S)$ by Lemma D.11(vi), a contradiction. $\square$

Proof of Proposition 15.1. By Lemmas 15.2(iv) and 15.5 the assumptions of Lemma 4.2 are satisfied. Application of Lemma 4.2 now implies the statements of Proposition 15.1. $\square$

16. The groups with $\mathbf{F^*(H) \cong P\Omega_8^+(3)}$ and $\mathbf{N_G(Q) \not\le H}$

In this section we address a special case of Theorem 3. We will prove the following proposition.

Proposition 16.1. *Assume Hypothesis* 6.1 *holds with $F^*(H) \cong \mathrm{P}\Omega_8^+(3)$. If $N_G(Q) \not\le H$, then $F^*(G) \cong \mathrm{F}_2$ or $\mathrm{M}(23)$.*

We fix the hypothesis of Proposition 16.1 throughout this section. The proof of Proposition 16.1 is more intricate than that of the other groups with $C_H(z)$ soluble for some $z \in Z(S)$ because of the exceptional structure of $\mathrm{Out}(F^*(H))$.

We continue with the notation introduced in Section 7. In particular, this means that $S_0 \in \mathrm{Syl}_3(H)$, $S = S_0 \cap F^*(H)$, $R = Z(S_0) = Z(S)$, $Q = O_3(C_{F^*(H)}(R))$ and $L = O^{3'}(N_{F^*(H)}(Q))$. We have that R has order 3 and

$$N_{F^*(H)}(R)/Q \sim (\mathrm{SL}_2(3) \circ \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)) : 2 \sim 2_-^{1+6}.3^3.2,$$

as can be seen from Lemma D.26. Furthermore Q is extraspecial of order 3^9 and exponent 3. By [**79**, Theorem 1] $N_G(Q)/Q$ is isomorphic to a subgroup of $\mathrm{GSp}_8(3)$. We also recall that by Lemma D.26 $H/F^*(H)$ embeds into $\mathrm{Out}(H) \cong \mathrm{Sym}(4)$. The action of $N_{F^*(H)}(R)$ on Q/R is as a tensor product of the natural $\mathrm{SL}_2(3)$-module with the four-dimensional orthogonal module for $\Omega_4^+(3)$ (see Lemma D.1). In particular, $N_{F^*(H)}(R)$ acts irreducibly on Q/R which has order 3^8. Furthermore by Proposition 7.1 we have $Q = O_3(N_G(R))$. In particular, as $R = Z(S)$, $N_G(S) \le N_G(Q)$.

In [**59**, Section 3], we introduced a subgroup Y of $\mathrm{GSp}_8(3)$, which is isomorphic to

$$(\mathrm{GL}_2(3) \circ \mathrm{GL}_2(3) \circ \mathrm{GL}_2(3)).\mathrm{Sym}(3).$$

There we described the action of Y on the natural $\mathrm{GSp}_8(3)$-module and showed that any subgroup of $\mathrm{GSp}_8(3)$ which is isomorphic Y is conjugate to Y in $\mathrm{GSp}_8(3)$. We may consider Y as a subgroup of $\mathrm{Out}(Q) \cong \mathrm{GSp}_8(3)$.

We summarize the above discussion in the following lemma.

LEMMA 16.2. *Suppose that $F^*(H) \cong \mathrm{P}\Omega_8^+(3)$. Then*

(i) *$Q = O_3(N_G(R))$ is extraspecial of order 3^9 of exponent 3 and $N_G(Q)/Q$ is isomorphic to a subgroup of $\mathrm{GSp}_8(3)$.*
(ii) *S/Q is elementary abelian of order 3^3 and $|S_0/S| \le 3$.*
(iii) *$N_{F^*(H)}(S)/S$ is elementary abelian of order 4.*
(iv) *$N_H(Q)/Q$ is isomorphic to a subgroup of Y containing*

$$C_{F^*(H)}(R)/Q \cong \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3).$$

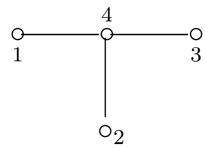
(v) *$Z(N_G(Q)/Q) = Z(N_H(Q)/Q)$ inverts Q/R.*

We shall need the following specific fact about the normalizer of an extraspecial 2-subgroup of $\mathrm{Sp}_8(3)$.

LEMMA 16.3. *Suppose that $K \cong \mathrm{Sp}_8(3)$ and W is an extraspecial subgroup of K of order 2^7. Then $N_K(W)/W \cong \Omega_6^-(2) \cong \mathrm{PSU}_4(2)$ and, in particular, $N_K(W)$ contains no elements which act as transvections on $W/Z(W)$.*

PROOF. This follows from [**37**, Proposition 4.6.9]. $\square$

Let P_1, P_2, P_3 and P_4 be the minimal parabolic subgroups of $F^*(H)$ containing $N_{F^*(H)}(S)$ with P_4 corresponding to the middle node.



Note that $N_{F^*(H)}(Q) = P_1P_2P_3$. For $1 \leq i \leq 3$ set

$$K_i = \langle P_4, P_j \mid j \neq i\rangle \text{ and } E_i = O_3(K_i).$$

Then, for $1 \leq i \leq 3$, $O^{3'}(K_i/E_i) \cong \Omega_6^+(3)$ with E_i elementary abelian of order 3^6 (we may suppose that K_1 normalizes a maximal singular subspace of the natural module for $F^*(H)$. Then use that K_1, K_2 and K_3 are conjugate by a diagram automorphism). Notice also that $E_i \not\leq Q$ as Q is extraspecial of order 3^8 and so has 3-rank 5. Put

$$\overline{C_G(R)} = C_G(R)/Q.$$

and set

$$X = O_{3,2}(N_H(Q)).$$

We have

$$\overline{X} \cong 2_-^{1+6} \text{ and } C_{F^*(H)}(R) = XS.$$

LEMMA 16.4. *For $1 \leq i \leq 3$,*

(i) *E_iQ is normalized by P_j, $1 \leq j \leq 3$, $i \neq j$;*
(ii) *$|\overline{E}_i| = 3$;*
(iii) *$S = E_1E_2E_3Q$; and*
(iv) *$E_i = C_{QE_i}(E_i \cap Q)$.*
(v) *if $\tau \in S \setminus Q$ is such that $C_Q(\tau)$ is elementary abelian of order 3^5, then there exists $1 \leq i \leq 3$ such that $\tau \in E_i$.*
(vi) *$N_G(S)$ permutes $\{E_1, E_2, E_3\}$.*

PROOF. As there is no edge in the diagram between i and j for $1 \leq i, j \leq 3$, we have $P_iP_j = \langle P_i, P_j\rangle$. Recall $K_i = N_{F^*(H)}(E_i)$ for $i \leq 3$ is the ith parabolic and $K_i = \langle P_j \mid j \neq i\rangle$. In particular $E_i \trianglelefteq P_j$ for $j \neq i$ and (i) holds.

It suffices to consider $i = 1$ as a triality automorphism of $F^*(H)$ can be chosen to H permute the set $\{E_1, E_2, E_3\}$. As $N_{F^*(H)}(E_iQ) = P_jP_k$ we have $N_{F^*(H)}(E_2E_3Q) = P_1$ and $E_2Q \neq E_3Q$, so as $|S : Q| = 3^3$ we have $|S/E_2E_3Q| = 3$. In particular, $|\overline{E_2}| = |\overline{E_3}| = 3$ and so also $|\overline{E_1}| = 3$. This proves (ii) and shows $S = E_1E_2E_3$, so that (iii) holds.

By (ii), $|E_1 \cap Q| = 3^5$ and so $E_1 \cap Q$ is a maximal elementary abelian subgroup of Q. Let $C = C_{E_1Q}(E_1 \cap Q)$. Then, as Q is extraspecial and by maximality of $E \cap Q$, $C \cap Q = E_1 \cap Q$ and, as E_1 is abelian, $E_1 \leq C$. Since $E_1Q/Q = CQ/Q$, it follows that $C = E_1$.

To see part (v) we note that $\overline{S}$ can be identified with the subgroup $D = \langle d_1, d_2, d_3 \rangle$ described in [**59**, Section 3]. Having done this, we use [**59**, Lemma 3.1] to see that $\overline{E_i}$ corresponds to d_1 and from as there are only 3 conjugates of $\langle d_1 \rangle$ in D, we obtain the result.

Finally, as $N_G(S)$ normalizes Q, $N_G(S)$ acts on the set

$$\{\tau \in S \mid C_Q(\tau) \text{ is elementary abelian of order } 3^5\}.$$

Therefore part (v) implies that $N_G(S)$ permutes $\{E_1, E_2, E_3\}$.

$\square$

Lemma 16.5. *Suppose that $\bar{i}$ is a non-central involution in $\overline{X}$. Then $\langle \bar{i}^{\overline{S}} \rangle = C_{\overline{X}}(\overline{E_j})$ for some $1 \leq j \leq 3$ and $C_{\overline{C_G(R)}}(\langle \bar{i}^{\overline{S}} \rangle)$ is isomorphic to a subgroup of* $\mathrm{GL}_2(3)$.

Proof. For $1 \leq j \leq 3$, set $X_j = [X, E_j]Q$. By Lemma 16.4 $\overline{P}_k\overline{P}_\ell \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$ centralizes $\overline{E}_j$, so $X = X_1X_2X_3$ and $\overline{X_j} \cong \mathrm{Q}_8$. Furthermore, $\overline{E_jX_j} \cong \mathrm{SL}_2(3)$. If $\bar{i}$ is a non-central involution in $\overline{X}$, then there is a 2-set $\{j,k\} \subseteq \{1,2,3\}$ such that $\bar{i} \in \overline{X}_j\overline{X}_k$ and there are elements $\bar{a} \in \overline{X_j}$ and $\bar{b} \in \overline{X_k}$ such that $\bar{a}\bar{b} = \bar{i}$. Now we see that $\bar{i}$ is centralized by $\overline{X}_\ell$ where $\ell \notin \{j,k\}$ and $\bar{i}^{\overline{S}}$ has size 9 and generates $\overline{X}_j\overline{X}_k$. This proves the first part of the claim. Next we note that (using Lemma 16.4(iv)) X_jX_k acts irreducibly on $(E_\ell \cap Q)/R$ and so as a $\overline{X_jX_k}$-module Q/R is a direct sum of two isomorphic absolutely irreducible modules of dimension 4. By [**22**, Chap. 3, Theorem 5.4 (iii)], $C_{\mathrm{GL}_8(3)}(\overline{X_jX_k}) \cong \mathrm{GL}_2(3)$. Hence $C_{\overline{C_G(R)}}(\overline{X_jX_k})$ is isomorphic to a subgroup of $\mathrm{GL}_2(3)$. $\square$

Lemma 16.6. *Suppose that $Q < T < S$ with $|\overline{T}| = 9$ and $\overline{E}_i \cap \overline{T} = 1$ for all $1 \leq i \leq 3$. Then $C_{\overline{C_H(R)}}(\overline{T}) \leq \overline{XS}$.*

Proof. Suppose that $C_{\overline{C_H(R)}}(\overline{T}) \not\leq \overline{SX}$. Since either $S_0 = S$ or $\overline{S_0}$ is non-abelian with $|Z(\overline{S_0}| = 3$, there exists $w \in C_{\overline{C_H(R)}}(\overline{T})$ such that $w \notin X$ and $w^2 \in X$. Using Lemma 16.3 and setting $\widetilde{X} = X/X'$, we have that $|\widetilde{X}/C_{\widetilde{X}}(w)| = |[\widetilde{X}, w]| = 4$ and that $\overline{T}$ acts in exactly the same way on $\widetilde{X}/C_{\widetilde{X}}(w)$ and $[\widetilde{X}, w]$. Hence $\overline{T_1} = C_{\overline{T}}([\widetilde{X}, w])$ has order 3 and so $C_{\widetilde{X}}(\overline{T_1})$ has order 16. But then $\overline{T_1}$ centralizes an involution in $\overline{X}$ and so $T_1 = E_iQ$ for some i by Lemma 16.5, a contradiction. $\square$

Lemma 16.7. *We have for $1 \leq i \leq 3$ $N_{F^*(H)}(E_i) = \langle Q^{N_G(E_i)} \rangle$ is normal in $N_G(E_i)$ and $N_G(E_i) = N_{N_G(E_i)}(S)N_{F^*(H)}(E_i)$.*

Proof. As $C_S(E_i) = E_i$, we have $R \leq E_i$. As $E_i = O_3(N_G(E_i))$, we get with Lemma 2.1 (ii) that $C_G(E_i) = E_i$.

Let $e \in E_i$ correspond to a non-singular point in E_i and assume that e is conjugate to $r \in R^\#$ in $N_G(E_i)$. Then $C_{N_{F^*(H)}(E_i)}(e)/E_i$ has a normal subgroup isomorphic to $\Omega_5(3) \cong \mathrm{PSp}_4(3)$. As $|\overline{S_0}| \leq 3^4$, we see that $E_i \leq O_3(C_G(e))$. But Q does not contain an elementary abelian group of order 3^6. So we have that $N_{F^*(H)}(E_i)$ controls fusion of the $N_G(E_i)$-conjugates of r in E_i and this yields $N_{F^*(H)}(E_i) = \langle Q^{N_G(E_i)}\rangle$. In particular $N_{F^*(H)}(E_i)$ is normal in $N_G(E_i)$ and $N_G(E_i) = N_{F^*(H)}(E_i)N_{N_G(E_i)}(S)$ as claimed. $\square$

Lemma 16.8. *Suppose that $F^*(H) \cong \mathrm{P}\Omega_8^+(3)$. Then $N_G(S) = N_H(S)$. In particular, for $1 \leq i \leq 3$, $N_G(E_i) \leq H$.*

Proof. Since $N_G(S)$ normalizes Q and permutes $\{E_1, E_2, E_3\}$ by Lemma 16.4 (vi), we have $N_G(S)$ normalizes

$$\langle N_{F^*(H)}(E_i) \mid i = 1, 2, 3\rangle = F^*(H)$$

by Lemma 16.7. As $H = N_G(F^*(H))$ it follows that $N_G(S) \leq H$. Now, for $1 \leq i \leq 3$, $N_G(E_i) \leq H$ by Lemma 16.7. $\square$

Lemma 16.9. *We have $N_G(Z_2(S)) = P_4N_G(S) \leq H$.*

Proof. From [**59**, Lemma 3.1 (v)] we have that $Z_2(S)$ has order 9. We consider P_4. Since $Z(O_3(P_4)) \neq R$, $Q \not\leq O_3(P_4)$. Let $h \in P_4$ with $Q^h \neq Q$. Then $\langle Q, Q^h\rangle$ covers $O^{3'}(P_4/O_3(P_4)) \cong \mathrm{SL}_2(3)$. As $Q \cap Q^h$ is elementary abelian, we have that $|Q \cap Q^h| \leq 3^5$. As $|Q \cap O_3(P_4)| = 3^8$, we now see that $|(Q \cap O_3(P_4))(Q \cap O_3(P_4))^h| \geq 3^{11}$. As $|S| = 3^{12}$ we have that $O^{3'}(P_4) = \langle Q, Q^h\rangle$. Furthermore $Z(O_3(P_4)) \leq Q \cap Q^h$ is equal to RR^h, which is $Z_2(S)$. Assume that $g \in G$ and $R^g \leq Z_2(S)$. Then $R^g = R^h$ for some $h \in P_4$ and therefore

$$O^{3'}(P_4) = \langle Q^g \mid g \in G, R^g \leq Z_2(S)\rangle$$

which means that $O^{3'}(P_4)$ is normal in $N_G(Z_2(S))$. Hence $N_G(Z_2(S)) = P_4N_G(S)$. Finally Lemma 16.8 yields then $N_G(Z_2(S)) \leq H$. $\square$

Our objective over the next few lemmas is to show that $N_G(Q) = N_G(X)$.

Lemma 16.10. *Either*

(i) $N_{C_G(R)}(X)/X \cong \mathrm{PSU}_4(2)$ *or* $3_+^{1+2}.\mathrm{SL}_2(3)$*; or*
(ii) $N_{C_G(R)}(X) = C_H(R)$.

PROOF. By Lemma 16.3, we have that $N_{C_G(R)}(X)/X$ is isomorphic to a subgroup of $\mathrm{PSU}_4(2)$ which has order divisible by 3^3. If $N_{C_G(R)}X/X$ normalizes SX/X, then

$$N_{C_G(R)}(X) \le N_G(S)X \le H$$

by Lemma 16.9. This is (ii). Now Lemma E.5 delivers the assertion. $\square$

LEMMA 16.11. $|C_G(R) : C_H(R)| \in \{1, 4, 7, 10, 13, 16, 25, 28, 40\}$.

PROOF. By Lemma 16.8 $C_H(R) \ge N_{C_G(R)}(S)$ and so

$$a = |C_G(R) : C_H(R)| \equiv 1 \pmod 3.$$

Since $N_G(Z_2(S)) = P_4N_H(S)$ by Lemma 16.9, we have $N_{C_G(R)}(Z_2(S)) = P_4N_H(S) \cap C_G(R) = N_{C_H(R)}(S)$. Since $N_X(S) = X'$ and $N_H(X) = XN_H(S)$, we have

$$|C_H(R) : N_{C_H(R)}(Z_2(S))| = 64.$$

Thus the number of $C_G(R)$-conjugates of $Z_2(S)$ in Q is $64a$. Since there are exactly $(3^8-1)/2$ cyclic subgroups in Q/R we have $64a \le (3^8-1)/2$. Hence $a \le 51$. Using this, $a \equiv 1 \pmod 3$ and a divides $|\mathrm{Sp}_8(3)|$ yields the result. $\square$

LEMMA 16.12. *Suppose* $N_{C_G(R)}(X)/X \cong \mathrm{PSU}_4(2)$. *Then* $C_G(R) = N_{C_G(R)}(X)$.

PROOF. From the structure of $\mathrm{PSU}_4(2)$, we have $|N_{C_G(R)}(X) : C_H(R)| = 40$ and so the result follows from Lemma 16.11. $\square$

LEMMA 16.13. *The subgroup* X *is weakly closed in* $N_{C_G(R)}(X)$.

PROOF. Suppose that there exists $g \in C_G(R)$ with $X^g \le N_{C_G(R)}(X)$ and $X^g \ne X$. Since $C_G(R) \ne N_{C_G(R)}(X)$, we have by Lemma 16.10 and Lemma 16.12 that $N_{C_G(R)}(X)/X \cong 3^{1+2}_+.\mathrm{SL}_2(3)$ or $N_{C_G(R)}(X) \le H$ and $N_{C_G(R)}(X)/X$ is isomorphic to a subgroup of 3^3:Sym(4). As $\overline{X}'$ is normal in $\overline{C_G(R)}$ we get X^gX/X is elementary abelian. Now we either have $|X^gX/X| = 2$ or 4. Since $\overline{X}^g$ centralizes $\overline{X}^g \cap \overline{X})/\overline{X}'$ Lemma 16.3 implies that $|X^gX/X| = 4$ and that every element $x \in (X^gX/X)^\#$ satisfies $C_{\overline{X}/\overline{X}'}(x) = C_{\overline{X}/\overline{X}'}(\overline{X}^g) = (\overline{X} \cap \overline{X}^g)/\overline{X}'$. But then Lemma 2.14 implies that X^gX/X centralizes SX/X. This contradicts Lemma 16.6 and proves the lemma. $\square$

LEMMA 16.14. *One of the following holds:*

(i) $N_{C_G(R)}(X) = C_H(R)$ *and* $|C_G(R) : C_H(R)| \in \{1, 7, 13, 25\}$; *or*

(ii) $N_{C_G(R)}(X)/X \cong 3^{1+2}_+.\mathrm{SL}_2(3)$ *and* $|C_G(R) : C_H(R)| \in \{4, 28\}$.

PROOF. By Lemma 16.13 X acts fixed-point-freely by conjugates on $X^{C_G(R)} \setminus \{X\}$. Hence $|C_G(R) : N_{C_G(R)}(X)|$ is odd. Thus Lemma 16.11 immediately gives (i). In case (ii), we require $|C_G(R) : C_H(R)|$ to be divisible by $|N_{C_G(R)}(X) : C_H(R)| = 4$ and this gives the result. $\square$

LEMMA 16.15. *If* $N_{C_G(R)}(X)/X \cong 3^{1+2}_+.\mathrm{SL}_2(3)$*, then* $C_G(R) = N_{C_G(R)}(X)$.

PROOF. By Lemma 16.14 (ii), if the claim is false then $|C_G(R) : N_{C_G(R)}(X)| = 7$. Let $N = \bigcap_{g \in C_G(R)} N_{C_G(R)}(X)$. Then $C_G(R)/N$ is isomorphic to a subgroup of $\mathrm{Sym}(7)$. As $X \cap N$ is normalized by $N_{C_G(R)}(X)$ and $N_{C_G(R)}(X)$ acts irreducibly on X/X', we see that XN/N either is trivial or $|XN/N| \geq 2^6$. Since $C_G(R)/N$ has Sylow 2-subgroups of order at most 16, we have $N \leq X$. Now using Lemma 16.13 and fact that X is normal in N we have X is normal in $C_G(R)$, a contradiction. $\square$

LEMMA 16.16. *We have* $N_G(Q) = N_G(X)$.

PROOF. Suppose $N_G(Q) > N_G(X)$. As $N_G(Q) = N_G(S)C_G(R)$, $C_G(R) > N_{C_G(R)}(X)$. Combining Lemmas 16.10, 16.12 and 16.15 yields

$$N_{C_G(R)}(X) = C_H(R)$$

and

$$|C_G(R) : C_H(R)| \in \{7, 13, 25\}$$

by Lemma 16.14. By considering the action of S on the set $X^{C_G(R)}$, we see that X is fixed and S has at least one orbit of length at most 3. Select $g \in C_G(R)$, $X^g \in X^{C_G(R)}$ so that $|(X^g)^S| \leq 3$ and let $T = N_S(X^g)$ with notation chosen so that $T \leq S^g$. Suppose that $T \geq E_iQ \neq Q$ for some $1 \leq i \leq 3$. Then E_iQ and Q are normalized by S^g. Since $(E_i \cap Q)/R = Z(QE_i)$, S^g normalizes $E_i \cap Q$ and so S^g normalizes $C_{E_iQ}(E_i \cap Q) = E_i$. Using Lemma 16.8 we have the $S^g \leq H$ and so $S^{gh} = S$ for some $h \in C_H(R)$ as $\overline{S_0}$ contains a unique abelian subgroup of order 3^3. But then $gh \in C_H(R) \leq N_H(X)$ by Lemma 16.8 and this means that $g \in N_H(X)$ so that $X = X^g$, a contradiction. Hence, for $1 \leq i \leq 3$, we have $T \cap E_iQ = Q$ and $|\overline{T}| = 9$.

Recall that $C_H(R)/X$ is isomorphic to a subgroup of $3^3 : \mathrm{Sym}(4)$. Consider now $(X^g \cap C_H(R))X/X$. This is a 2-group which is normalized by TX/X. Since SX is normalized by $C_H(R)X/X$,

$$[X^g \cap C_H(R), T] \leq X^g \cap SX \leq X^g \cap X$$

and so TX/X is centralized by $(X^g \cap C_H(R))X/X$. Therefore $X^g \cap C_H(R) = X^g \cap X$ by Lemma 16.6. Hence using $|X^{C_G(R)}| \in \{7, 13, 25\}$ and $X^{(XS)^g} = 3|X^g/(X \cap X^g)|$, we see $|\overline{X^g \cap X}| \geq 2^4$ and $|X^{C_G(R)}| \in \{13, 25\}$. As $|\overline{X \cap X^g}| \geq 2^4$, there exists an involution $\overline{i} \in \overline{X \cap X^g}$ with

$\bar{i} \notin \overline{X'}$. Thus, up to change of notation, Lemma 16.5 implies we may assume that $\bar{i}$ is centralized by $\overline{E_1}$ and

$$\begin{aligned}\overline{X^g \cap X} &= \langle \bar{o}^{\overline{T}} \rangle = \langle \bar{i}^{\overline{E_1 T}} \rangle \\ &= \langle \bar{i}^{\overline{S}} \rangle = C_{\overline{X}}(\overline{E_1}) \cong \mathrm{Q}_8 \circ \mathrm{Q}_8.\end{aligned}$$

By Lemma 16.5 $C_{\overline{C_G(R)}}(\overline{X \cap X^g})$ is isomorphic to a subgroup of $\mathrm{GL}_2(3)$. But then $\overline{XE_1}$ has index at most 2 in $C_{\overline{C_G(R)}}(\overline{X \cap X^g})$ and therefore $X \cap X^g$ has index at most 2 in X^g, a contradiction. Hence $N_G(Q) = N_G(X)$. $\square$

The proof of Proposition 16.1. By Lemma 16.16, X is normal in $N_G(Q)$. As $N_G(Q) \not\leq H$ and $N_H(Q)$ contains an element which inverts $Z(Q)$, Lemma 16.10 indicates that $N_G(Q)/Q$ is an extension of X by $3_+^{1+2}.\mathrm{GL}_(3)$ or $\mathrm{PSU}_4(2){:}2$. Now an application of Lemma 3.8 and Lemma 3.9 yield the assertion. $\square$

17. The case when $\mathbf{p = 3}$, the centralizer of some 3-central element of $\mathbf{H}$ is soluble and $\mathbf{N_G(Q) \not\leq H}$

In this section we continue by investigating the groups which satisfy

Hypothesis 17.1. *Hypothesis 6.1 holds with $p = 3$, $F^*(H)$ is a group of Lie type in characteristic 3 and $C_H(z)$ is soluble for some $z \in Z(S_0)^\#$. In addition, assume that $N_G(O_3(C_G(t))) \not\leq H$ for some $t \in Z(S_0)^\#$.*

The main result of this section is

Proposition 17.2. *Suppose that Hypothesis* 17.1 *holds. Then one of the following holds*

(i) *the pair* $(F^*(G), F^*(H))$ *is one of* $(\mathrm{F}_4(2), \mathrm{PSL}_4(3))$, $(\mathrm{PSU}_6(2), \mathrm{PSU}_4(3))$, $(\mathrm{McL}, \mathrm{PSU}_4(3))$ $(\mathrm{Co}_2, \mathrm{PSU}_4(3))$, $({}^2\mathrm{E}_6(2), \mathrm{P}\Omega_7(3))$, $(\mathrm{M}(22), \mathrm{P}\Omega_7(3))$, $(\mathrm{M}(23), \mathrm{P}\Omega_8^+(3))$, $(\mathrm{F}_2, \mathrm{P}\Omega_8^+(3))$*; or*

(ii) $F^*(H) \cong \mathrm{PSL}_3(3)$.

Assume that Hypothesis 17.1 holds and continue the notation of Section 6. As for some $z \in Z(S_0)^\#$, $C_H(z)$ is soluble, Lemma D.15 implies that $F^*(H)$ is one of the groups

$$\mathrm{PSL}_3(3^e), \mathrm{G}_2(3^e), \mathrm{PSp}_4(3), \mathrm{PSL}_4(3), \mathrm{PSU}_4(3), \mathrm{P}\Omega_7(3), \text{ and } \mathrm{P}\Omega_8^+(3)$$

Recall that, if $F^*(H)$ is $\mathrm{PSL}_3(3^e)$, then Proposition 9.7 implies that (ii) holds. Proposition 15.1 contradicts Hypothesis 17.1 when $F^*(H) \cong \mathrm{G}_2(3^e)$. Hence Hypothesis 6.2 holds, Q is extraspecial and $R = Z(S) =$

$Z(S_0) = \langle z \rangle$ has order 3. By Hypothesis 17.1, $N_G(O_3(C_G(z))) \not\leq H$ and Proposition 7.1 gives $Q = O_3(C_{F^*(H)}(z)) = O_3(C_G(z))$.

If $F^*(H) \cong \mathrm{P}\Omega_8^+(3)$, then Proposition 16.1 shows that (i) holds. Thus the work in this section focuses on the groups

$$\mathrm{PSp}_4(3), \mathrm{PSL}_4(3), \mathrm{PSU}_4(3), \text{ and } \mathrm{P}\Omega_7(3).$$

In all cases Q is extraspecial of exponent 3. Hence, by [**79**, Theorem 1], $N_G(Q)/Q$ is a subgroup of $\mathrm{GSp}(Q/R)$.

Lemma 17.3. *We have* $F^*(H) \not\cong \mathrm{PSp}_4(3)$.

Proof. We have that Q is extraspecial of order 27. Therefore $\mathrm{Out}(Q) \cong \mathrm{GL}_2(3)$ and so $N_H(R)$ has index at most 2 in $N_G(Q)$. In particular, $N_H(Q) = N_H(R)$ is normal in $N_G(Q)$. Thus

$$N_G(Q) = N_{N_G(Q)}(S) N_H(Q)$$

and we have that $N_{N_G(Q)}(S)$ normalizes the unique abelian subgroup E of S of order 3^3. From the structure of $\mathrm{PSp}_4(3)$, we get $N_{F^*(H)}(E)/E \cong \mathrm{Alt}(4)$ and $C_G(E) = E$ as E is normal in S and G has parabolic characteristic 3. Thus $\langle N_{F^*(H)}(E), N_{N_G(Q)}(S) \rangle$ embeds into $\mathrm{GL}_3(3)$ and has Sylow 3-subgroups of order 3 and non-trivial Sylow 2-subgroups. Now Lemma E.3 shows that $N_{N_G(Q)}(S)$ normalizes $N_{F^*(H)}(E)$. But then $N_{N_G(Q)}(S)$ normalizes

$$\langle N_{F^*(H)}(E), N_{F^*(H)}(Z) \rangle = F^*(H).$$

Therefore $N_G(Q) \leq H = N_G(F^*(H))$, which is a contradiction to Hypothesis 17.1. $\square$

Proposition 17.4. *Suppose* $F^*(H) \cong \mathrm{PSL}_4(3)$ *or* $\mathrm{PSU}_4(3)$. *Then either* $F^*(H) \cong \mathrm{PSL}_4(3)$ *and* $F^*(G) \cong \mathrm{F}_4(2)$ *or* $F^*(H) \cong \mathrm{PSU}_4(3)$ *and* $F^*(G) \cong \mathrm{PSU}_6(2)$, McL *or* Co_2.

Proof. By Theorem A.10, $F^*(H) = O^{2'}(H)$ and so $S = S_0$ and $O_3(C_H(R)) \cong 3_+^{1+4}$. Using Lemma D.28 we have $E = J(S)$ is elementary abelian of order 3^4 and

$$N_{F^*(H)}(E)/E \cong \begin{cases} (\mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)){:}2 & \text{if } F^*(H) \cong \mathrm{PSL}_4(3) \\ \mathrm{PSL}_2(9) \cong \mathrm{Alt}(6) & \text{if } F^*(H) \cong \mathrm{PSU}_4(3) \end{cases}.$$

In both cases an inspection of the maximal subgroups of $\mathrm{GL}_4(3)$ [**14**, Table 8.8 and Table 8.9] yields

(17.4.1) $O^{3'}(N_H(E)) \trianglelefteq N_G(E)$ and $N_G(E) = N_G(S) O^{3'}(N_H(E))$.

We have that $N_G(Q)/Q$ is isomorphic to a subgroup of $\mathrm{GSp}_4(3)$, which has a Sylow 3-subgroup of order 3. Furthermore, independently of the isomorphism type of H, we have $C_H(R)/Q \cong \mathrm{SL}_2(3)$ and Q/R is a direct sum of two natural $\mathrm{SL}_2(3)$-modules for $C_H(R)$ by Lemma D.1. Employing [**53**, Lemma 4.21] we get that one of the following holds:

(1) $N_G(Q)/Q \cong 2_-^{1+4}.\mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Sym}(5)$;
(2) $E(N_G(Q)/Q) \cong \mathrm{SL}_2(5)$; or
(3) $|N_G(Q)/Q| = 2^a \cdot 3$ for some a.

If case (1) occurs, then, as R is not weakly closed in S with respect to G, Lemma 3.4 yields $G \cong \mathrm{Co}_2$.

Suppose we have possibility (2). Assume further that $F^*(H) \cong \mathrm{PSL}_4(3)$. We will show $N_G(S) \leq H$.

We know that $N_G(S)$ normalizes E and by (17.4.1) also normalizes $O^{3'}(N_H(E))$. Let $E_1 \leq S$ be the group of transvections to a point and $E_2 \leq S$ the group of transvections to a hyperplane containing this point. Then $O^{3'}N_{F^*(H)}(E_i)/E_i \cong \mathrm{SL}_3(3)$. Furthermore, $N_{F^*(H)}(E_i)$ acts transitively on the subgroups of E_i of order 3. Thus, as $Q = O_3(C_G(Z))$, for $i = 1, 2$, we have

$$\begin{aligned} U_i &= \langle O_3(C_G(R^g)) \mid g \in G, Z(Q)^g \leq E_i \rangle \\ &= \langle O_3(C_H(R^g)) \mid g \in G, Z(Q)^g \leq E_i \rangle = O^{3'}(N_{F^*(H)}(E_i)). \end{aligned}$$

We also calculate that E_1E/E and E_2E/E are the two subgroups of order three in S/E, which act quadratically on E. In particular $N_G(S)$ permutes the set $\{E_1E, E_2E\}$. We have that $O^{3'}(N_H(E))$ contains an involution x which inverts E and centralizes S/E. Let $M = N_{N_G(S)}(E_1E)$. We factor $M = C_M(x)E$. Then, for $i = 1, 2$, M normalizes $Z(E_iE) = E_i \cap E$ which has order 3^2. Now $E_i = C_{E_iE}(x)(E_i \cap E)$ is normalized by $C_M(x)$. Since E normalizes E_i, we infer that E_i is normalized by M. Therefore $N_G(S)$ permutes $\{E_1, E_2\}$ and normalizes $\langle U_1, U_2 \rangle = F^*(H)$. Hence by assumption we then have that $N_G(S) \leq N_G(F^*(H)) = H$.

Now generally if (2) holds, then we have that

$$N_G(Q) = \langle N_H(Q), N_{N_G(Q)}(S) \rangle,$$

as $N_H(Q)/Q \cap E(N_G(Q)/Q) \cong \mathrm{SL}_2(3)$ and $N_{E(N_G(Q)/Q)}(S/Q) \sim 3:4$ and together these groups generate $E(N_G(Q)/Q)$. Hence as $N_G(Q) \not\leq H$ and $N_G(S) \leq H$ when $H \cong \mathrm{PSL}_4(3)$, we get that $F^*(H) \cong \mathrm{PSU}_4(3)$ and so $E(N_G(E)/E) \cong \mathrm{Alt}(6)$. Finally, using Lemma 3.5 this yields $F^*(G) \cong \mathrm{McL}$.

So we may assume that we have possibility (3). The Frattini Argument delivers

$$N_G(O^{3'}(N_H(Q))) = N_{N_G(O^{3'}(N_H(Q)))}(S) O^{3'}(N_H(Q)).$$

By (17.4.1) we see that $N_{N_G(O^{3'}(N_H(Q)))}(S)$ normalizes $O^{3'}(N_H(E))$ and then

$$\langle O^{3'}(N_H(E)), O^{3'}(N_H(Q))\rangle = F^*(H).$$

Therefore the group $N_G(O^{3'}(N_H(Q)))$ normalizes $F^*(H)$ and so is contained in H. Thus $N_{N_G(Q)}(O^{3'}(N_H(Q))) \leq H$. Using this information with help from Lemma E.5 we obtain

$$O^{3'}(N_G(Q)/Q)N_H(S) \leq U \cong (\mathrm{Q}_8 \times \mathrm{Q}_8).\mathrm{Sym}(3).$$

Hence, as $N_G(Q) \not\leq H$, we get that U is isomorphic to a subgroup of $N_G(Q)/Q$. In particular we have that $N_G(Q)/Q$ is a subgroup of the subgroup of $\mathrm{GSp}_4(3)$ which preserves a decomposition of the natural 4-dimensional symplectic space over GF(3) into a perpendicular sum of two non-degenerate 2-spaces. We further see that $O^{3'}(N_G(Q)/Q)$ is isomorphic to a subgroup of $\mathrm{Sp}_2(3) \times \mathrm{Sp}_2(3)$ and projects non-trivially on to both symplectic groups. In particular it contains a normal subgroup isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$.

In both cases $F^*(H) \cong \mathrm{PSU}_4(3)$ and $F^*(H) \cong \mathrm{PSL}_4(3)$ we have that $Z(Q)$ is not weakly closed in S. Hence the assertion follows from Lemma 3.3. □

Proposition 17.5. *Suppose that $F^*(H) \cong \mathrm{P}\Omega_7(3)$. Then $F^*(G) \cong {}^2\mathrm{E}_6(2)$ or* M(22).

Proof. Again $S = S_0$ and $R = Z(S)$ has order 3. By Lemma D.1

$$N_{F^*(H)}(R) \sim 3_+^{1+6}.(\mathrm{SL}_2(3) \times \Omega_3(3)).2.$$

Furthermore, as a module for this group Q/R is the tensor product of the natural $\mathrm{SL}_2(3)$-module with the 3-dimensional orthogonal $\Omega_3(3)$-module and this is an irreducible action. By Proposition 7.1 we know $Q = O_3(N_{F^*(H)}(R)) = O_3(C_G(R))$. Application of Lemma E.6 shows that $N_G(Q)/Q$ can be identified as a subgroup of

$$U = (\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)) : 2$$

with $O_2(N_{F^*(H)}(Q)/Q) \geq \Omega_1(O_2(U))$.

Let $Q < T \leq S$ be such that $C_{O_2(N_{F^*(H)}(Q)/Q)}(T) \cong \mathrm{Q}_8$. Then T/Q does not centralize $\Omega_1(O_2(U))$ and so permutes the base group of U transitively. It follows that either

- $O_2(N_G(Q)/Q) \geq O_2(U)$;
- $O_2(N_G(Q)/Q) \cap O_2(U)$ has order 2^7; or
- $O_2(N_G(Q)/Q) \cap O_2(U) = O_2(N_{F^*(H)}(Q)/Q)$.

Therefore either the assumptions of Lemmas 3.6 or 3.7 are satisfied or $|N_G(Q) : N_{F^*(H)}(Q)| = 2$. Since Lemmas 3.6 or 3.7 identify ${}^2\mathrm{E}_6(2)$ or

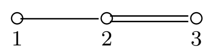
M(22) we have to consider the possibility that $|N_G(Q) : N_{F^*(H)}(Q)| = 2$. In this case

$$N_G(Q)/Q \cong \mathrm{GL}_2(3) \times \mathrm{Sym}(4)$$

and

$$N_G(Q) = N_G(S)N_{F^*(H)}(Q) = N_G(S)O^3(N_G(Q))$$

where $O^3(N_G(Q)) = O^3(N_H(Q))$. We have that $N_{F^*(H)}(R)$ is the parabolic subgroup in $F^*(H)$, which corresponds to the two end nodes of the Dynkin diagram



Let P_1 be the parabolic subgroup of $F^*(H)$ containing S, which corresponds to the A_2-subdiagram. Then by Lemma D.27 we have $O_3(P_1)$ is of order 3^6, $P_1/O_3(P_1) \cong \mathrm{SL}_3(3)$ and $E = Z(P_1) = \Phi(O_3(P_1))$ is elementary abelian of order 3^3. Now we set $P = N_{N_G(Q)}(QO_3(P_1))$. Then $P/O_3(P) \cong \mathrm{GL}_2(3) \times 2$ and $O_3(P) = QO_3(P_1)$ is normalized by $N_G(S)$. Since $|O_3(P_1)Q/Q| = 3$, $E = \Phi(O_3(P_1)) \leq Q$ and so

$$E/R \leq C_{Q/R}(O_3(P_1)Q).$$

As $O_3(P)/Q$ corresponds to the Sylow 3-subgroup of $\Omega_3(3) \cong \mathrm{Alt}(4)$ above, we see that $|C_{Q/Z(Q)}(O_3(P))| = 9$. Hence

$$E/R = C_{Q/R}(O_3(P_1)Q).$$

and so E is normalized by $N_G(S)$. As E is normal in S, we have that $O_3(N_G(E)) = O_3(N_{F^*(H)}(E))$. This yields that $N_{F^*(H)}(E)$ is normal in $N_G(E)$, as $N_{F^*(H)}(E) = O^{3'}(N_G(E))$. Then $N_G(S)$ normalizes $\langle N_{F^*(H)}(E), N_{F^*(H)}(Z(Q))\rangle = F^*(H)$. But then by assumption $N_G(Q) = N_G(S)N_{F^*(H)}(Q) \leq H$. This proves the proposition. $\square$

Proof of Proposition 17.2. Suppose Hypothesis 17.1 holds. As $C_H(z)$ is soluble for some 3-central element of H, Lemma D.15 yields $F^*(H) \cong \mathrm{PSL}_3(3^e)$, $\mathrm{G}_2(3^e)$, $\mathrm{PSp}_4(3)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSU}_4(3)$, $\mathrm{P}\Omega_7(3)$, or $\mathrm{P}\Omega_8^+(3)$. We have already mentioned that Proposition 9.7, Propositions 15.1 and 16.1 focus on the cases $F^*(H) \cong \mathrm{PSL}_3(3^e)$, $\mathrm{G}_2(3^e)$ or $\mathrm{P}\Omega_8^+(3)$ respectively. By Lemma 17.3, $F^*(H) \not\cong \mathrm{PSp}_4(3)$. Propositions 17.4 and 17.5 handle the remaining three cases. Together these results prove the proposition. $\square$

18. Proof of Theorem 2 and Theorem 3

In this short section we prove Theorems 2 and 3.

Proof of Theorem 2. The hypothesis of Theorem 2 is that Hypothesis 6.1 holds and that $N_H(O_p(C_G(z)))$ is not soluble for all $z \in Z(S_0)^\#$ as well as $N_G(O_p(C_G(t)) \not\leq H$ for some $t \in Z(S_0)^\#$.

Suppose first that Hypothesis 6.2 holds. Then Q is large by Lemma 7.2 and, by Proposition 7.1, $Q = O_p(C_G(z))$ for all $z \in Z(S_0)$. So, by assumption, $N_G(Q) \not\leq H$. Now Proposition 8.2 implies that $p = 5$ and $H \cong \mathrm{G}_2(5)$. If on the other hand, Hypothesis 6.2 does not hold. Then Lemma D.15 shows that

$$F^*(H) \cong \begin{cases} \mathrm{Sp}_{2n}(2^e) & n \geq 2, e \geq 1 \text{ and } (n, 2^e) \neq (2,2), (3,2) \\ {}^2\mathrm{F}_4(2^{2e+1}) & e \geq 1 \\ \mathrm{F}_4(2^e) & e \geq 1 \\ \mathrm{G}_2(3^e) & e > 2. \end{cases}$$

Now combining Propositions 13.8 and 15.1 yields a contradiction to the assumptions of Theorem 2.

In the case that $F^*(H) \cong \mathrm{G}_2(5)$, suppose that G is in addition a $\mathcal{K}_2$-group. Then Proposition 8.3 states that $G \cong \mathrm{LyS}$. $\square$

Proof of Theorem 3. The hypothesis of Theorem 3 states that Hypothesis 6.1 holds and that there exist $z, t \in Z(S_0)^\#$ such that $N_H(O_p(C_G(z)))$ is soluble and $N_G(O_p(C_G(t))) \not\leq H$. So Hypothesis 6.1 holds with $C_H(z)$ soluble for some $z \in Z(S_0)^\#$. Using Lemma D.15 we have $F^*(H) \cong \mathrm{PSL}_3(p^e)$ with p odd or $p \in \{2,3\}$. If $F^*(H) \cong \mathrm{PSL}_3(p^e)$, then Proposition 9.7 shows that $p^e \in \{3,5,7,13\}$ and Theorem 3 (iii) holds. Proposition 17.2 provides the statement of Theorem 3 (ii) in the special case that $p = 3$. So suppose that $p = 2$. In this case, Proposition 14.2 provides a complete determination of the groups which satisfy Hypothesis 6.1 with $C_H(z)$ soluble and $G \neq H$. Using Proposition 14.3 we see that the pairs $(F^*(G), F^*(H))$ with $F^*(H) \cong \mathrm{PSL}_4(2)$ or $\mathrm{PSU}_4(2)$ do not satisfy the hypothesis of Theorem 3. The remaining pairs are all listed in the statement of Theorem 3 (i). This concludes the proof of Theorem 3. $\square$

19. Groups which satisfy Hypothesis 6.2 with $\mathbf{N_G(Q) \leq H}$ and some p-local subgroup containing S not contained in H

We continue the notation introduced in Section 6. In particular, R is a root group in $Z(S)$ and $Q = O_p(N_{F^*(H)}(R))$. Our working hypothesis for this section is:

Hypothesis 19.1. *The group G is a $\mathcal{K}_p$-group which satisfies Hypothesis* 6.2 *and in addition*

(i) $N_G(Q) = N_H(Q)$;
(ii) *there exists a p-local subgroup M containing S such that $M \not\leq H$.*

Our intention is to prove

PROPOSITION 19.2. *If Hypothesis* 19.1 *holds, then $F^*(G) = \mathrm{PSL}_4(3)$ and $F^*(H) = \mathrm{PSU}_4(2)$.*

First we recall that if $p = 2$ and $C_H(z)$ is soluble for some 2-central involution z, then Proposition 14.2 classifies all the possible pairs $(F^*(G), F^*(H))$ which satisfy Hypothesis 6.1 and, in particular, the only pair which satisfies Hypothesis 19.1 is $(F^*(G), F^*(H)) = (\mathrm{PSL}_4(3), \mathrm{PSU}_4(2))$. Henceforward we therefore assume that when $p = 2$, $C_H(z)$ is not soluble for all 2-central involutions z.

Recall that when $F^*(H)$ is as in Hypothesis 6.2, then the results from Section 7 are available. In particular, Q is semi-extraspecial, and Q is large by Lemma 7.2. Therefore, in addition to $N_G(Q) = N_H(Q)$, we also know that

(L2) if $1 \neq U \leq G$ and $[U, Q] = 1$, then $N_G(U) \leq N_G(Q)$

and Q is weakly closed in S_0 with respect to G.

Our first lemma concerns the structure of over-groups K of $QQ_p(M)$ in M which are not in H. Since $K \geq O_p(M)$ and $C_G(O_p(M)) \leq O_p(M)$ by Lemma 2.3(iv), we have

$$C_G(O_p(K)) \leq O_p(M) \leq O_p(K)$$

and so K has characteristic p.

LEMMA 19.3. *Suppose that $QO_p(M) \leq K \leq M$ and $K \not\leq H$. Set $Y_K = \langle \Omega_1(Z(S_0))^K \rangle$ and $\widetilde{K} = K/C_K(Y_K)$. Then*

(i) $Q \not\leq O_p(K)$;
(ii) $Y_K \leq Z(O_p(K)) \leq O_p(K)$;
(iii) $R < Y_K \cap Q \leq Y_K$ *and* $C_{Y_K}(Q) = R$;
(iv) $C_K(Y_K) \leq H$;
(v) Y_K *is an irreducible K-module;*
(vi) *if $Y_K \leq Q$, then, as a $\widetilde{K}$-module, Y_K is a dual F-module with dual offender $\widetilde{Q}$; and*
(vii) *if $Y_K \not\leq Q$, then, as a $\widetilde{K}$-module, Y_K is dual to a $2F$-module with 2-offender $\widetilde{Q}$ acting strictly cubically.*

PROOF. If $Q \leq O_p(K)$, then, as Q is weakly closed in S_0, we have $K \leq N_G(Q) \leq H$ which is a contradiction. Hence (i) holds.

Because K has characteristic p and $\Omega_1(Z(S_0)) \le Q \le K$, we have $\Omega_1(Z(S_0)) \le Z(O_p(K))$ and therefore $Y_K \le Z(O_p(K)) \le O_p(K)$ which is (ii).

If $Y_K \le R$, then, as $R = Z(Q)$ and Q is large, we have

$$K \le N_G(Y_K) \le N_G(Q) \le H$$

by (L2), which is not the case. Hence $Y_K \not\le R$ and, in particular, $Y_K \ne R$. If $Y_K \le Q$, then, as Q is semi-extraspecial and $Y_K \not\le R$, Lemma D.16 implies that $[Q, Y_K] = R$. If $Y_K \not\le Q$, then $[Q, Y_K] \le Q \cap Y_K$ is not contained in R because $N_G(Q)/Q$ acts faithfully on Q/R. Hence $R = [Y_K, Q, Q] \le Y_K$ again by Lemma D.16. Finally, as $R \le Y_K$ and Y_K is abelian we have

$$R \le C_{Y_K}(Q) \le C_G(Q) \le C_Q(Q) = R.$$

Thus (iii) holds.

Since $R \le Y_K$ we have $C_K(Y_K)$ centralizes R and is therefore contained in H by (L2). So part (iv) holds.

Suppose that U is a non-trivial K-invariant subgroup of Y_K. If U is centralized by Q, then $U \le R$ and we have $K \le H$ by L(2). Since $K \not\le H$, we conclude U is not centralized by Q. If $U \le Q$, then $[U, Q] = R \ge \Omega_1(Z(S_0))$ and so $Y_K = U$ by the definition of Y_K. If $U \not\le Q$, then $[U \cap Q, Q] = R$ and again we have $U = Y_K$. Thus Y_K is irreducible as a K-module. Hence (v) holds.

To prove (vi), suppose $Y_K \le Q$. Then $[Y_K, Q, Q] = 1$ and $Y_K \not\le R$ by (iii), so we have $|Q : C_Q(Y_K)| \ge |R|$ and $[Q, Y_K] = R$ by Lemma D.16. Thus $|\widetilde{Q}| \ge |[Y_K, Q]|$ which means that Y_K is dual to an F-module with dual offender $\widetilde{Q}$.

Now for part (vii). Assume that $Y_K \not\le Q$. Then $[Y_K, Q] \not\le R$ and so Lemma D.16 implies that $[Y_K, Q, Q] = R$ and

$$[Y_K, Q, Q, Q] \le [Q, Q, Q] = 1.$$

Hence Q operates strictly cubically on Y_K.

Set $|Y_K \cap Q| = p^{x+e}$. Then, as Q is semi-extraspecial,

$$|\widetilde{Q}| = |Q : C_Q(Y_K)| \ge |Q : C_Q(Y_K \cap Q)| \ge p^x.$$

Denoting the dual of Y_K by Y_K^*, noting that $p^x \ge p^e$ by Lemma 2.10 and using Lemma 2.19, we have

$$|Y_K^* : C_{Y_K^*}(Q)| = |[Y_K, Q]| \le |Y_K \cap Q| = p^{e+x} \le p^{2x} \le |\widetilde{Q}|^2.$$

Thus Y_K is a dual $2F$-module for $\widetilde{K}$ with $\widetilde{Q}$ a strictly cubic 2-offender on Y. This proves (vii). $\square$

Select $P \leq M$ of minimal order subject to $P \geq SO_p(M)$ and $P \not\leq H$. Notice that the results of Lemma 19.3 are available for P.

Lemma 19.4. *The following hold:*

(i) *$P \cap H$ is the unique maximal subgroup of P which contains $SO_p(M)$;*
(ii) *if $S_1 \in \mathrm{Syl}_p(P)$ with $SO_p(M) \leq S_1$, then $N_P(S_1) \leq H$; and*
(iii) *P is a p-minimal group.*

Proof. By the minimal choice of P, $P \cap H$ is a maximal subgroup of P. Assume that P_1 is a maximal subgroup of P which contains $SO_p(M)$. The minimal choice of P implies that $P_1 \leq H$. Therefore $P \cap H$ is the unique maximal subgroup of P containing $SO_p(M)$. This is (i).

Since $Q \leq S$ and Q is weakly closed in S_0 with respect to G, we have that $N_G(S_1) \leq N_G(Q) \leq H$. So (ii) holds.

If $\langle S_1^P \rangle < P$, then $\langle S_1^P \rangle \leq H$ by (i). In addition, $P = \langle S_1^P \rangle N_P(S_1)$ by the Frattini Argument. By (ii) $P \leq H$, a contradiction. We conclude that $P = \langle S_1^P \rangle$ and with (i) this shows that P is p-minimal. Hence (iii) holds. $\square$

For the remainder of this section we fix the following notation

- $S_1 \in \mathrm{Syl}_p(P)$ with $SO_p(M) \leq S_1 \leq S_0 \in \mathrm{Syl}_p(H)$.
- $Y = Y_P = \langle \Omega_1(Z(S_0))^P \rangle$.
- $\overline{P} = P/C_P(Y)$.

Since P is p-minimal by Lemma 19.4, the structure of P is generally portrayed by Lemma 2.7. By Lemma 19.3 (iv), $C_P(Y)$ is contained in H and, since $P \cap H$ is the unique maximal subgroup of P containing S_1, it follows from Lemma 2.7 (ii) and (iv) that $O_p(P) \in \mathrm{Syl}_p(C_P(Y))$ and $\overline{C_P(Y)}$ is nilpotent. Remember also that $O_p(M) \leq O_p(P) \leq C_P(Y)$ so that P is of characteristic p and even though P is not a p-local subgroup it is a $\mathcal{K}$-group as G is a $\mathcal{K}_p$-group.

Lemma 19.5. *Let $|Y| = 16$ and $|[Q,Y]| = 8$, $|C_Y(Q)| = 2$, then $Y \not\leq Q$ but $Y \leq S$.*

Proof. As $[Y,Q,Q] \neq 1$, we have $Y \not\leq Q$. Assume now $Y \not\leq S$. As $[Y,S] = [Y,Q]$ we see that YQ/Q centralizes S/Q. By Lemma D.25 we have that Y induces a group of inner automorphisms on the Levi complement of $N_{F^*(H)}(R)$. Set $L = O^{2'}(N_{F^*(H)}(R)/Q)$. Then there is some $1 \neq \tilde{y}$ with $\tilde{y}^2 \in Q$ which induces an outer automorphism on $F^*(H)$ with $[L,\tilde{y}] = 1$. Assume $F^*(H) \not\cong \mathrm{PSL}_n(2)$. Then by Lemma D.1 we have that L acts irreducibly on Q/R and so $[\tilde{y},Q] \leq R$ (recall $\mathrm{PSU}_4(2) \cong \Omega_6^-(2)$ is excluded as $C_H(z)$ is soluble for a 2-central involution of H.) This implies that there is some $1 \neq y_1$, $y_1^2 \in R$ such that $[S, y_1] = 1$ and y_1 induces an outer automorphism on $F^*(H)$ which contradicts Lemma D.25.

We are left with $F^*(H) \cong \mathrm{PSL}_n(2)$. We now have $n \geq 5$ and so there are exactly two non-trivial, non-isomorphic L-modules involved in Q/R, which then again implies that there is y_1 which centralizes S, again a contradiction. $\square$

LEMMA 19.6. *Suppose that $E(\overline{P}) = 1$. Then $p \in \{2, 3\}$, $Y \leq Q$ has order p^2 and $\overline{P} \cong \mathrm{SL}_2(p)$.*

PROOF. Suppose that $E(\overline{P}) = 1$. We will show that $\overline{P} \cong \mathrm{SL}_2(2)$ or $\mathrm{SL}_2(3)$ and $Y \leq Q$ has order 4 or 9 respectively.

Since $E(\overline{P}) = 1$, $F(\overline{P}) \neq 1$. Let $K_0 \geq C_P(Y)$ be such that $\overline{K}_0 = F(\overline{P})$. Then $S_1 \cap K_0 \in \mathrm{Syl}_p(K_0)$ and $(S_1 \cap K_0)C_P(Y)$ is a normal subgroup of P. Thus

$$P = N_P(S_1 \cap K_0)(S_1 \cap K_0)C_P(Y) = N_P(S_1 \cap K_0)C_P(Y).$$

Since $S_1 \cap K_0$ is normalized by S_1, and $C_P(Y) \leq P \cap H$, we deduce that $P = N_P(S_1 \cap Y)$ and $S_1 \cap K_0 = O_p(P) \leq C_P(Y)$. Hence $\overline{K}_0$ is a p'-group. Since, $C_P(\overline{K_0}) \leq \overline{K_0}$, we now have $C_{\overline{Q}}(\overline{K}_0) = 1$. Define

$$K = [K_0, Q]C_P(Y).$$

Then, as $Q \not\leq O_p(P)$, $\overline{K} \neq 1$ and $C_{\overline{Q}}(\overline{K}) = 1$ by coprime action.

Suppose that $\overline{Q}$ is not cyclic. Then

$$\overline{K} = \langle C_{\overline{K}}(\overline{J}) \mid |\overline{Q} : \overline{J}| = p\rangle.$$

Since $[\overline{K}, \overline{Q}] = \overline{K}$, there exists a maximal subgroup J of Q such that $\overline{K_J} = C_{\overline{K}}(J)$ is not centralized by $\overline{Q}$. Let K_J be the preimage of $\overline{K_J}$. Then K_J normalizes $[Y, J, J] \leq R$.

Suppose that $1 < U \leq R < Y$ is K_J-invariant. Then $K_J \leq N_G(U) \leq N_G(Q)$ by $(L2)$. Now $[K_J, Q]$ is a p-group and so $[\overline{K}_J, \overline{Q}] = 1$, a contradiction to our selection of J. Thus no such U exists.

If $[Y, J, J] \neq 1$, then, setting $U = [Y, J, J] \leq [Q, Q] = R$, we have a contradiction. Therefore J operates quadratically on Y. In particular, $[Y, J]$ is centralized by J. If $[Y, J] \leq R$, we set $U = [Y, J]$ and have a contradiction. Hence

$$[Y, J] \not\leq R \text{ and } Y \not\leq Q.$$

Since $[Y, J]$ is centralized by J which has index p in Q we have $[Y, J, Q] = R$ is of order p and Lemma 2.10 implies that Q is extraspecial and $[Y, J]$ has order p^2. Now $\overline{K}_J\overline{Q}$ acts non-trivially on $[Y, J]$ and so $\overline{K}_J\overline{Q}$ maps into $\mathrm{GL}_2(p)$. By Dickson's list of subgroups of $\mathrm{GL}_2(p)$ [**33**, Satz 8.27], we see that a p-group acts non-trivially on a p'-group only for $p = 2$ or 3. Hence we have that

$$p \in \{2, 3\}.$$

Because $\overline{Q}$ is not cyclic, $C_{\overline{K}}(J) \neq \overline{K}$ and so there is a further maximal subgroup J_1 of Q with $C_{\overline{K}}(J_1)$ not centralized by Q. We have

$$R \leq R[Y, J \cap J_1] \leq [Y, J] \cap [Y, J_1].$$

As $|[Y, J]| = |[Y, J_1]| = p^2$, we either have $[Y, J] = [Y, J_1]$ or $[Y, J \cap J_1] \leq R$. Option one is impossible as $[Y, J] = [Y, J_1]$ is then centralized by $JJ_1 = Q$ which means that $[Y, Q] \leq R$ and delivers $Y \leq Q$, a contradiction. Thus $U = [Y, J \cap J_1] \leq R$. If $U \neq 1$, then K_J normalizes U which we have already seen is impossible. Therefore $[Y, J \cap J_1] = 1$. In particular, $|\overline{Q}| = p^2$. Since J operates quadratically on Y as an element of order p, we have $|Y/C_Y(J)| \cong [Y, J]$ as $C_K(J)Q$-modules. In particular, $|Y : C_Y(J)| = |Y : C_Y(J_1)| = p^2$ and $|Y/C_Y(J)C_Y(J_1)| = |Y/[Y, Q]| = p$. Since $R = C_Y(J) \cap C_Y(J_1)$, we have

$$|Y| = p^4.$$

Suppose that $p = 3$. Then $\overline{P} = \overline{\langle S_1^P \rangle}$ is isomorphic to a subgroup of $\mathrm{GL}_4(3)$ which is contained in $\mathrm{SL}_4(3)$. The only nilpotent subgroups in $\mathrm{SL}_4(3)$ on which an elementary abelian subgroup of order 9 can act faithfully are isomorphic to $\mathrm{Q}_8 \times \mathrm{Q}_8$ or $\mathrm{Q}_8 \circ \mathrm{Q}_8$ (recall that $\mathrm{SL}_4(3)$ does not contain elementary abelian subgroups of order 16). Now we have that $\overline{Q} = \overline{S}$ and $\overline{P} = \overline{K} N_{\overline{P}}(\overline{Q})$. In particular $\overline{K} \not\leq \overline{(H \cap P)}$. But then $\overline{Q} C_{\overline{K}}(J)$ and $\overline{Q} C_{\overline{K}}(J_1)$ are in different maximal subgroups of $\overline{P}$, a contradiction. Therefore

$$p = 2.$$

Because $p = 2$, $\overline{P}$ is contained in $\mathrm{GL}_4(2)$. This time the only nilpotent group of odd order in $\mathrm{GL}_4(2)$, on which a fours group acts faithfully is a Sylow 3-subgroup. This means that $\overline{P}$ is isomorphic to a subgroup of $\mathrm{O}_4^+(2)$. As $\overline{P}$ is 2-minimal, we get

$$\overline{P} \cong \mathrm{O}_4^+(2) \sim 3^2.\mathrm{Dih}(8).$$

Since R has order 2, $F^*(H)$ is a group of Lie type satisfying Hypothesis 6.2 with $p^e = 2$. Furthermore, using $|Y/[Y, Q]| = 2$ and $Y \not\leq Q$, we determine that YQ/Q has order 2 and is normalized by S/Q and $[Y, Q]/R$ has order 4. Hence by Lemma 19.5 we have $Y \leq S$ and then by Lemma D.19 $F^*(H) \cong \mathrm{PSU}_n(2)$ or $\mathrm{PSL}_n(2)$. But then $[Q, Y, S] = R$. This shows $[Q, Y] = Z_2(S)$. Now we see that $C_Q(Z_2(S)) = Q \cap O_2(P)$

and then $Y \leq C_S(C_Q(Z_2(S)))$ which gives that $Y = V(Q,S)$ and contradicts Proposition 7.7. Hence

$$\overline{Q} \text{ is cyclic.}$$

Because $\overline{Q}$ is cyclic, we have $C_Q(Y)$ has index p in Q. Since $Y \cap Q \not\leq R$ by Lemma 19.3 (iii), we have $p^e = p$. Suppose that $Y \not\leq Q$. Then, by Lemma 19.3 (vii), $C_Y(Q) = [Y,Q,Q] = R$ has order p, and so we infer that Q has exactly one Jordan block when it acts on Y. As Q acts cubically we get $|Y| = p^3$. Furthermore, Y operates as a transvection on Q/R and therefore, by Lemma D.20, $F^*(H) \cong \mathrm{PSp}_{2n}(p)$ with p odd and $S = S_0$. Now

$$O_p(P) = C_S(Y) = C_S(Z_2(S)) = O_p(P(S,L))$$

and so, by Lemma D.23, $\Omega_1(Z(O_p(P))) = V(Q,S)$ is normalized by P and this contradicts Proposition 7.7 and $P \not\leq H$.

On the other hand, if $Y \leq Q$, we obtain $|Y| = p^2$ and so $\overline{KQ}$ is a subgroup of $\mathrm{SL}_2(p)$. Since two distinct cyclic subgroups of order p in $\mathrm{GL}_2(p)$ generate $\mathrm{SL}_2(p)$ we deduce that $\overline{P} = \overline{KQ} \cong \mathrm{SL}_2(p)$ and so $p \in \{2,3\}$ in this case. This proves the result. □

Lemma 19.7. *Suppose that $E(\overline{P}) \neq 1$. Then $\overline{Q}$ normalizes every component of $\overline{P}$.*

Proof. Assume $E(\overline{P}) = \overline{J_1} \cdots \overline{J_k}$ with $\overline{J}_i$ components of $\overline{P}$ and, for $1 \leq i \leq k$, let $J_i \geq C_P(Y)$ be the preimage of $\overline{J_i}$. Recall that S_1 permutes $\overline{J}_1, \ldots, \overline{J}_l$ transitively and so $\overline{J}_i \cong \overline{J}_j$ for $1 \leq i \leq j \leq k$.

Aiming for a contradiction we may assume that $\{\overline{J_1}, \ldots, \overline{J_\ell}\} = J_1^Q$ is a Q-orbit on the components of $\overline{P}$ with $\ell \geq 2$ a power of p. As $R \leq O_p(P)$, we have that $\overline{Q}$ is elementary abelian. Let

$$S^1 = S_1 \cap J_1.$$

Then

$$[S^1, Q] \leq Q.$$

Let $w \in \overline{Q}$ with $\overline{J_1}^w = \overline{J_2}$. Then $\overline{Q} \geq [\overline{S^1}, w] \cong \overline{S^1}$, so $\overline{S^1}$ is elementary abelian (recall that $Z(\overline{J_1})$ is a p-prime group, so $\overline{S_1}$ is a Sylow p-subgroup of $\overline{J_1}/Z(\overline{J_1})$ as well.) and as $[\overline{Q}, [\overline{S_1}, w]] = 1$, we have

$$|(S^1)^Q| = 2.$$

In particular,

$$\ell = p = 2.$$

Let

$$\overline{S_2} = \overline{S_1} \cap E(\overline{P})$$

and let

$$N = N_{J_1 \cdots J_k}(S_2).$$

Then, as $\overline{J}_1$ has elementary abelian Sylow 2-subgroups and $\overline{J}_1$ does not have a normal 2-complement, N contains $N_{J_1}(S^1) > S^1$. Since $NSO_p(M) \neq P$, we have

$$N \leq H$$

by Lemma 19.4. Using $Q \not\leq O_2(NS)$, we see that $S \not\leq O_2(NS)$. We further have that $[N, Q]S \leq F^*(H)$ is normalized by a parabolic subgroup of $F^*(H)$ which contains S (see [**27**, Theorem 2.6.7]). Since S is not normal in $[N, Q]S$ and since $[N, Q]S$ is soluble (recall $p = 2$ and N/S_2 has odd order), we infer

$$p^e = 2 \text{ and } N/S_2 \text{ is an elementary abelian 3-group.}$$

In particular,

$$|R| = 2.$$

Recall that $J_1^w = J_2$ and then define $\overline{D} = C_{\overline{J}_1\overline{J}_2}(w)$. We have $\overline{D}/Z(\overline{D}) \cong \overline{J}_1/Z(\overline{J}_1)$, $\overline{D} \geq [\overline{S^1}, w] \cong \overline{S^1}$. Now $[Y, w] \leq Q$ and, as $\overline{Q}$ is abelian, $[Y, w]$ is normalized by $\overline{Q}$ as well as by $\overline{D}$. Since $|R| = 2$, $R \leq [Y, w]$ and

$$[Y, w, [\overline{S^1}, w]] = [Y, w, Q] \leq R.$$

Suppose that $\overline{D}$ does not centralize $[Y, w]$. Then $[Y, w]$ is an F-module for $\overline{D}$ with a Sylow 2-subgroup of $\overline{D}$ acting as a GF(2)-transvection group. Therefore, Lemma C.21 implies that $\overline{D} \cong \mathrm{PSL}_2(5) \cong \mathrm{Alt}(5)$. But this group contains no GF(2)-transvections (as an involution inverts an element of order 5 for example). This contradiction shows that $\overline{D}$ centralizes $[Y, w]$ and hence also centralizes R. But then $\overline{D} \leq N_G(R) = N_G(Q) \leq H$ and so $\overline{D}$ normalizes $\overline{Q} \geq [\overline{S^1}, w]$ and this is nonsense as $[\overline{S^1}, w] \in \mathrm{Syl}_2(\overline{D})$. Thus Lemma 19.7 is proved. $\square$

Until the last results of this section, we focus on the cases when $E(\overline{P}) \neq 1$. For this purpose we fix once and for all a component $\overline{J}$ of $\overline{P}$ and a subgroup $J \geq C_P(Y)$ mapping to $\overline{J}$. Then

$$E(\overline{P}) = \langle \overline{J}^{\overline{S}_1} \rangle = F^*(\overline{P})$$

and $\overline{J}N_{\overline{S}_1}(\overline{J})$ is p-minimal by Lemma 2.8. Lemma 19.7 implies that Q normalizes J and we define

$$K = JQ.$$

Since

$$K \geq C_P(Y) \geq O_p(P) \geq O_p(M),$$

K is characteristic p and, as $P = \langle J^{S_1}\rangle S_1$, we have $K \not\leq H$. Therefore the results of Lemma 19.3 apply to K. In particular, we recall that

$$Y_K = \langle \Omega_1(Z(S_0))^K\rangle.$$

LEMMA 19.8. *Define $X = F^*(K/C_K(Y_K))$. Then the possibilities for X and p are as follows:*

(i) $X/Z(X) \cong \mathrm{PSL}_2(p^a)$ *all* $p \geq 2$;
(ii) $X/Z(X) \cong \mathrm{PSU}_3(p^a)$ *all* $p \geq 2$;
(iii) $p = 2$ *and* $X/Z(X) \cong {}^2\mathrm{B}_2(2^{2a+1})$;
(iv) $p = 2$ *and* $X/Z(X) \cong \mathrm{PSL}_3(2^a)$ *and* $N_{S_1}(J)J$ *has an element which swaps the two maximal parabolic subgroups of* $F^*(K/C_K(Y_K))$ *containing* $(S_1 \cap J)C_K(Y_K)/C_K(Y_K)$;
(v) $p = 2$ *and* $X \cong \mathrm{Sp}_4(2^a)'$ *and* $N_{S_1}(J)J$ *has an element which swaps the two maximal parabolic subgroups of* $F^*(K/C_K(Y_K))$ *containing* $(S_1 \cap J)C_K(Y_K)/C_K(Y_K)$;
(vi) $p = 2$ *and* $X \cong \mathrm{Alt}(2^a + 1)$ *with* $a \geq 3$ *(two possible actions on* Y *for* $\mathrm{Alt}(9)$ *both with* $|Y| = 2^8$*)*;
(vii) $p = 3$ *and* $X \cong \mathrm{Alt}(9)$ *or* $\mathrm{Alt}(3^a + 1)$ *with* $a \geq 2$;
(viii) $p = 3$ *and* $X \cong 2{\cdot}\mathrm{Alt}(9)$;
(ix) $p = 3$ *and* $X \cong \mathrm{Sp}_6(2)$;
(x) $p = 3$ *and* $X \cong 2{\cdot}\mathrm{Sp}_6(2)$.

PROOF. By Lemma 2.8, $\overline{K}N_S(\overline{K})$ is p-minimal. If $\overline{K}$ is a group of Lie type in characteristic p we get cases (i)-(v) by Lemma A.18. In the remaining cases we have, by Lemma 19.3 (vi) and (vii), that Y_K is either a dual F-module or a dual of a cubic $2F$-module. As a dual F-module in particular is also a dual $2F$-module by Lemma C.19 we may apply Theorem C.24. The cases (vi)-(x) follow from Theorem C.24. The fact that $X \not\cong {}^2\mathrm{G}_2(3^{2a+1})$ comes from Lemma C.25. □

The candidates for X given in Lemma 19.8 give us the possibilities for $\overline{J}$ and we now consider these in turn.

LEMMA 19.9. *We have* $\overline{J} \not\cong \mathrm{PSL}_3(2^a)$, $\mathrm{SL}_3(2^a)$ *or* $\mathrm{Sp}_4(2^a)'$.

PROOF. Assume that we have such a component. Then, by Lemma 19.8 (iv) and (v), $N_{S_1}(J)$ has an element which exchanges the two maximal parabolic subgroups of $\overline{J}$ containing $\overline{S_1 \cap J}$. Suppose for a moment that $\overline{J} \not\cong \mathrm{Alt}(6)$. Since $\overline{Q}$ is elementary and normalized by $N_{\overline{S}_1}(\overline{J})$, Lemmas D.2, D.3 and D.4 imply that $\overline{Q}$ is contained in $O_2(\overline{U})$ where $\overline{U}$ projects to a maximal 2-local subgroup of $\overline{JQ}/C_{\overline{Q}}(\overline{J}) \cong \mathrm{PSL}_3(2^a)$, $\mathrm{SL}_3(2^a)$ or $\mathrm{Sp}_4(2^a)$. Let U be the preimage of $\overline{U}$. Since Q is weakly closed in S_0, we infer that $U \leq N_G(Q) \leq H$ and this is a contradiction as $\langle U^{S_0}\rangle = K$. Now for $\overline{J} \cong \mathrm{Alt}(6)$, we have $\overline{S_1 \cap J} \cong \mathrm{Dih}(8)$ and $\overline{Q}$ must normalize both parabolic subgroups of $\overline{J}$ for otherwise $\overline{Q} \cap \overline{S_1 \cap J}$ has an element of order 4. Thus $\overline{QJ} = \overline{J}$ or $\overline{QJ} \cong \mathrm{Sp}_4(2) \cong \mathrm{Sym}(6)$. In any case $\overline{Q}$ is normalized by a parabolic subgroup of $\overline{J}$ and so the above argument goes through unchanged. This completes the lemma. $\square$

LEMMA 19.10. *Suppose p is odd and $\overline{J}/Z(\overline{J})$ is not a simple group of Lie type defined in characteristic p. Then $P = JS_1$, $O_p(P)$ is elementary abelian and $[O_p(P), O^p(J)] \le Y$. In particular S normalizes J.*

PROOF. By Lemma 19.8, we have $p = 3$ and $\overline{J}$ is one of the groups listed in parts (vii)-(x) of Lemma 19.8. By construction $Y_K \le Y$ and Y_K is an irreducible K-module by Lemma 19.3 (v).

Since $[O_p(K), Q, Q] \le R < Y_K$, Q acts quadratically or trivially on $O_p(K)/Y_K$.

By definition we have $J/C_P(Y)$ is quasisimple and we know that $C_P(Y) \le N_H(Q)$. Hence $[J,Q]O_p(K)/O_p(K)$ is quasisimple. Suppose there is a non-central KQ-chief factor in $O_p(K)/Y_K$. Then by Lemma C.12

$$|QO_p(K)/O_p(K)| = |Q : C_Q(Y_K)| = 3.$$

Thus, as K is non-soluble, Y_K does not have order 9,

$$Y_K \not\le Q, \text{ and } Y_K \text{ induces transvections on } Q/R.$$

Hence, by Lemma D.20,

$$F^*(H) \cong \mathrm{PSp}_{2n}(3) \text{ and } |Y_K : Y_K \cap Q| = 3 = |R|.$$

As Y_K centralizes a subgroup of index 3 in Q, we get that $Y_K \cap Q$ has order 9. Hence Y_K has order 3^3 and we have a contradiction as $\mathrm{SL}_3(3)$ has order coprime to 5 but 5 divides $|\overline{J}|$.

So we have shown that $O^p(J)$ acts trivially on $O_p(K)/Y_K$. As Y_K is an irreducible K-module, we now have $Y_K \cap \Phi(O_p(P)) = 1$. As $R \le Y_K$ and $C_{O_p(K)}(K)$ is normalized by Q, we have that $C_{O_p(K)}(K) = 1$ which implies $\Phi(O_p(K)) = 1$. Since $O_p(K)$ is abelian, $O_p(P) \le O_p(K)$ and P has characteristic p, $O_p(P) = O_p(K)$. Since $J \cap Q \not\le O_p(P)$, for $s \in S_1$, $(K \cap Q)^s \not\le O_p(P) = O_p(K)$. If $\overline{J} \ne \overline{J}^s$, this is impossible. Hence $J^{S_1} = J$ and we are done. $\square$

LEMMA 19.11. *Suppose p is odd. Then $\overline{J}/Z(\overline{J})$ is a simple group of Lie type defined in characteristic p.*

PROOF. Otherwise, again $p = 3$ and $\overline{J}$ is one of the groups in listed in parts (vii)-(x) of Lemma 19.8. Consider $N = JS_1 \cap F^*(H) \ge S$ by Lemma 19.10. Then, by [**27**, Theorem 2.6.7], N is normal in a parabolic subgroup of $F^*(H)$. If $\overline{J}/Z(\overline{J}) \cong \mathrm{Sp}_6(2)$ or $\mathrm{Alt}(3^a+1)$ with $a \geq 2$, then $\overline{N}/Z(\overline{N}) \cong \mathrm{PSp}_4(3)$ or $\mathrm{Alt}(3^a)$ respectively. Since $\overline{N}$ must be a group of Lie type in characteristic 3 defined over a field of order at least 3^e, we see that $\overline{J}/Z(\overline{J}) \cong \mathrm{Sp}_6(2)$ and that $e = 1$. The only other candidate is $\overline{J}/Z(\overline{J}) \cong \mathrm{Alt}(9)$ and in this case $\overline{N}S$ is soluble but not 3-closed. Hence again we get that $e = 1$. In particular

$$|R| = 3 \text{ and } \overline{J}/Z(\overline{J}) \cong \mathrm{Sp}_6(2) \text{ or } \mathrm{Alt}(9).$$

In $\overline{J}$ we can find a subgroup $2^3{:}7$ and so Lemma 2.23 implies that $|Y| \geq 3^7$. From the structure of $\overline{KS}$, we see that $S/O_p(K) \cong 3 \wr 3$. Hence, as $JS/C_P(YS) \cong \overline{J}$, we have

$$|S| \geq |Y||S/O_p(K)| = 3^7 \cdot 3^4 = 3^{11}.$$

Furthermore,

$$|Q : Q \cap O_3(K)| = |QO_3(K)/O_3(K)| \leq 27$$

and so, as $O_3(K)$ is abelian by Lemma 19.10 and Q is extraspecial, Q/R has order at most 3^6. If $|Q| = 3^5$, then, as $|\mathrm{Sp}_4(3)|_3 = 3^4$, we obtain $|S| \leq 3^9$, a contradiction to $|S| \geq 3^{11}$. Therefore Q has order 3^7, $|O_3(K) \cap Q| = 3^4$ and $|S/Q| \geq 3^4$. Now Lemmas A.2 and D.1 show that $F^*(H) \cong \mathrm{PSp}_8(3)$ and $|S| = 3^{16}$. Since Y centralizes $O_3(K) \cap Q$, we have $|YQ/Q| \leq 3^6$. Hence $|YQ| \leq 3^{13}$. By Lemma 19.10 $O_3(K)/Y \leq Z(S/Y)$. Hence $O_3(K)Q/QY \leq Z(S/QY)$. As $N_{\mathrm{PSp}_8(3)}(QY)/QY$ contains $\mathrm{SL}_3(3)$, we see that $|Z(S/QY)| = 3$ and so $|O_3(K)/Y|$ has order at most three. As $O_3(K)Q$ has index 3 in S this implies $|S| \leq 3^{15}$, a contradiction as $3^{16} = |S|$. □

LEMMA 19.12. *We cannot have $p = 2$ and $\overline{J} \cong \mathrm{Alt}(2^a + 1)$, $a \geq 3$.*

PROOF. We have $\overline{J \cap H} \cong \mathrm{Alt}(2^a)$ and so using the fact that $(K^S \cap F^*(H))S$ is normal in a parabolic subgroup of $F^*(H)$ by [**27**, Theorem 2.6.7], we deduce

$$a = 3 \text{ and } p = 2 = |R|.$$

Furthermore, we see that S normalizes $K \cap F^*(H)$ and so S normalizes K. Thus $Y_K = Y_{KS}$ and $|Y_K| = 2^8$ by Lemma 19.3 (v).

If $Y_K \leq Q$, then $[Y_K, Q] = R$ has order 2. Hence the non-trivial composition factor of Y_K is the natural K-module by Lemma C.21 and so Q operates as a transposition. Thus $[Y_K, Q]$ is normalized by S and $C_{\overline{J}}(\overline{Q}) \cong \mathrm{Alt}(7)$. But then the preimage N of $C_{\overline{J}}(\overline{Q})$ is contained in H, a contradiction as $J = \langle N, S_1 \cap J\rangle$. Thus $Y_K \not\leq Q$, and additionally $C_{Y_K}(Q) = R$ and $[Y_K, Q, Q] = R$ by Lemma 19.3(vii). Thus Lemma C.29 implies that $|[Y_K, Q]| > 2|\overline{Q}|$. On the other hand, we have

$|\overline{Q}| = |Q : Q \cap O_2(K)|$ and $Q \cap Y_K \leq Z(Q \cap O_2(K))$. Hence, as Q is extraspecial, we have

$$|[Y_K, Q]| \leq |Y_K \cap Q| \leq 2|Q : Q \cap O_2(K)| = 2|\overline{Q}| < |[Y_K, Q]|,$$

which is a contradiction. □

Summarising, by Lemmas 19.8, 19.12 and 19.9, $\overline{J}$ is a group of Lie type defined in characteristic p. This means

$$\overline{J} \cong \mathrm{SL}_2(p^a), \mathrm{PSL}_2(p^a), \mathrm{SU}_3(p^a), \mathrm{PSU}_3(p^a) \text{ or } {}^2\mathrm{B}_2(2^a).$$

LEMMA 19.13. *If $E(\overline{P}) \neq 1$, then $E(\overline{P}) = \overline{J}$ is quasisimple. In particular, $P = JS_1$.*

PROOF. Assume the result is false. Let $S_2 = S_1 \cap \langle J^{S_1} \rangle$ and define

$$N = N_{\langle J^{S_1} \rangle}(S_2 C_P(Y)).$$

Then $\overline{S}_2 \in \mathrm{Syl}_p(E(\overline{P}))$ and $\overline{N} = N_{E(\overline{P})}(\overline{S_2})$. Furthermore $(N \cap J)/(S_2 \cap J)$ is a cyclic group of order $p^a - 1$, $(p^a - 1)/2$, $p^{2a} - 1$, $(p^{2a} - 1)/3$ or $2^a - 1$ according to Lemma A.19 and $NS < P$. This forces $NS \leq H$. Since $[N, S]S \leq F^*(H)$, we have that $[N, S]S$ is normal in a parabolic subgroup of $F^*(H)$ in which S is not normal. Since $[N, S]S/S_2$ is soluble, we deduce $p = 2$ or $p = 3$ and $|R| = p$. Furthermore, $[N, S]S_2/S_2$ is either an elementary abelian 3-group or an elementary abelian 2-group when $p = 2$ or 3 respectively. If $p = 3$, we get $3^a - 1 = 2$, $(3^a - 1)/2 = 2$, $3^{2a} - 1 = 2$ or $(3^{2a} - 1)/3 = 2$, respectively, which all have no solution. Let $p = 2$, then as $\overline{J}$ is non-soluble we have $a > 1$. Now some of $2^a - 1$ or $2^{2a} - 1$ must be equal to 3. We deduce that $a = 2$ and

$$\overline{J} \cong \mathrm{SL}_2(4) \text{ with } p = 2.$$

In this case Q induces a group of order at most 2^2 on $\overline{J}$. Set $S_J = J \cap S_2$ and $Q_J = Q \cap S_J$. Suppose that $Q \not\leq S_2$. Then $\overline{Q}_J \neq 1$. If $Q \leq S_2$ then as Q is weakly closed in S, N normalizes Q and as a consequence, $\overline{Q} = \overline{\Omega_1(Z(S_2))}$. In both cases $\overline{Q}_J \neq 1$.

Let $\overline{J}^*$ be a component of $\overline{P}$ with $\overline{J}^* \neq \overline{J}$ and let J^* be its preimage. Now we consider $[Y, Q_J] \leq Q$. As Q normalizes $[Y, Q_J]$ and $|R| = 2$, we see that $R \leq [Y, Q_J]$ and, as J^* normalizes $[Y, Q_J]$ so $Y_{J^*} \leq [Y, Q_J] \leq Q$. Since

$$|QC_{QJ^*}(Y_{J^*})/C_{QJ^*}(Y_{J^*})| \leq 2^2,$$

we see that $|Q : Q \cap C_Q(Y_{J^*})| \leq 2^2$ and so $|Y_{J^*}| \leq 2^3$. But then $\overline{J}^*$ cannot act on such a group, a contradiction. Hence $\overline{P}$ has a unique component. □

LEMMA 19.14. *We do not have $\overline{J}/Z(\overline{J}) \cong \mathrm{PSU}_3(p^a)$ or ${}^2\mathrm{B}_2(2^{2a+1})$.*

Proof. Suppose that $\overline{J} \cong \mathrm{SU}_3(p^a)$, $\mathrm{PSU}_3(p^a)$, ${}^2\mathrm{B}_2(2^{2a+1})$. Assume that $\overline{Q} \not\leq \overline{J}$. Then $\overline{Q}$ contains some element g, which induces an outer automorphism on $\overline{J}$. Then, by Theorem A.11, $\overline{J}/Z(\overline{J}) \cong \mathrm{PSU}_3(p^a)$ and g induces a field automorphism on $\mathrm{GF}(p^{2a})$. If p is odd, then g does not act quadratically on $Z(\overline{S_1 \cap J})$, which contradicts the fact that $\overline{Q}$ is abelian. Hence $p = 2$ and $\overline{J}/Z(\overline{J}) \cong \mathrm{PSU}_3(2^a)$. Now $[\overline{S_1 \cap J}, \overline{Q}] \leq \Omega_1(\overline{S_1 \cap J}) = Z(\overline{S_1 \cap J})$ and so $\overline{Q}$ centralizes $(\overline{S_1 \cap J})/Z(\overline{S_1 \cap J})$, but in fact it induces an automorphism of order 2. Hence $\overline{Q} \leq \overline{J}$.

So we have that $Q \leq S \cap J$ and consequently $J \cap H$ normalizes Q as Q is weakly closed in S. As, by Lemma A.19 (ii) and (iii), $J \cap H$ acts irreducibly on $(\overline{S_1 \cap J})/Z(\overline{S_1 \cap J})$ and $\overline{S_1 \cap J}$ is non-abelian, we see that $\overline{Q} = Z(\overline{S_1 \cap J})$. If $p = 2$, using Lemmas C.26 and C.27 we obtain $[Y, Q, Q] = 1$. But then $[Y, Q] = R$ and $Y \leq Q$ which means that Y is a dual F-module with offender $\overline{Q} = Z(\overline{S_1 \cap J})$. We have a contradiction using Lemma C.21 and then C.23. Hence p is odd and $\overline{J}/Z(\overline{J}) \cong \mathrm{PSU}_3(p^a)$. Since p is odd, $\overline{Q}$ commutes with an involution $i \in \overline{J}$. Thus $Y = [Y, i] \times C_Y(i)$ is a Q-invariant decomposition. Since $i \notin Z(\overline{J})$, we have $Y > [Y, i] \neq 1$. Hence $[Y, i] \cap R \neq 1 \neq C_Y(i) \cap R$. If $C_Y(i) \not\leq Q$, then $[C_Y(i), Q] \not\leq R$ and $C_Y(i) > [C_Y(i), Q, Q] = R$, a contradiction. Similarly $[Y, i] \leq Q$. But then $Y \leq Q$, a contradiction as Y is not a dual F-module by Lemma C.23. □

Lemma 19.15. *Neither of the following configurations can occur.*

(a) $\overline{J}/Z(\overline{J}) \cong \mathrm{PSL}_2(p^a)$*; or*
(b) $p \in \{2, 3\}$*,* $Y \leq Q$ *has order* p^2 *and* $\overline{P} \cong \mathrm{SL}_2(p)$*.*

Proof. Deny the claim and assume that either (a) or (b) holds.

We first show that in case (a), we have $\overline{Q} \leq \overline{J}$, $Y \leq Q$ and Y is the natural $\overline{J}$-module. After this we go on to handle both cases (a) and (b) simultaneously.

Suppose that $\overline{Q} \not\leq \overline{J}$. Then, as $\overline{Q}$ acts quadratically on a Sylow p-subgroup of $\overline{J}$ and some element from $\overline{Q}$ acts as a field automorphism on $\overline{J}$, we must have $p = 2$. Furthermore, $[J \cap H, Q]S$ is a normal subgroup in a parabolic subgroup of $F^*(H)$ and so $[\overline{J \cap H}, \overline{Q}]/(S \cap J)$ has order 3 and $|R| = 2$. Since $\overline{J} \cong \mathrm{SL}_2(2^{2b})$, we have $|[\overline{J \cap H}, \overline{Q}]| = 2^b + 1 = 3$ and hence $b = 1$. Thus $\overline{J} \cong \mathrm{SL}_2(4)$ and $\overline{JQ} = \overline{JS_1} \cong \mathrm{Sym}(5)$. As Y is irreducible by Lemma 19.3, we also have $|Y| = 16$. If Y is the permutation module (otherwise known as the $\mathrm{O}_4^-(2)$-module), then, as $\overline{Q} \not\leq \overline{J}$ and $\overline{Q}$ is elementary abelian, we reach the contradiction

$$2 = |R| = |C_Y(Q)| = 4$$

using Lemma 19.3 (iii). So we have that Y is the natural $\mathrm{SL}_2(4)$-module for $\overline{J}$. Let $N = N_J(S_1 \cap J)$. Then $[N, Q]Q \leq H$ and, in particular, $(Q \cap J)^N \leq S$. Thus $\overline{S} \cong \mathrm{Dih}(8)$. Since $|\overline{Q}| = 4$, Y centralizes a subgroup of index four in $Q/Z(Q)$ and so by Lemma D.19 we have $F^*(H) \cong \mathrm{PSL}_n(2)$ or $\mathrm{PSU}_n(2)$ and furthermore $[Y, Q] = Z_2(S)$ by Lemma D.21. Now $[Y, Q, O_2(P)S] = R$, whereas when we calculate in Y as a $\mathrm{GL}_2(4)$-module we see that $[Y, Q, S]$ has order 4. This shows that

$$\overline{Q} \leq \overline{J}.$$

Because $\overline{Q} \leq \overline{J}$, Q is weakly closed in S_0 and $N_{\overline{J}}(\overline{S_1 \cap J})$ acts irreducibly on $\overline{S_1 \cap J}$, we have that $\overline{Q}$ is a Sylow p-subgroup of $\overline{J}$.

Suppose that $Y \not\leq Q$. Then Lemma 19.3 (vii) shows that $\overline{Q}$ is a strictly cubic 2-offender on Y. Thus we may apply Lemma C.28. As on the natural module a Sylow p-subgroup acts quadratically, we obtain that either $|Y| = p^{2a}$ and Y is the orthogonal $\Omega_4^-(p^{a/2})$-module for $\overline{J} \cong \mathrm{SL}_2(p^a)$ or $|Y| = p^{3a}$, p odd and Y is the $\Omega_3(p^a)$-module for $\overline{J}/Z(\overline{J}) \cong \mathrm{PSL}_2(p^a)$. In both cases $R = [Y, Q, Q]$ has order p^e. Hence $\overline{J} \cong \mathrm{SL}_2(p^{2e})$ in the first case and $\overline{J}/Z(\overline{J}) \cong \mathrm{PSL}_2(p^e)$ in the second case. Furthermore, as $\overline{S}$ normalizes $\overline{J}$ and centralizes R, we either have

$$\overline{S} = \overline{Q} \leq \overline{J} \text{ or } p = 2 \text{ and } |SJ/J| = 2.$$

Suppose first that $[Y, Q] = Z_2(S)$. Then, as Y is either the $\Omega_4^-(p^e)$-module or the $\Omega_3(p^e)$-module, we have $|[Y, Q]/R| = |Z_2(S)|/R = |\overline{Q}|$. Hence $Q \cap O_p(P) = C_Q(Z_2(S))$. But then $Y \leq C_{S_0}(C_Q(Z_2(S))))$.

If $Y \not\leq S$, then Y induces some automorphism on $F^*(H)$, which centralizes R. By Theorem A.11 we get that it has to induce a graph automorphism in case of $F^*(H) \cong \mathrm{PSL}_n(p^e)$ and a field automorphism in case of $F^*(H) \cong \mathrm{PSU}_n(p^e)$. But in both cases Y would not centralize $Z_2(S)$. Hence we even have that $Y \leq C_S(C_Q(Z_2(S))))$ and we conclude that $Y = V(Q, S)$. This contradicts Proposition 7.7. Thus we have

$$[Y, Q] \neq Z_2(S).$$

Suppose that $\overline{S} = \overline{Q}$. Then $[Y, Q, Q] = [Y, S, S] = R$ and so $[Y, S] \leq Z_2(S)$. Then, as $|[Y, S]| \geq p^{2e}$, by Lemma D.21 we must have $F^*(H) = \mathrm{PSU}_n(p^e)$ or $\mathrm{PSL}_n(p^e)$ and this contradicts Lemma D.1, as $|[Y, Q/R]| < p^{2e}$. Hence we have shown $\overline{S} \neq \overline{Q}$ and $[Y, Q] \neq Z_2(S)$. Furthermore, we have $|Y| = 2^{4e}$ as well as $|[Y, Q]| = 2^{3e}$.

Since $[N_J(S\cap J),S]S \le F^*(H)$ and $|[N_J(S\cap J),S] : S\cap J| = 2^e+1$ is inverted by S. The structure of parabolic subgroups of $F^*(H)$ implies that $2^e+1=3$. In particular, $e=1$, $\overline{J}\cong \mathrm{SL}_2(4)$ and $\overline{JS}\cong \mathrm{O}_4^-(2)\cong \mathrm{Sym}(5)$. By Lemma 19.5 we have that $Y\le S$. Then $1\neq YQ/Q \le Z(S/Q)$ and $[Y,Q]/R$ has order 4. Thus Lemma D.19 implies that $F^*(H)\cong \mathrm{PSL}_n(2)$ or $\mathrm{PSU}_n(2)$. But then $[Y,Q]=Z_2(S)$, a contradiction. Therefore $Y\le Q$. Now Lemma 19.3 (vi) implies that Y is an F-module with offender $\overline{Q}$ and so Lemma C.28 shows that Y is the natural $\overline{J}$-module.

We now continue assuming that both (a) and (b) hold. Since $\overline{Q}\in \mathrm{Syl}_p(\overline{J})$ and $[Y,Q]=R=C_Y(S)$, we have $\overline{S}=\overline{Q}$. In particular, $p^e=|R|=[Y,Q]=p^a$, $|Y|=p^{2e}$ and $Y\le Z_2(S)$. Suppose that $Y=Z_2(S)$. Then as $P\not\le H$ we have that $R^P\neq R^H\cap Y$. Application of D.22 shows that $F^*(H)\cong \mathrm{PSp}_{2n}(p^e)$, p odd. In particular by Theorem A.11 we have that $C_{S_0}(Z_2(S))=C_S(Z_2(S))$ and so $O_p(P)\le S$. Then

$$O_p(P)\le C_S(Z_2(S))=O_p(P(S,L)).$$

On the other hand, $C_{S_0}(Z_2(S))\le C_P(Y)$ and, as $O_p(P)\in \mathrm{Syl}_p(C_P(Y))$, we have $O_p(P)=O_p(P(S,L))$. We conclude from Lemma D.23 that

$$V(Q,S)=Z(O_p(P(S,L)))$$

is normalized by P. Thus $P\le N_G(V(Q,S))\le H$ by Proposition 7.7, a contradiction. Hence $Y\neq Z_2(S)$. It follows from Lemma D.21 that $|Z_2(S)|=p^{3e}$ and that

$$F^*(H)\cong \mathrm{PSL}_n(p^e) \text{ or } \mathrm{PSU}_n(p^e).$$

Let $y\in Y\setminus R$. Then, as P acts transitively on the elements of $Y^\#$, $y=r^x$ for some $x\in P\setminus(P\cap H)$. If there is some $h\in H$ with $y^h\in R$. Then $xh\in H$ and we obtain $P\le \langle x, P\cap H\rangle \le H$, a contradiction. In particular, we have $N_{F^*(H)}(Y)\le N_{F^*(H)}(R)$. Since $Y^x=Y$, $R\le Y\le Q^x$ and so $C_{Q^x}(Y)\le N_H(Q)$. However $Q^x=O_p(C_G(y))$ is normalized by $C_H(y)$ and so $C_{Q^x}(Y)=Q^x\cap H$ is also normalized by $C_H(y)$. We have $Y=Z(C_Q(Y))$, therefore $Y=Z(C_{Q^x}(Y))$ is normalized by $C_H(y)$. We now have $C_H(y)\le N_G(R)$. Now note that $y\in V(Q,S)$ which is an orthogonal module for $P(S,L)/O_p(P(S,L))\cong \Omega_4^\pm(p^e)$. Since y is not H-conjugate to an element of R, it certainly is not $P(S,L)$-conjugate to an element of R. Hence y corresponds to a singular vector in $V(Q,S)$. Now we see that $C_{P(S,L)}(y)$ does not normalize R and we have a contradiction. $\square$

Proof of Proposition 19.2. Assume that Hypothesis 19.1 holds. If $p = 2$ and $C_H(z)$ is soluble for 2-central involution in H, then as we have already remarked $F^*(G) \cong \mathrm{PSL}_4(3)$ and $F^*(H) \cong \mathrm{PSU}_4(2)$. So assume that if $p = 2$, then $C_H(z)$ is soluble is not soluble for all $z \in Z(S)$. If $E(\overline{P}) = 1$, then Lemma 19.6 implies that the Lemma 19.15 holds, a contradiction. Thus $E(\overline{P}) \neq 1$. The possibilities for $\overline{J}/Z(\overline{J})$ are enumerated by Lemma 19.8. The candidates in parts (iv) and (v) are eliminated in Lemma 19.9, those in parts (vii) to (x) in Lemma 19.11, and those in (vi) by Lemma 19.12. All that remains are the rank one groups of Lie type in parts (i), (ii) and (iii) of Lemma 19.8. These are shown to be impossible in Lemmas 19.14 and 19.15. In conclusion, Hypothesis 19.1 is only satisfied in the exceptional $p = 2$ configuration. □

20. Proof of Theorem 4

In this section we prove Theorem 4.

Proof of Theorem 4. Suppose that G satisfies the hypothesis of Theorem 4. Then $C_H(z) = C_G(z)$ for every p-central element of H. Since H is a group of Lie type in characteristic p, $C_H(z)$ has characteristic p. It follows from Lemma 2.1 (iii) that G is of parabolic characteristic p. Furthermore G satisfies Hypothesis 6.1. If G satisfies Hypothesis 6.2, then Hypothesis 19.1 holds and we may apply Proposition 19.2 to obtain the first possibility in Theorem 4(i). So assume that Hypothesis 6.2 is not satisfied. Then $F^*(H) \cong {}^2\mathrm{F}_4(2^{2e+1})$, $\mathrm{F}_4(2^e)$, $\mathrm{Sp}_{2n}(2^e)$, $n \geq 3$, $\mathrm{G}_2(3^e)$, $\mathrm{PSL}_3(p^e)$ or $\mathrm{PSL}_4(2)$. The first three types are examined in Proposition 13.8 and so yield a contradiction. When $F^*(H) \cong \mathrm{G}_2(3^e)$, then Proposition 15.1 yields a contradiction. Finally, if $F^*(H) \cong \mathrm{PSL}_3(p^e)$, p odd, Proposition 9.1 yields Theorem 4 (ii).

Suppose now that $F^*(H) \cong \mathrm{PSL}_4(2)$. Then Proposition 14.3 implies that $F^*(G) \cong \mathrm{Alt}(9)$ or $\mathrm{Alt}(10)$. The second case is included in Theorem 4(i). The case $F^*(H) \cong \mathrm{Alt}(9)$ does not show up, as in this case H is the only maximal subgroup of G which contains S_0, which contradicts the assumption of Theorem 4 that there is a 2-local subgroup M of G containing S_0, which is not contained in H. □

21. Proof of Theorem 1

To complete the proof of Main Theorem 2, Theorems 2, 3 and 4 imply that we have to consider the situation in which, for all $1 \neq E \trianglelefteq S_0$, we have that $N_G(E) \leq H$. Ideally we would like to show that H is strongly p-embedded in G in this situation, however we can only do this at the present time under the hypothesis that G is of local characteristic p. Thus the objective of this section is to prove Theorem

1. Because of the results in Theorem 4.7, we may assume that $F^*(H)$ is a group of Lie type in odd characteristic p and of Lie rank two. For $F^*(H) \cong \mathrm{PSL}_3(p^e)$, p odd, and $F^*(H) \cong \mathrm{G}_2(3^e)$ we have shown in Proposition 9.1, Proposition 15.1, respectively, that Theorem 1 holds. So we only have to treat the cases with $F^*(H)$ one of

$$\mathrm{PSp}_4(p^e), \mathrm{PSU}_4(p^e), \mathrm{PSU}_5(p^e), {}^3\mathrm{D}_4(p^e), p \text{ odd, or } \mathrm{G}_2(p^e), p \geq 5.$$

The overall development of this section to a certain extent follows the proof of Theorem 4.7 in [**66**], but, because of the very restricted structure of $F^*(H)$, at various stages we can adopt more elementary arguments.

To make things precise, in this section we work under the following hypothesis.

Hypothesis 21.1. *Assume that Hypothesis* 6.1*(i), (ii) and (iii) hold and in addition assume*

(i) *G is of local characteristic p;*
(ii) *for all $1 \neq E \trianglelefteq S_0$, we have that $N_G(E) \leq H$; and*
(iii) *$F^*(H) \cong \mathrm{PSp}_4(p^e)$, $\mathrm{PSU}_4(p^e)$, $\mathrm{PSU}_5(p^e)$, ${}^3\mathrm{D}_4(p^e)$, p odd, or $\mathrm{G}_2(p^e)$, $p \geq 5$.*

We begin by repeating some of the general setup as developed in [**66**]. For a p-subgroup U of H we define the set

$$\mathcal{M}(U) = \{M \mid M \not\leq H, O_p(M) \neq 1, U \leq M\}.$$

Thus Hypothesis 21.1 (ii) states that

$$\mathcal{M}(S_0^h) = \emptyset \text{ for all } h \in H.$$

Also, if $N_G(U) \not\leq H$ for some non-trivial p-subgroup $U \leq H$, then $\mathcal{M}(U) \neq \emptyset$.

We define a relation on $\mathcal{M}(U)$ as follows: let J, K be in $\mathcal{M}(U)$, then $J \sqsubset K$ if and only if there is Sylow p-subgroup T of $K \cap H$ such that $T \cap J$ is a Sylow p-subgroup of $J \cap H$ but $T \cap J \neq T$. Notice that $\sqsubset$ is not a partial order. Nevertheless, we say that $K \in \mathcal{M}(U)$ is maximal with respect to $\sqsubset$ provided there are no members $L \in \mathcal{M}(U)$ with $K \sqsubset L$. Now define

$$\mathcal{M}_{\max}(U) = \{K \mid K \in \mathcal{M}(U) \text{ and } K \text{ is maximal with respect to } \sqsubset\}.$$

Further we define

$$\mathcal{P}(U) = \{K \mid K \in \mathcal{M}_{\max}(U) \text{ and } K \text{ minimal by inclusion}\}.$$

If $\mathcal{M}(U)$ is non-empty for some non-trivial p-subgroup $U \le H$, then also $\mathcal{P}(U)$ is non-empty. Set

$$\mathcal{P} = \bigcup_{\substack{1 \ne U \le H \\ U \text{a } p\text{-group}}} \mathcal{P}(U).$$

We will show that up to the two exceptional cases in Theorem 1 we have $\mathcal{P} = \emptyset$. To demonstrate this, it is enough to show that $\mathcal{P}(C_G(t)) = \emptyset$ for all $t \in H$, $o(t) = p$ for then $C_G(t) \le H$ and so, as $N_G(S_0) \le H$, H is strongly p-embedded by [**26**, Proposition 17.11].

We first present some general results about the structure of $K \in \mathcal{P}$. The first result should be compared with [**66**, Lemma 1.2].

Lemma 21.2. *Suppose that $K \in \mathcal{M}_{\max}(U)$ and $T \in \mathrm{Syl}_p(H \cap K)$ with $U \le T$. Then*

(i) *$N_G(T) = N_H(T)$, T is a Sylow p-subgroup of K and $T \notin \mathrm{Syl}_p(G)$.*
(ii) *If V is a non-trivial normal p-subgroup of K, then K contains a Sylow p-subgroup of $N_G(V)$.*
(iii) *If $1 \ne C$ is characteristic in T, then $N_G(C) \le H$.*
(iv) *If $K \in \mathcal{P}(U)$, then K is a p-minimal group.*

Proof. We first prove (i). By Hypothesis 21.1 (ii), T is not a Sylow p-subgroup of H. Hence $|N_H(T)|_p > |T|$. Assume $N_G(T) \not\le H$. Then $N_G(T) \in \mathcal{M}(U)$. Let $T_1 \in \mathrm{Syl}_p(N_H(T))$. Then $T_1 \cap K = T_1 \cap K \cap H = T < T_1$. This shows $K \sqsubset N_G(T)$, a contradiction. Hence $N_G(T) \le H$ and, in particular, $T \in \mathrm{Syl}_p(K)$.

For the proof of part (ii), set $M = N_G(V)$. Then $K \le M$, and so $M \in \mathcal{M}(U)$. Furthermore, $T \in \mathrm{Syl}_p(N_H(V))$ for otherwise $K \sqsubset N_G(V)$ which is impossible. Suppose $M_1 \in \mathcal{M}(U)$ with $M \sqsubset M_1$. Then there is some $T_1 \in \mathrm{Syl}_p(H \cap M_1)$ such that $T_1 \cap N_H(K) < T_1$ and $T_1 \cap N_H(V) \in \mathrm{Syl}_p(N_H(V))$. As $T \in \mathrm{Syl}_p(N_H(V))$, there exists $g \in N_H(V)$ such that $T = T_1^g \cap N_H(V)$. Then $T_1^g \in \mathrm{Syl}_p(H \cap M_1^g)$ and $T = T_1^g \cap N_H(V) < T_1^g$. Hence $K \sqsubset M_1^g$, a contradiction. This implies $M \in \mathcal{M}_{\max}(U)$ and so $T \in \mathrm{Syl}_p(M)$ by part (i).

Next we prove (iii). As C is characteristic in T and $T \notin \mathrm{Syl}_p(H)$, we have that $|N_G(C)|_p > |T|$. Suppose $N_G(C) \not\le H$. Then $N_G(C) \in \mathcal{M}(U)$. By (i), $N_{N_G(C)}(T) \le N_H(C)$. Now we choose $T_1 \in N_H(C)$, with $T \le T_1$. Then $T_1 > T$. Furthermore $T_1 \cap K = T < T_1$ and so $K \sqsubset N_G(C)$, a contradiction. Hence $N_G(C) \le H$.

Finally, we prove part (iv). By (i), T is not normal in K. Let $T \le L_1 < K$. We show $L_1 \le H$. Otherwise $L_1 \in \mathcal{M}(U)$. Assume that there exists $L_2 \in \mathcal{M}(U)$ with $L_1 \sqsubset L_2$. Then there is a Sylow p-subgroup T_2

of $L_2 \cap H$ such that $T_1 = T_2 \cap H \cap L_1$ is a Sylow p-subgroup of $H \cap L_1$ and $T_1 < T_2$. By Sylow's Theorem, there exists $g \in L_1 \cap H$ with $T = T_1^g$. Then we get that $K \sqsubset L_2^g$, contradicting $K \in \mathcal{M}_{\max}(U)$. Hence we have that $L_1 \in \mathcal{M}_{\max}(U)$. As $K \in \mathcal{P}(U)$, this is not possible. We have shown that $L_1 \leq H$ and so $H \cap K$ is the unique maximal subgroup of K containing T, this means that K is a p-minimal group. $\square$

Lemma 21.3. *Let $K \in \mathcal{P}$, then $F^*(K) = O_p(K)$.*

Proof. By Lemma 21.2(ii), K contains a Sylow p-subgroup of $M = N_G(O_p(K))$. In particular, $O_p(M) \leq K$ and so $O_p(K) = O_p(M)$. As G is of local characteristic p, we have $C_G(O_p(M)) \leq O_p(M)$ and so $F^*(K) = O_p(K)$. $\square$

The next lemma is taken from [**66**, Lemma 1.4].

Lemma 21.4. *Let $K \in \mathcal{P}(U)$ and T be a Sylow p-subgroup of $K \cap H$ with $U \leq T$. If $V \leq T$, then $K \in \mathcal{P}(V)$.*

Proof. Obviously, $K \in \mathcal{M}(V)$. If $K \notin \mathcal{M}_{\max}(V)$, there is $K_1 \in \mathcal{M}(V)$ and $T_1 \in \mathrm{Syl}_p(H \cap K_1)$ such that $T_1 \cap K \in \mathrm{Syl}_p(H \cap K)$ with $T_1 > T_1 \cap K$. Let $g \in H \cap K$ such that $(T_1 \cap K)^g = T$. Then also $K_1^g \in \mathcal{M}(V)$. As $X \leq T \leq K_1^g$, we have that $K_1^g \in \mathcal{M}(U)$. But $T_1^g > T$ and so $K \sqsubset K_1^g$, a contradiction. So we have that $K \in \mathcal{M}_{\max}(V)$ and then $K \in \mathcal{P}(V)$. $\square$

For the structure of $B(T)$-blocks we refer the reader to Definition 2.18. We now apply the Bundy-Hebbinghaus-Stellmacher $C(G,T)$-Theorem [**15**].

Lemma 21.5. *Suppose that $K \in \mathcal{P}$. For $T \in \mathrm{Syl}_p(K \cap H)$ let*

$$\{X_1, X_2, \ldots, X_f\}$$

be the set of maximal $B(T)$-blocks in K. Then $X_1 \cdots X_f$ is a normal subgroup of K,

$$K = T(X_1 \cdots X_f),$$

T acts transitively by conjugation on $\{X_1, X_2, \ldots, X_f\}$ and $[X_i, X_j] = 1$ for $1 \leq i < j \leq f$.

Proof. This follows from Lemmas 21.3, 21.2 and [**15**, Corollary 1.9]. $\square$

Notation 21.6. *For the remainder of this section, whenever $K \in \mathcal{P}$, we fix the following notation (which depends on the choice of P). We choose $S_0 \in \mathrm{Syl}_p(H)$, so that T is a Sylow p-subgroup of $H \cap K$ with $T \leq S_0$. We write $K = XT$ where*

$$X = X_1 \cdots X_f = \langle X_1^T \rangle,$$

with, for $1 \le i \le f$, X_i a $B(T)$-block of K and $X_i/O_p(X_i) \cong \mathrm{SL}_2(p^d)'$. Further, for $1 \le i \le f$,

(i) $Y_i = [O_p(K), O^p(X_i)]$ *and* $Y = Y_1 \cdots Y_f$.
(ii) $W_i = [Z(O_p(K)), O^p(X_i)]$ *and* $W = W_1 \cdots W_f$.
(iii) F_i *is such that* $F_i/Y_i = Z(X_i/Y_i)$ *and* $F = F_1 \cdots F_f$.
(iv) $I \in \mathrm{Syl}_2(F)$ *and* $I_i = I \cap F_i$.
(v) R *is a long root group in* $Z(S)$ *and* $Q = O_p(C_{F^*(H)}(R))$.

Notice that if X_i is not an exceptional bock then $Y_i = W_i$ and $|W_i| = p^{2d}$.

We also relax our notation by setting

$$q = p^e$$

and, once $K \in \mathcal{P}$ is given,

$$r = p^d.$$

Recall that

$$Q \text{ is semi-extraspecial}$$

by Lemma D.16.

LEMMA 21.7. *$C_G(t) \le H$ for all $1 \ne t \in Z(Q)$.*

PROOF. The statement is true if $t \in Z(S_0)$ by Hypothesis 21.1 (ii). As $Z(Q) = R$, we have by Lemma A.4 that all elements in R are H-conjugate into $Z(S_0)$. □

LEMMA 21.8. *Let $K \in \mathcal{P}$. Then for all $w \in (Z(Q) \cap O_p(K))^\#$ and all $1 \le i \le f$, $[w, X_i] \ne 1$. If $Z(Q) \cap X_j \ne 1$ for some j, then $f = 1$. In particular, if $Z(Q) \cap W_j \ne 1$ for some j, then $f = 1$.*

PROOF. Assume that $w \in (Z(Q) \cap O_p(K))^\#$ and $[w, X_i] = 1$ for some $1 \le i \le f$. Then by Lemma 21.7 we have $X_i \le H$. But then, as T acts transitively on $\{X_1, \dots, X_f\}$, $K = \langle X_1^T \rangle T \le C_G(w) \le H$, a contradiction.

Now, if $w \in (Z(Q) \cap X_j)^\#$ and $f > 1$, then for $i \ne j$, $[X_i, X_j] = [X_i, w] = 1$ which is a contradiction. Hence, if $Z(Q) \cap W_j \ne 1$ for some j, then $f = 1$. □

Since $K \cap H$ is the unique maximal subgroup of K which contains T, we have

$$FT \le H.$$

We use this fact to show that $Y \le F^*(H)$.

LEMMA 21.9. *Let $K \in \mathcal{P}$. Then*

(i) $Y \le S \le F^*(H)$;
(ii) $R \le Z(S) \le Z(O_p(K))$;

(iii) RW *is normal in* K*; and*
(iv) $r \geq q$.

PROOF. Suppose (i) is false. Then, for some $1 \leq i \leq f$, there exist non-trivial elements of $Y_i F^*(H)/F^*(H)$ which induce outer automorphisms of H of order p. Since $Y_i = [Y_i, I_i]$, we see that the involution in I_i inverts some non-trivial element of $Y_i F^*(H)/F^*(H)$ of order p. This contradicts Theorem A.11 (i) and (ii). Hence (i) holds.

Since $Y \leq S$ by (i), $Z(S) \leq N_{S_0}(Y) = T$ by Lemma 21.2 (ii). Therefore, as $C_K(Y) \leq O_p(K)$,

$$R \leq O_p(K).$$

As $[O_p(K)', X] = 1$, we have that $R \cap O_p(K)' = 1$. As R is normal in $O_p(K)$ this implies $R \leq Z(O_p(K))$.

For (iii), we note that $W = [Z(O_p(K)), X]$ and so RW is normalized by $K = XT$.

Finally, as $R \leq O_p(K)$, R projects into each W_i faithfully by Lemma 21.8. Hence by Lemma E.7 we have that $r \geq |R| = q$. This proves (iv). $\square$

LEMMA 21.10. *Assume that* $K \in \mathcal{P}$*. Then there exists* $\omega \in W$ *such that* $C_G(\omega) \not\leq H$*. In particular, if* $C_G(t) \leq H$ *for all* $t \in F^*(H)$ *with* t *of order* p*, then* H *is strongly* p*-embedded in* G*.*

PROOF. Since $K = \langle T^K \rangle \not\leq H$, there exists $k \in K$ such that $T^k \not\leq H$. Select $\omega \in C_W(T^k)$. Then $T^k \leq C_G(\omega) \not\leq H$.

Since $W \leq S$ by Lemma 21.9, if $C_G(t) = C_H(t)$ for all $t \in F^*(H)$, then we must have $\mathcal{P} = \emptyset$ and this means that H is strongly p-embedded in G. $\square$

Next we treat the case $F^*(H) \cong \mathrm{PSp}_4(q)$. Here by Lemma C.15 R is weakly H-closed in Q and all elements in $Q \setminus R$ are conjugate.

LEMMA 21.11. *Suppose that* $F^*(H) \cong \mathrm{PSp}_4(q)$*. If* H *is not strongly* p*-embedded in* G*, then* $p = 5$*,* $H = \mathrm{Aut}(\mathrm{PSp}_4(5))$ *and for any element* $\omega \in H$ *with* $o(\omega) = 5$ *such that* 5^3 *divides* $|C_H(\omega)|$ *we have* $C_G(\omega) = C_H(\omega)$*.*

PROOF. Suppose false. Since H is not strongly p-embedded in G, $\mathcal{P} \neq \emptyset$. Bearing in mind Lemma 21.10, select $K \in \mathcal{P}$ and $t \in T$ so that $|C_S(t)|$ has maximal order. By Lemma 21.9 (iii), RW is normal in K and so, as $R = Z(S)$, $Z_2(S) \leq N_S(RW)$ and by Lemma 21.2 (ii)

$$Z_2(S) \leq T \leq K.$$

If $[Z_2(S), W] = 1$, we have by Lemma D.22 that $W \leq C_S(Z_2(S)) = J(S)$, which is elementary abelian of order q^3. Since $J(S)$ centralizes

W, $J(S) \leq O_p(K)$ and so $J(S_0) = J(S) \leq O_p(K)$. By Lemma 21.2 we have that $K \leq N_G(J(S_0))$. But then $N_G(J(S_0)) \in \mathcal{M}(S_0) = \emptyset$, a contradiction.

So we have that $[Z_2(S), W] \neq 1$. Then $R \geq [W_1, Z_2(S)] \neq 1$ and so by Lemma 21.8

$$f = 1.$$

Furthermore, as, for $x \in Z_2(S) \setminus O_p(K)$, we have $|[W, x]| = r$ we obtain so $q = |R| \geq r$. Thus Lemma 21.9 (iv) implies that $|R| = q = r$. In particular $|Y| \leq r^4$ and so X is not exceptional.

Since $R = [W, Z_2(S)] \leq W$, we have $W \not\leq J(S)$ and $W \cap J(S) = C_W(Z_2(S)) = R$. If $t \in Q$, by Lemma 21.7 then $t \in Q \setminus R$. As H induces $SL_2(q)$ on Q/R, the maximal choice implies $t \in Z_2(S)$ and $C_S(t) \geq J(S)$. We obtain $J(S) \leq T$ and $T \geq WC_S(t) = WJ(S) = S$, which is a contradiction as W is not normal in S. Thus $t \notin Q$. If $W \cap Q > R$, then there exists $w \in (W \cap Q) \setminus R$ such that $C_G(w) \not\leq H$ and $|C_{F^*(H)}(w)|_p \geq q^3$, we must have $|C_S(t)| \geq q^3$. Since $|C_Q(t)| \leq q^2$, we again get $S = QC_S(t) \leq T$, a contradiction. Hence $W \cap Q = R$ and $T \cap Q = Z_2(S)$. It follows that

$$T \cap S = WZ_2(S).$$

Application of [**62**, Theorem 2.9] implies $F^*(H) \cong \mathrm{PSp}_4(5)$, $O^{p'}(K) \sim 5^2{:}\mathrm{SL}_2(5)$, which gives $|C_S(t)| = 25$. In particular by the choice of t for any element $\omega \in H$ with $o(\omega) = 5$ such that 5^3 divides $|C_H(\omega)|$ we have $C_G(\omega) = C_H(\omega)$. Furthermore that $N_{XB(T)}(R)/B(T)$ is cyclic of order 4 and so $N_H(R)/Q \cong \mathrm{GL}_2(5)$, which gives $H = \mathrm{Aut}(\mathrm{PSp}_4(5))$.

Finally, using Lemma 21.10 shows that either H is strongly p-embedded in G or we have the exceptional configuration as described. □

From now on we have

$$F^*(H) \cong \mathrm{G}_2(q),\ \mathrm{PSU}_4(q),\ \mathrm{PSU}_5(q) \text{ or } {}^3\mathrm{D}_4(q).$$

Lemma 21.12. *Suppose that $K \in \mathcal{P}$ and put $Q_1 = K \cap Q$. Then $Q_1 \leq N_K(X_i)$, $Q_1 \not\leq O_p(K)$ and $[W_i, Q_1, Q_1] = 1$ for all $1 \leq i \leq f$.*

Proof. Let $t \in Q_1$ and assume that $X_1^t = X_2$. Then $[T \cap X_1, t] \leq Q$. As p is odd, we have that t induces an orbit of length at least three on the X_i, so $1 \neq [T \cap X_1, t, t] \leq R$. Now $R_1 = [W_1, [T \cap X_1, t, t]] \neq 1$ and $R_1 \leq R$. Application of Lemma 21.8 gives the contradiction $f = 1$. Hence $Q_1 \leq N_K(X_1)$ and so $Q_1 \leq N_K(X_i)$ for $1 \leq i \leq f$.

Again let $t \in Q_1$ and assume that $[(T \cap X_1)/O_p(X_1), t] \neq 1$. Then t induces a field automorphism on $X_1/O_p(X_1)$ and on $(T \cap X_1)/O_p(X_1)$. Since $[T \cap X_1, Q_1, Q_1] \leq [Q_1, Q_1] \cap X_1 \leq R \leq O_p(X_1)$, this is impossible.

Hence Q_1 induces automorphisms of W_1, which are induced by some element from $\mathrm{SL}_2(r)$.

Suppose that $Q_1 \leq O_p(K)$. Then Q_1Y is normalized by K as $Q_1Y \leq T \leq S_0$, Q_1Y normalizes Q. If $Q_1 < Q$, then $N_Q(Q_1Y) > Q_1$ and we have a contradiction to Lemma 21.2 (ii). Thus $Q_1 = Q$ and $QY \leq O_p(K)$. But then $R = Q' \leq O_p(K)' \leq C_T(X)$ and so $K \leq N_G(R) \in \mathcal{M}(S_0)$, which is a contradiction. Therefore $Q_1 \not\leq O_p(K)$.

Finally as Q_1 induces elements from $\mathrm{SL}_2(r)$ on W_1, Q_1 acts quadratically on W_1 and hence also on W_i. □

LEMMA 21.13. *We have $\mathcal{M}(Q) = \emptyset$.*

PROOF. Assume that $\mathcal{M}(Q) \neq \emptyset$ and choose $K \in \mathcal{P}(Q)$. Let $D = C_{O_p(K)}(O^p(K))$. If $D \neq 1$, then there is $d \in D^{\#}$ with $[Q, d] = 1$. But then $d \in Z(Q)$ and so by Lemma 21.7 $K \leq H$, a contradiction. Hence $D = 1$ and, in particular, X_1 is not exceptional. Furthermore $O_p(K) = W$ is elementary abelian.

By Lemma 21.12 we have that $[W_1, Q, Q] = 1$. Hence $1 \neq [W_1, Q] \leq Z(Q)$, so by Lemma 21.8 we have

$$f = 1.$$

So

$$|O_p(K)| = |W| = r^2.$$

Since Q acts quadratically on W by Lemma 21.12 and Q does not centralize W, we now have $X_1Q/W \cong \mathrm{SL}_2(r)$. Especially, $|Q| \leq r^3$.

As $Z(Q) = Z(S)$ has order q and Q acts quadratically on W by Lemma 21.12, we have $[W, Q] \leq Z(Q) = R$ and hence $r = |[W, Q]| \leq |R| = q$ and so Lemma 21.9 (iv) implies that $q = r$. This shows $|Q| = q^3$ which is a contradiction. Hence $\mathcal{P}(Q) = \emptyset$. □

LEMMA 21.14. *Let $K \in \mathcal{P}$. Then $W \not\leq Q$.*

PROOF. Suppose that $W \leq Q$. By Lemma 21.9 (iii), RW is normal in K and is also normalized by Q. But then $N_G(RW) \in \mathcal{M}(Q) = \emptyset$. □

LEMMA 21.15. *Let $K \in \mathcal{P}(U)$. Then $O_p(K) \cap Q$ is not a maximal elementary abelian self-centralizing subgroup of Q.*

PROOF. Set $A = O_p(K) \cap Q$ and assume that A is a maximal elementary abelian self-centralizing subgroup of Q. Then by Lemma 2.21 $[W, Q] \leq A$ and so $Q \leq N_G(WA)$. But WA also is normalized by K, which contradicts Lemma 21.13. □

LEMMA 21.16. *Let $K \in \mathcal{P}(X)$. Then X_1 is not of exceptional type.*

PROOF. Suppose that X_1 is of exceptional type. Then $p = 3$. By Lemma 21.12, we have that $Q_1 = Q \cap K$ acts on Y as a subgroup of $XB(T)$ and $Q_1 \not\leq O_3(K)$. Therefore, the Definition 2.18 (iii) states that Q_1 does not act quadratically on Y. Therefore, Lemma 21.12 shows that

$$1 \neq [Y_1, Q_1, Q_1] \leq R$$

and so $R \cap C_{Y_1}(X_1)W \neq 1$. Application of Lemma 21.8 gives $f = 1$.

Now $[Y, Q_1] \leq Y \cap Q$ and $[Y, Q_1] \not\leq Z(Y)$. Thus $1 \neq [Y, Q_1, Y] \leq Y' \cap Q$. Let $t \in [Y, Q_1, Y]^{\#}$. Since $Y' \leq C_Y(X)$, Lemma 21.2(ii) implies that $C_Q(t) \leq T$. By Lemma 2.10, $|Q : C_Q(t)| = q$. In addition, $|C_Q(t)O_3(K)/O_3(K)| \leq |Q_1O_3(K)/O_3(K)| \leq r$. Now $Q_1 \cap O_3(K)$ has index at most qr in Q and

$$(Q \cap O_3(K))' \leq O_3(K)' \cap Q' = C_{O_3(K)}(X) \cap R = 1.$$

Since Q is semi-extraspecial this gives

$$q^5 \leq |Q| \leq r^2q^3.$$

Suppose that $|Q| = q^5$. Then $F^*(H) \cong \mathrm{PSU}_4(q)$ and $|S| = q^6$. As $C_Q(t)Y \leq S$ has order at most r^6 and $r \geq q$ by Lemma 21.9 we get $r = q$. Then $Q \leq S = C_Q(t)Y \leq T \leq K$ contrary to Lemma 21.13.

Hence $|Q| > q^5$ which means that $|Q| \geq q^7$. Therefore $r^2 \geq q^4$. On the other hand, $q^3 \geq |YQ/Q| \geq r^2$ and so we have the absurd situation

$$q^3 \geq q^4,$$

which is a contradiction. $\square$

Because of Lemma 21.16, we now have $Y = W$.

LEMMA 21.17. *Let $t \in Q^{\#}$ be an element of order p. Then $N_G(\langle t\rangle) \leq H$.*

PROOF. Assume the statement is false. Then, by Lemma 21.7, $t \in Q \setminus Z(Q)$ and $|Q : C_Q(t)| = q$ by Lemma 2.10. Choose $K \in \mathcal{P}(C_{S_0}(t))$. By Lemma 21.4 $K \in \mathcal{P}(C_S(t))$. Application of Lemmas 21.9 and 21.14 gives $R \leq O_p(K)$ and $W \not\leq Q$. So we may assume that $W_1 \not\leq Q \cap K$. By the transitive action of T on the X_i we get that $W_i \not\leq Q \cap K$ for all $1 \leq i \leq f$.

Assume that $R_1 = R \cap \Phi((Q \cap O_p(K))) \neq 1$. Then $[R_1, X_1] = 1$ and so by Lemma 21.7, we get $K \leq H$, a contradiction. This shows

$$Q \cap O_p(K) \text{ is elementary abelian.}$$

By Lemma 21.12 we have that $1 \neq [W_1, Q \cap K] \leq Z(Q \cap K)$. Furthermore as T normalizes $Q \cap K$, we get $1 \neq [W_i, Q \cap K] \leq Z(Q \cap K)$

for all i. As $|[W_1, Q \cap K]| = r$, we get that $|Z(Q \cap K)| \geq r^f$. Using Q is semi-extraspecial we have $|Z(Q \cap K)| \leq q^2$. Therefore, by Lemma 21.9

$$r^f \leq q^2 \leq r^2.$$

As f is a power of p, we obtain

$$f = 1 \text{ and } r \leq q^2.$$

By Lemma 21.12, $C_Q(t)O_p(K)/O_p(K) \leq B(T)X/O_p(K)$ and so $|C_Q(t) : C_Q(t) \cap K| \leq r \leq q^2$. It follows that $|Q : C_Q(t) \cap K| \leq q^3$ and, as $Q \cap O_p(K)$ is abelian and Q is semi-extraspecial, this yields $|Q| \leq q^7$. In particular, $F^*(H) \not\cong {}^3\mathrm{D}_4(q)$. Furthermore, if $|Q| = q^7$, then $Q \cap O_p(K)$ is a maximal abelian self-centralizing subgroup of Q and this violates Lemma 21.15. Hence $F^*(H) \not\cong \mathrm{PSU}_5(q)$.

We now know that $F^*(H) \cong \mathrm{PSU}_4(q)$ or $\mathrm{G}_2(q)$ with $p \geq 5$ and $|S/Q| = q$.

Choose $w \in W \setminus [W, Q \cap K]$, then $[w, Q \cap K] \neq 1$. Hence if $w \in Q$, we get $[w, Q \cap K] \leq R$ and so $R = [W, Q \cap K]$, which shows $q = r$. If $[W, Q \cap K] = W \cap Q$, then, as $W \leq S$, $|W : W \cap Q| = r \leq q$ and again $q = r$. Hence we have

$$q = r.$$

Therefore, $|Q : Q \cap O_p(K)| \leq q^2$ and again we contradict Lemma 21.15. □

Lemma 21.18. *If $t \in S$ is an element of order p, then $N_G(\langle t \rangle) \leq H$ or $F^*(H) \cong \mathrm{G}_2(7)$ and $C_{S \cap F^*(H)}(t)$ is of order 49.*

Proof. Suppose false. By Lemma 21.17 $t \notin Q$. Choose t with $|C_S(t)|$ maximal. Select $K \in \mathcal{P}(C_S(t))$. By Lemma 21.13 we have $Q \not\leq K$ and, by Lemma 21.14, $W \not\leq Q$. Using Lemma 21.12, we have $[W_1, Q \cap K] \neq 1$ and so application of Lemma 21.17 yields

$$f = 1.$$

Furthermore, Lemmas E.7 and 21.17 imply that $W \cap Q = C_W(T \cap XB(T))$ has order r. If $R \not\leq W \cap Q$, then, as $R \leq Z(O_p(K))$ by Lemma 21.9, $C_{Z(O_p(K))}(O^p(K)) \cap Q \neq 1$ and this is also against Lemma 21.17. Hence $R \leq W \cap Q$. Since $R \leq C_W(T \cap XB(T))$, R is normalized by F and therefore so is Q. Hence $Q \cap O_p(K) = C_{Q \cap O_p(K)}(I)[Q \cap O_p(K), I]$. Since $C_{Q \cap O_p(K)}(I) \leq C_{O_p(K)}(O^p(K))$, Lemma 21.17 implies that

$$Q \cap O_p(K) = [Q \cap O_p(K), I] = Q \cap W = C_W(T \cap XB(T))$$

has order r. In particular, $|K \cap Q| \leq r^2$.

Since $R = Z(S) \leq W$, we have

$$Z_2(S) \leq N_S(W) = T \leq K$$

by Lemma 21.2.

Assume that $q = r$. Then $|K \cap Q| \leq r^2 = q^2$. Since $Z_2(S) \leq Q \cap K$, it follows that $|Z_2(S)| \leq q^2$ and, as $C_Q(t) \leq XB(T)$ by Lemma 21.12, $|C_Q(t)| \leq q^2$. Now Lemma D.24 shows that $F^*(H) \not\cong {}^3\mathrm{D}_4(q)$ and Lemma D.21 shows that $F^*(H) \not\cong \mathrm{PSU}_4(q)$ or $\mathrm{PSU}_5(q)$.

Hence we are left with $F^*(H) \cong \mathrm{G}_2(q)$ with $p \geq 5$. As $O_p(K) \leq C_{S_0}(R) = S$, we have that $|O_p(K) : O_p(K) \cap Q| \leq |S : Q| = q$. But $|W : W \cap Q| = q$, so $O_p(K) = W$ and then application of [**62**, Theorem 2.9] gives $F^*(H) = \mathrm{G}_2(7)$, the assertion. In particular we have that $O^{7'}(K) \sim 7^2{:}\mathrm{SL}_2(7)$ and so $|C_S(t)| = 7^2$.

For the remainder of the proof we may assume

$$q \neq r.$$

Since $W \cap Q$ has order r, we have $q < r = |WQ/Q| \leq |S/Q|$. Hence $F^*(H) \cong \mathrm{PSU}_5(q)$ or ${}^3\mathrm{D}_4(q)$. If $Z_2(S) \not\leq O_p(K) = W$, $R \geq [W, Z_2(S)]$, and we have $r \leq q$, a contradiction. Hence $Z_2(S) \leq W \cap Q$. Assume that $F^*(H) \cong \mathrm{PSU}_5(q)$. Then, by Lemma D.21, $|Z_2(S)| = q^3 \leq r$ and $q^3 \leq |WQ/Q| \leq |S/Q| \leq q^3$, whence $WQ/Q = S/Q$ is abelian, a contradiction. Hence $F^*(H) \cong {}^3\mathrm{D}_4(q)$ in which case $|Z_2(S)| = q^2$ by Lemma D.24. In particular,

$$q^2 \leq r.$$

Since $Z_2(S) \leq W$, $Z_3(S)$ normalizes W and so by Lemma 21.2 $Z_3(S) \leq K$. Furthermore by Lemma D.24 we have that $|Z_3(S)| \geq q^5$ and $Z_3(S) \leq Q$ by Lemma D.24 (iv). As $|W : W \cap Q| = r$, and $|S/Q| = q^3$ we have $r \leq q^3$. In particular $|Q \cap W| = r \leq q^3$ and so $Z_3(S) \not\leq W \cap Q = O_p(K) \cap Q$. In particular $Z_3(S) \not\leq O_p(K)$. Hence we have $[W, Z_3(S)] \neq 1$. As $[Z_3(S), W] \leq Z_2(S) \leq W \cap Q$, which is of order q^2, we get $r \leq q^2$. But then

$$q^5 \leq |Z_3(S)| \leq |Q \cap K| \leq r^2 \leq q^4,$$

which is a contradiction. Hence Lemma 21.18 holds. $\square$

LEMMA 21.19. *Let* $F^*(H) \cong \mathrm{G}_2(q)$, $\mathrm{PSU}_4(q)$, $\mathrm{PSU}_5(q)$ *or* ${}^3\mathrm{D}_4(q)$. *If* H *is not strongly* p*-embedded and* $G \neq H$, *then* $p = 7$, $F^*(H) \cong \mathrm{G}_2(7)$ *and for any element* $\omega \in H$ *with* $o(\omega) = 7$ *such that* 7^3 *divides* $|C_H(\omega)|$ *we have* $C_G(\omega) = C_H(\omega)$.

PROOF. The assertion follows from Lemmas 21.18 and 21.10. $\square$

LEMMA 21.20. *Let* $F^*(H) \cong \mathrm{PSp}_4(5)$ *and assume that* H *is not strongly* 5*-embedded in* G. *Then* H *controls* G*-fusion of involutions in* H. *Furthermore* H *has four conjugacy classes of involutions.*

Proof. By Lemma 21.11 we have $|H : F^*(H)| = 2$ and $H \cong \mathrm{Aut}(\mathrm{PSp}_4(5))$. Therefore the second assertion follows from [**27**, Table 4.5.1].

According to Lemma E.2, for all $\omega \in H$ of order 5 with $|C_H(\omega)|$ even, we have 5^3 divides $|C_H(\omega)|$. Therefore, Lemma 21.11 implies that $N_G(\langle\omega\rangle) \leq H$. Another application of Lemma E.2 shows that there are three classes of such groups in H, and the orders of their normalizers are pairwise distinct. So we have that H controls G-fusion of these subgroups of order 5.

By Lemma E.2 5 divides $|C_H(i)|$ for all involutions $i \in H$. Let U be a Sylow 5-subgroup of $C_H(i)$ and $U_1 \leq C_G(i)$ with $|U_1 : U|$ divides 5. Then U_1 centralizes some $1 \neq \omega \in U$ and therefore $U_1 \leq H$. We have that $C_H(i)$ contains a Sylow 5-subgroup of $C_G(i)$.

Let i, j be involutions in H and assume that $i^g = j$ for some $g \in G$. Choose $\omega \in C_H(i)$ some element of order 5. Then $\omega^g \in C_G(j)$ But as $C_H(j)$ contains a Sylow 5-subgroup of $C_G(j)$ we may assume that $\omega^g \in C_H(j)$. As H controls G-fusion of ω there is some $h \in H$ with $\omega^g = \omega^h$. Then, as $N_G(\langle\omega\rangle) \leq H$, we have $g \in H$. This shows that H controls fusion of involutions in H. □

Lemma 21.21. *Suppose that H is not strongly p-embedded in H and $F^*(H) \cong \mathrm{PSp}_4(5)$ or $\mathrm{G}_2(7)$. Then $O_{p'}(G) = 1$.*

Proof. Let $E \leq Q$ with $|E| = p^2$. Then by Lemmas 21.11 and 21.19, $C_G(e) \leq H$ for all $e \in E^\#$. In particular, $O_{p'}(C_G(e)) = 1$ for all $e \in E^\#$. Therefore,

$$O_{p'}(G) = \langle C_{O_{p'}(G)}(e) \mid e \in E^\# \rangle = 1,$$

as claimed. □

Lemma 21.22. *Suppose that H is not strongly p-embedded in H and $F^*(H) \cong \mathrm{PSp}_4(5)$ or $\mathrm{G}_2(7)$. Let i be a 2-central involution in H. If $C_G(i) \leq H$, then $G = H$.*

Proof. Assume $G \neq H$. If $F^*(H) = \mathrm{G}_2(7)$, then $F^*(H) = H = \mathrm{G}_2(7)$ and by [**27**, Table 4.5.1] we have that H has just one conjugacy class of involutions. So $F^*(H)$ controls fusion of involutions. If $F^*(H) \cong \mathrm{PSp}_4(5)$, then $|H : F^*(H)| = 2$ and so $F^*(H)$ controls fusion of 2-central involutions according to Lemma 21.20.

Let i be a 2-central involution of H. Then by assumption $C_G(i) = C_H(i)$. Because of Lemma 21.21, we may apply Lemma 4.4 and, as $F^*(H) \ncong \mathrm{Alt}(n)$, we have shown $G = H$. □

LEMMA 21.23. *Suppose that H is not strongly p-embedded in H and $F^*(H) \cong \mathrm{PSp}_4(5)$ or $\mathrm{G}_2(7)$. Assume further that G is a $\mathcal{K}_2$-group. Then for a 2-central involution i in H we have that $C_G(i) \le H$.*

PROOF. Let R be a root subgroup in $F^*(H)$. Then by Lemmas D.1 and Lemma D.10 we get

$$O^{p'}(C_{F^*(H)}(R)) \sim \begin{cases} 5_+^{1+2}{:}\mathrm{SL}_2(5) & H \cong \mathrm{Aut}(\mathrm{PSp}_4(5)) \text{ or} \\ 7_+^{1+4}{:}\mathrm{SL}_2(7) & H \cong \mathrm{G}_2(7). \end{cases}$$

Let i be an involution in $O^{p'}(C_{F^*(H)}(R))$. Then i centralizes R and also a subgroup of H which is isomorphic to $\mathrm{SL}_2(p)$, $p = 5$ or 7. Furthermore i inverts $O_p(C_{F^*(H)}(R))/R$. If $F^*(H) \cong \mathrm{G}_2(7)$, then there is exactly one conjugacy class of involutions in H, so i is a 2-central one. If $F^*(H) \cong \mathrm{PSp}_4(5)$, we have $i \in F^*(H)$ and by Lemma E.2 we get that just the 2-central involutions of $F^*(H)$ have a centralizer in H of order divisible by 25. So in both cases

$$i \text{ is a 2-central involution of } H.$$

Application of [**27**, Table 4.5.1] shows

$$F^*(C_H(i)) = K_1 \circ K_2,$$

where $K_1 \cong K_2 \cong \mathrm{SL}_2(p)$, $p = 5$ or 7 and $\langle i \rangle = K_1 \cap K_2$. In particular $N_H(R)$ contains a Sylow p-subgroup of $C_H(i)$. As i inverts $Z_2(S)/R$ and centralizes R, we find that any p-element in $C_H(i) \cap N_H(R)$ centralizes $Z_2(S)$, hence the centralizer of such a p-element has order divisible by p^3. By Lemmas 21.11 and 21.19, for $\omega \in C_H(i)$ of order p, we have

$$N_G(\langle \omega \rangle) \le H.$$

Assume that $C_G(i) \neq C_H(i)$. Let E be a Sylow p-subgroup of $C_H(i)$ which contains R. Then

$$\langle N_G(\langle e \rangle) \mid e \in E^{\#} \rangle = \langle N_H(\langle e \rangle) \mid e \in E^{\#} \rangle \ge N_H(R).$$

As $N_H(R)$ is a maximal subgroup of H, we have that $\langle N_G(\langle e \rangle) \mid e \in E^{\#} \rangle = N_H(R)$ or H. Since $\langle N_G(\langle e \rangle) \mid e \in E^{\#} \rangle$ is normalized by $N_G(E)$, we find that $N_G(E)$ normalizes either Q or H. In either case, $N_G(E) = N_H(E)$. Hence $N_{C_G(i)}(E) \le C_H(i)$ and we conclude that

$$C_H(i) \text{ is strongly } p\text{-embedded in } C_G(i).$$

As E is a Sylow p-subgroup of $C_G(i)$ we can apply coprime action to receive

$$O_{p'}(C_G(i)) = \langle C_{O_{p'}(C_G(i))}(e) \mid e \in E^{\#} \rangle \le H.$$

In particular $O_{p'}(C_G(i)) \le O_{p'}(C_H(i)) \le Z(C_H(i)) = \langle i \rangle$. So

$$O_{p'}(C_G(i)) = \langle i \rangle.$$

Furthermore, $O_p(C_G(i)) = O_p(C_G(i)) \cap E \leq O_p(C_G(i)) \cap K_1K_2 = 1$. Hence $F(C_G(i)) = \langle i \rangle$. Since $i \in K_1K_2$, we get $F^*(C_G(i)) = E(C_G(i))$. Set $K = E(C_G(i))$. If K has components L_1 and L_2, then $E \cap L_1$ is centralized by L_2 and L_1 centralizes $E \cap L_2$. Thus $K = E(C_H(i)) = K_1K_2$ and this means that $C_G(i) = KN_{C_G(i)}(E) \leq H$, a contradiction to the assumption $C_G(i) \neq C_H(i)$. Therefore K is quasisimple. Moreover, by the Schreier property [**27**, Theorem 7.1.1 (a)], $K_1K_2 \leq K$.

Since K is quasisimple and $m_p(K) > 1$, we can apply [**27**, Theorem 7.6.1]. We consider each of the candidates for K given in [**27**, Theorem 7.6.1]. Recalling that $E \leq K_1K_2 \leq K$, and the subgroup K_3 of K generated by the normalizers of the subgroups of order p in E is contained in $C_H(i) \geq K_1K_2$. Hence $F^*(K_3)/\langle i \rangle \cong \mathrm{PSL}_2(p) \times \mathrm{PSL}_2(p)$. Now we can apply [**27**, Theorem 7.6.2]. The only possibility which fits with our K_3 then will be

$$p = 5 \text{ and } K \cong \mathrm{Alt}(10).$$

In $C_H(i)$ we see that $N_{C_H(i)}(E) = N_{C_G(i)}(E)$ is of order $2^5.5^2$. This is the same order for the normalizer of a Sylow 5-subgroup in K. This shows that $C_G(i) \cong 2{}^{\cdot}\mathrm{Alt}(10)$. But in this group the involutions of $K/Z(K)$ which are products of two transpositions will become elements of order 4 in $C_G(i)$. Therefore $C_G(i)$ has at most three conjugacy classes of involutions. On the other hand $C_H(i)$, as i is a 2-central involution in H, contains a Sylow 2-subgroup of H and so by Lemma 21.20 contains four G-classes of involutions, we have a contradiction to the assumption $C_G(i) \neq C_H(i)$. □

We now prove Theorem 1.

The proof of Theorem 1. Suppose that $G \neq H$ and that H is not strongly p-embedded in G. Then by Theorem 4.7, $F^*(H)$ is a rank 2 group of Lie type and p is odd. If $F^*(H) \cong \mathrm{PSL}_3(p^e)$, then Proposition 9.1 gives (id) of Theorem 1. Proposition 15.1 shows that $F^*(H) \not\cong \mathrm{G}_2(3^e)$. Thus we may assume that Hypothesis 21.1 holds and so Lemmas 21.11 and 21.19 show that (ib) and (ic) hold. Under the additional assumption that G is a $\mathcal{K}_2$-group, Lemma 21.23 with Lemma 21.22 shows that (ib) and (ic) lead to the contradiction $G = H$. This yields Theorem 1 (ii). Finally, assuming that G is a $\mathcal{K}_{\{2,p\}}$-group in the case that p is odd, Propositions 4.5 and 4.6 yield Theorem 1(iii). □

22. Proof of Main Theorem 1 and Main Theorem 2

In this section we now will prove both of our Main Theorems.

Main Theorem 1 follows immediately from Theorems 2, 3 and 4.

The proof of Main Theorem 2. In addition to our standard assumption that G is almost a group of Lie type in characteristic p suppose that G is a $\mathcal{K}_{\{2,p\}}$-group, $G \neq H$ and $F^*(H) \not\cong \mathrm{PSL}_3(p)$, p an odd prime.

Since Propositions 4.5 and 4.6 imply that H is not strongly p-embedded in G, Main Theorem 1 and Theorem 1 combine to give that $F^*(G)$ and p are as follows:

- $p = 2$ and $F^*(G) \cong \mathrm{Mat}(11)$, $\mathrm{Mat}(23)$, $\mathrm{PSL}_4(3)$, $\mathrm{Alt}(10)$, $\mathrm{G}_2(3)$ or $\mathrm{P}\Omega_8^+(3)$; or
- $p = 3$ and $F^*(G) \cong \mathrm{PSU}_6(2)$, $\mathrm{F}_4(2)$, ${}^2\mathrm{E}_6(2)$, Co_2, $\mathrm{M}(22)$, $\mathrm{M}(23)$, McL or F_2; or
- $p = 5$ and $G \cong \mathrm{LyS}$.

As $G = \mathrm{PSL}_4(3)$ contains an involution j with $F^*(C_G(j)) \cong \mathrm{PSL}_2(9)$ (see Lemma D.28) and in $G = \mathrm{Alt}(10)$ for $j = (12)(34)$ we have $F^*(C_G(j)) \cong \mathrm{Alt}(6)$, both groups are not of local characteristic 2. According to Lemma D.26 we have an involution i in $\mathrm{P}\Omega_8^+(3)$ such that $F^*(C_G(i)) \cong 2{\cdot}\mathrm{PSU}_4(3)$, so G is not of local characteristic 2. Suppose that $p = 3$. Then McL has local characteristic 3 whereas all the other groups are not. For example there is an element ρ of order three in G such that $E(C_G(\rho)) \cong \mathrm{PSU}_4(2)$, $\mathrm{PSp}_6(2)$, $\mathrm{PSU}_6(2)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$, $\mathrm{P}\Omega_7(3)$, $\mathrm{M}(22)$. For $\mathrm{PSU}_6(2)$ this is in [**50**, Lemma 22], for $\mathrm{F}_4(2)$ we get the result with [**60**, Lemma 8.2], and for $\mathrm{M}(22)$ with [**55**, Lemma 7.1]. For the sporadic simple groups this follows from [**27**, Table 5.3]. This proves Main Theorem 2. $\square$

A. Properties of finite simple groups of Lie type

We take the monograph [**27**] as our fundamental source of data about the finite simple groups of Lie type. Thus, following [**27**, Definition 2.2.1], for a prime p, we let $\overline{\mathrm{GF}(p)}$ be the algebraic closure of $\mathrm{GF}(p)$, $\overline{K}$ be a semisimple $\overline{\mathrm{GF}(p)}$-algebraic group and σ be a Steinberg endomorphism of $\overline{K}$. Then, as in [**45**, Definition page 2], we make the following definition.

Definition A.1. *A* genuine group of Lie type in characteristic p *is a group isomorphic to* $O^{p'}(C_{\overline{K}}(\sigma))$ *and a* simple group of Lie type in characteristic p *is a non-abelian composition factor of a genuine group of Lie type in characteristic* p.

The simple groups of Lie type in characteristic p, which are not genuine groups of Lie type in characteristic p are $\mathrm{Sp}_4(2)'$, $\mathrm{G}_2(2)'$, ${}^2\mathrm{F}_4(2)'$ and ${}^2\mathrm{G}_2(3)'$. Almost always, we will use classical notation for those simple groups of Lie type which have an alternative classical description.

Thus, for simple groups, we have $\mathrm{A}_n(p^e) = \mathrm{PSL}_{n+1}(p^e)$, ${}^2\mathrm{A}_n(p^e) = \mathrm{PSU}_{n+1}(p^e)$, $\mathrm{B}_n(p^e) = \mathrm{P}\Omega_{2n+1}(p^e)$, $\mathrm{C}_n(p^e) = \mathrm{PSp}_{2n}(p^e)$, $\mathrm{D}_n(p^e) = \mathrm{P}\Omega^+_{2n}(p^e)$ and ${}^2\mathrm{D}_n(p^e) = \mathrm{P}\Omega^-_{2n}(p^e)$. The groups ${}^2\mathrm{B}_2(2^{2e+1})$ are called *Suzuki groups* and the groups ${}^2\mathrm{F}_4(2^{2e+1})$ and ${}^2\mathrm{G}_2(3^{2e+1})$ are called *Ree groups*. Collectively we call them *Suzuki–Ree* groups.

Let $\overline{T}$ be a maximal torus of $\overline{K}$, Σ a root system with respect to $\overline{T}$ and Π a set of fundamental roots in Σ. Taking a σ-stable Borel subgroup $\overline{B} = \overline{UT}$ with $\overline{T}^\sigma = \overline{T}$, $\overline{N} = N_{\overline{K}}(\overline{T})$ and $W = \overline{N}/\overline{T}$, we define

$$\widetilde{W} = C_W(\sigma) = C_W(\tau)$$

where τ is the symmetry induced on the Dynkin diagram of $\overline{K}$ by application of σ. Then, by [**27**, Proposition 2.3.2], $\widetilde{W}$ is a *Weyl group* with respect to a root system $\widetilde{\Sigma}$. The *rank* of K is then defined to be $\dim\langle\widetilde{\Sigma}\rangle$ (see [**27**, page 42]) and the *untwisted rank* of K is $\dim\langle\Sigma\rangle$.

For example, the simple groups of rank one are $\mathrm{PSL}_2(p^e)$, $\mathrm{PSU}_3(p^e)$, ${}^2\mathrm{B}_2(2^e)$ and ${}^2\mathrm{G}_2(3^e)'$ and our main theorems refer to the remaining simple groups of Lie type.

For $\alpha \in \Sigma$, let $\overline{X}_\alpha = \{x_\alpha(t) \mid t \in \overline{\mathrm{GF}(r)}\}$ be a $\overline{T}$ root subgroup of $\overline{K}$. Then, for $\widetilde{\alpha} \in \widetilde{\Sigma}$, $X_{\widetilde{\alpha}}$ is the subgroup of $\langle \overline{X}_\alpha^{\langle\sigma\rangle}\rangle$ in K centralized by σ. Then $X_{\widetilde{\alpha}}$ is called a *root subgroup* of K. The structure of root subgroups is given in [**27**, Table 2.4] and $K = \langle X_{\widetilde{\alpha}} \mid \widetilde{\alpha} \in \widetilde{\Sigma}\rangle$. For a long root $\widetilde{\alpha} \in \widetilde{\Sigma}$, by a *long root subgroup*, we mean

$$\Omega_1(Z(X_{\widetilde{\alpha}})).$$

In [**27**, page 103], the authors define long root subgroups slightly differently: they choose a σ-invariant long root $\alpha \in \Sigma$ and then define the corresponding long root subgroup to be $C_{\overline{X}_\alpha}(\sigma)$. For this definition they exclude the Suzuki-Ree case, see definition above. With this exception these two definitions coincide. The order of the *field of definition* of K is $|X|$ for X a long root subgroup. Thus, for example, the field of definition of ${}^2\mathrm{A}_n(q) \cong \mathrm{PSU}_{n+1}(q)$ is $\mathrm{GF}(q)$. The standard definition of a *parabolic subgroup* is taken from [**27**, Definition 2.6.4] and we take the definition of a *Levi factor* or *Levi complement* from [**27**, Definition 2.6.6]. If L is a Levi factor then we call $O^{p'}(L/Z(L))$ a *Levi section*. If K is a simple group of Lie type, then to cover the non-genuine groups we define a parabolic subgroup to be the intersection of a parabolic subgroup of the corresponding genuine group of Lie type with K. The lattice of parabolic subgroups containing a fixed Borel subgroup of K is congruent to the lattice of subsets of a set of size the rank of K.

Lemma A.2. *Suppose that X is a genuine group of Lie type defined over* $\mathrm{GF}(p^e)$. *Then $|X|_p = (p^e)^m$ where m is given in the following* *table.*

$X/Z(X)$	$\mathrm{PSL}_n(p^e)$	$\mathrm{PSU}_n(p^e)$	$\mathrm{PSp}_{2n}(p^e)$	$\mathrm{P\Omega}_{2n+1}(p^e)$	$\mathrm{P\Omega}^-_{2n}(p^e)$
m	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	n^2	n^2	$n(n-1)$

$X/Z(X)$	$\mathrm{P\Omega}^+_{2n}(p^e)$	${}^2\mathrm{B}_2(2^e)$	${}^3\mathrm{D}_4(p^e)$	$\mathrm{E}_6(p^e)$	${}^2\mathrm{E}_6(p^e)$
m	$n(n-1)$	2	12	36	36

$X/Z(X)$	$\mathrm{E}_7(p^e)$	$\mathrm{E}_8(p^e)$	$\mathrm{F}_4(p^e)$	${}^2\mathrm{F}_4(2^e)$	$\mathrm{G}_2(p^e)$	${}^2\mathrm{G}_2(3^e)$
m	63	120	24	12	6	3

PROOF. This is taken from [**27**, Table 2.2]. $\square$

LEMMA A.3. *Suppose that X is a genuine group of Lie type defined in characteristic p and U is a Sylow p-subgroup of X. Then either $Z(U)$ is a root group or $X \cong \mathrm{Sp}_{2n}(2^e)$, $\mathrm{F}_4(2^e)$ or $\mathrm{G}_2(3^e)$, where $Z(U)$ is a product of two root groups one long and one short.*

PROOF. See [**27**, Theorem 3.3.1]. $\square$

LEMMA A.4. *Let X be a group of Lie type over $\mathrm{GF}(p^e)$ and R be a long root subgroup of X.*

(i) *If X is not $\mathrm{PSL}_2(p^e)$ or a Suzuki–Ree group, then for $g \in X$, either $\langle R, R^g\rangle$ is p-group or $\langle R, R^g\rangle \cong \mathrm{SL}_2(p^e)$. Moreover, both cases occur.*

(ii) *If $X = {}^2\mathrm{F}_4(2^{2e+1})$, then there is a conjugate R^g of R such that $\langle R, R^g\rangle \cong {}^2\mathrm{B}_2(2^{2e+1})$.*

(iii) *If $g \in X$ and $R^g \cap R \neq 1$, then $g \in N_X(R)$*

In particular in any case $N_X(R)$ acts irreducibly on R. Furthermore if U is a Sylow p-subgroup of $\mathrm{Aut}(X)$ with $R \leq Z(U \cap X)$, then any element in R is conjugate into $Z(U)$.

PROOF. Part (i) is [**27**, Theorem 3.2.9] and (ii) can be found in [**27**, Example 3.2.5, page 102] and (iii) is explained on page 103 [**27**].

As in both groups $\mathrm{SL}_2(p^e)$ and ${}^2\mathrm{B}_2(2^e)$ the normalizer of a Sylow p-subgroup acts irreducibly on the centre of the Sylow p-subgroup the follow up statement holds as well. Furthermore in ${}^2\mathrm{B}_2(2^e)$ the normalizer acts transitively on the non-trivial elements of R. In the typical case where $\langle R, R^g\rangle \cong \mathrm{SL}_2(p^e)$ there can be two orbits on the subgroups of order p when p is odd whereas there is only one if $p = 2$. This shows that for a Sylow p-subgroup U of $\mathrm{Aut}(X)$ such that $R \leq Z(U \cap X)$ we have that any subgroup of order p in R is conjugate into $Z(U)$. $\square$

LEMMA A.5. *Let X be a group of Lie type over $\mathrm{GF}(p^e)$ which is not $\mathrm{PSL}_2(p^e)$ or a Suzuki-Ree group. Assume that R, R_1, R_2 and R_3 are long root subgroups in X and set $A = \langle R_1, R_2\rangle$. If $A \cong \mathrm{SL}_2(p^e)$ and $[A, R_3] = 1$, then $R \cap AR_3 \subset A \cup R_3$. In particular, if $R \cap R_1R_3 \neq 1$, then $R = R_1$ or $R = R_3$.*

Proof. Suppose that $x \in (R \cap AR_3)^{\#}$ and assume $x \notin A \cup R_3$. Let $y \in A$ be the projection of x onto A. Then we may suppose that y does not normalize R_1. Hence $\langle R, R_1\rangle \geq \langle x, R_1\rangle > \langle R_1, R_1^x\rangle = A$. Using Lemma A.4 (i) gives a contradiction. Hence $R \cap AR_3 \subset A \cup R_3$. Now, if $R \cap R_1R_3 \neq 1$, then either $R \cap R_1 \neq 1$ or $R \cap R_3 \neq 1$. Lemma A.4 (iii) now gives $R = R_1$ or $R = R_3$. □

Lemma A.6. *Let $X \cong \mathrm{PSU}_3(p^e)$, $p^e > 2$, and R be a root subgroup of X. Then X is generated by three conjugates of R.*

Proof. By Lemma A.4 (i) there is an X-conjugate R^g of R such that $Y = \langle R, R^g\rangle \cong \mathrm{SL}_2(p^e)$. We use [**27**, Theorem 6.5.3], to see that if $R^h \not\leq Y$, then $G = \langle R, R^g, R^h\rangle$. □

Lemma A.7. *Let X be a group of Lie type isomorphic to one of $\mathrm{Sp}_{2n}(2^e)$, $\mathrm{F}_4(2^e)$ or $\mathrm{G}_2(3^e)$ and R be a short root subgroup of X. Then $N_X(R)$ acts irreducibly on R and, if $R^g \cap R \neq 1$ for some $g \in X$, then $g \in N_X(R)$.*

Proof. If X is either $\mathrm{F}_4(2^e)$ or $\mathrm{G}_2(3^e)$, then by [**27**, Theorem 3.3.1 (c)] there is an automorphism of X mapping R to a long root subgroup. If $X \cong \mathrm{Sp}_{2n}(2^e)$, then we use the fact that $X \cong \Omega_{2n+1}(2^e)$ by [**27**, Theorem 1.15.9] and that in this incarnation R becomes a long root subgroup. Hence the result follows from Lemma A.4. □

Lemma A.8. *Suppose that X is a genuine group of Lie type defined in characteristic p and U is a Sylow p-subgroup of X. Let R be a root subgroup in $Z(U)$. Then $C_X(r) = C_X(R)$ for all $r \in R^{\#}$.*

Proof. By Lemma A.3 either R is a long root subgroup or $X \cong \mathrm{Sp}_{2n}(2^e)$, $\mathrm{F}_4(2^e)$ or $\mathrm{G}_2(3^e)$. Set $P = N_X(R)$. By Lemma A.4(iii) and Lemma A.7, R is a TI-subgroup of X, so $R \trianglelefteq C_X(r)$, and hence $C_X(r) = C_P(r)$. We know that P is a parabolic subgroup of X and as $C_X(R) \geq O^{p'}(P)$, we have $N_X(R) = N_X(U)C_X(R)$. As $N_X(U)/U$ is abelian, $P/C_X(R)$ is abelian. In particular, $C_P(r)$ is normal in P. Since P acts irreducibly on R by Lemmas A.4 (iii) and A.7, we deduce that $C_P(r) = C_P(R)$. □

Suppose that K is a genuine group of Lie type signified by the symbol ${}^d\Sigma(q)$ as in [**27**]. In [**27**] they adopt two distinct notations for graph automorphisms one in [**27**, Theorem 2.5.1] and a different one in [**27**, Definition 2.5.13]. We have elected to adopt the former notation which follows Steinberg's Yale notes [**72**, Section 10]. This decision means that we have to be extremely careful when we apply results about automorphisms from later sections of [**27**] on the other hand it does mean that Theorem A.10 below remains valid. Here is a definition which is extracted from [**27**, Theorem 2.5.1].

DEFINITION A.9. *Suppose that K is a genuine group of Lie type defined over* $\mathrm{GF}(p^e)$.

(i) *A* diagonal automorphism *of K is an automorphism d which is induced by conjugation by an element $h \in N_{\overline{T}}(K)$, so that for all $\alpha \in \Sigma$,*

$$x_\alpha(t)^d = x_\alpha(\alpha(h)t).$$

If K is untwisted, this gives the action of d on each X_α. If K is a twisted group, then d normalizes every $X_{\tilde{\alpha}}$;

(ii) *A* field *automorphism f of K is one arising from the restriction of an automorphism φ of $\overline{\mathrm{GF}(p)}$, and carrying the generators*

$$x_\alpha(t)^f = x_\alpha(t^\varphi);$$

and

(iii) *A* graph *automorphism of K is trivial unless K is untwisted, and then is defined as follows:*

(a) *If Σ has one root length, then for some isometry ρ of Σ carrying Π to Π,*

$$x_\alpha(t)^g = x_{\alpha^\rho}(\epsilon_\alpha t)$$

for all $\alpha \in \Sigma$, $t \in \mathrm{GF}(p^e)$, where the $\epsilon_\alpha = \pm$ are signs and $\epsilon_\alpha = 1$ if $\alpha \in \Pi$ or $-\alpha \in \Pi$; or

(b) *If $\Sigma = \mathrm{B}_2$, F_4, or G_2, with $p = 2$, 2, or 3, respectively then*

$$x_\alpha(t)^g = \begin{cases} x_{\alpha^\rho}(t) & \text{if } \alpha \text{ is long} \\ x_{\alpha^\rho}(t^p) & \text{if } \alpha \text{ is short} \end{cases}$$

where ρ is the unique angle-preserving, length-changing bijection from Σ to Σ carrying Π to Π.

The fundamental theorem about the automorphism group of a genuine group of Lie type is as follows:

THEOREM A.10. *Every automorphism of K is a product $idfg$ where $i \in \mathrm{Inn}(K)$, d is a diagonal automorphism, f is a field automorphism and g is a graph automorphism of K.*

PROOF. This is [**72**, (3.2)] (see also [**27**, Theorem 2.5.1]). $\square$

We define

$$\begin{aligned}
\mathrm{Diag}_K &= \mathrm{Inn}(K)\langle d \mid d \text{ a diagonal automorphism of } K\rangle\\
\Phi_K &= \langle f \mid f \text{ a field automorphism of } K\rangle \text{ and}\\
\Gamma_K &= \langle g \mid g \text{ a graph automorphism of } K\rangle.
\end{aligned}$$

Thus Diag_K consists of all the inner and diagonal automorphisms of K.

THEOREM A.11. *Suppose that K is a genuine group of Lie type with $K \cong {}^d\Sigma(q)$ and $Z(K) = 1$. Identify K with $\mathrm{Inn}(K)$. Then the following hold.*

(i) $\mathrm{Aut}(K)$ *is the semidirect product of* Diag_K *by* $\Phi_K\Gamma_K$ *where* $\Phi_K\Gamma_K$ *is abelian.*
(ii) Diag_K/K *has order coprime to* p.
(iii) $\Phi_K \cong \mathrm{Aut}(\mathrm{GF}(q^d))$.
(iv) *If K is twisted, then* $\Gamma_K = 1$.
(v) *If* $\Gamma_K \neq 1$ *and* $K \ncong \mathrm{P}\Omega_8^+(q)$, *then either*
  (a) Γ_K *has order 2 and* $K \cong \mathrm{PSL}_n(q)$, $n \geq 3$, $\mathrm{P}\Omega^+_{2n}(q)$, $n \geq 4$, *or* $\mathrm{E}_6(q)$.
  (b) Γ_K *has order* $2e_2$, $|\Phi_K\Gamma_K : \Phi_K| = 2$ *and* $K \cong \mathrm{Sp}_4(2^e)$, $\mathrm{F}_4(2^e)$ *or* $\mathrm{G}_2(3^e)$.
(vi) *If* $K \cong \mathrm{P}\Omega_8^+(q)$, *then* $\Gamma_K \cong \mathrm{Sym}(3)$ *and* $\mathrm{Diag}_K\Gamma_K/K \cong \mathrm{Sym}(4)$.

PROOF. See [**27**, Theorem 2.5.12]. □

LEMMA A.12. *If* $K \cong \mathrm{Sp}_4(2^e)$, $\mathrm{F}_4(2^e)$ *or* $\mathrm{G}_2(3^e)$, *then* $\mathrm{Out}(K)$ *is cyclic.*

PROOF. This can be taken from [**27**, Theorem 2.5.12]. □

LEMMA A.13. *Let* $K = {}^2\mathrm{F}_4(q)'$, $q = 2^{2e+1}$ *with* $e \geq 0$. *If* $e > 0$, *then* $\mathrm{Out}(K)$ *has odd order. If* $e = 0$, *then* $\mathrm{Aut}(K) \cong {}^2\mathrm{F}_4(2)$ *and there is no involution in* $\mathrm{Aut}(K) \setminus K$.

PROOF. These results follow from [**27**, Theorems 2.5.12, 2.5.15 and 3.3.2]. □

LEMMA A.14. *Let K be a genuine group of Lie type in characteristic p and $U \in \mathrm{Syl}_p(K)$. Identify K with* $\mathrm{Inn}(K)$. *Suppose that* $E \leq \mathrm{Aut}(K)$ *normalizes all the parabolic subgroups of K which contain $N_K(U)$. Then* $E \leq N_{\mathrm{Diag}_K}(U)\Phi_K$. *In particular,* $\mathrm{Aut}(K) = KN_{\mathrm{Diag}_K}(U)\Phi_K$ *if and only if* $\Gamma_K = 1$.

PROOF. This follows from Definition A.9 and Theorem A.10. □

We shall be interested in automorphisms of K of order p, the defining characteristic of K. We distinguish two cases.

Lemma A.15. *Suppose that K is a genuine group of Lie type over a field of odd characteristic p, $Z(K) = 1$ and $\alpha \in \Phi_K$ has order p. If $K \not\cong \mathrm{P}\Omega_8^+(3^e)$ or ${}^3\mathrm{D}_4(3^e)$, then all elements of order p in the coset Kx are Diag_K-conjugate.*

Proof. This is [**27**, Proposition 4.9.1]. $\square$

Lemma A.16. *Suppose that K is a genuine group of Lie type over a field of characteristic 2 of type ${}^d\Sigma(q)$ with $Z(K) = 1$ and identify K with $\mathrm{Inn}(K)$. Let U be a Sylow 2-subgroup of K, $\alpha \in N_{\mathrm{Aut}(K)}(U)$ have order 2 and let $z \in Z(U\langle\alpha\rangle)^\#$. Denote the image of a subgroup or element of $\mathrm{Out}(K)$ by adding a hat. Then $\widehat{\alpha}$ is conjugate into $\widehat{\Phi_K\Gamma_K}$ and furthermore*

(i) *If $\widehat{\alpha} \in \widehat{\Phi_K}$, then one of the following holds*
  (a) *K is not twisted, α is Diag_K-conjugate to an element of Φ_K and $O^{p'}(C_K(\alpha)) \cong \Sigma(q^{\frac{1}{2}})$.*
  (b) *$K \cong \mathrm{PSU}_n(q)$, n odd, α is Diag_K-conjugate to an element of Φ_K and $C_K(\alpha) \cong \mathrm{PSp}_{n-1}(q)$.*
  (c) *$K \cong \mathrm{PSU}_n(q)$, n even, or ${}^2\mathrm{E}_6(q)$, α is Diag_K-conjugate to an element of $\langle z\rangle\Phi_K$. If $\alpha \in \Phi_K$, then $C_K(\alpha) \cong \mathrm{PSp}_n(q)$, or $\mathrm{F}_4(q)$ in the respective cases and*
$$C_K(z\alpha) = C_K(\alpha) \cap C_K(z) = C_{C_K(\alpha)}(z).$$

(ii) *If $\widehat{\alpha} \in \widehat{\Gamma_K}$, then K is untwisted and*
  (a) *$K \cong \mathrm{PSL}_n(q)$, n odd, α is Diag_K-conjugate to an element of Γ_K and $C_K(\alpha) \cong \mathrm{PSp}_{n-1}(q)$;*
  (b) *$K \cong \mathrm{PSL}_n(q)$, n even, or $\mathrm{E}_6(q)$, α is Diag_K-conjugate to an element of $\langle z\rangle\Gamma_K$. If $\alpha \in \Gamma_K$, then, in the respective cases, $C_K(\alpha) \cong \mathrm{PSp}_n(q)$, or $\mathrm{F}_4(q)$ and*
$$C_K(z\alpha) = C_K(\alpha) \cap C_K(z) = C_{C_K(\alpha)}(z).$$
  (c) *$K \cong \mathrm{PSp}_4(2^e)$ or $\mathrm{F}_4(2^e)$ and α is K-conjugate to an element of Γ_K. In the respective cases, if e is odd, then $C_K(\alpha) \cong {}^2\mathrm{B}_2(2^e)$ or ${}^2\mathrm{F}_4(2^e)$ whereas, if e is even, then $C_K(\alpha) \cong \mathrm{PSp}_4(2^{e/2})$ or $\mathrm{F}_4(2^{e/2})$.*

(iii) *If $K \cong \mathrm{P}\Omega_{2n}^\pm(2^e)$ and $\alpha \in \widehat{\Gamma_K} \cup \widehat{\Phi_K}$, then either $\alpha \in \Gamma_K \cup \Phi_K$ and $C_K(\alpha) \cong \mathrm{P}\Omega_{2n-1}(2^e) \cong \mathrm{Sp}_{2n-2}(2^e)$ or $F^*(C_K(\alpha))$ is a 2-group.*

(iv) *If $\widehat{\alpha} \in \widehat{\Phi_K\Gamma_K} \setminus (\widehat{\Phi_K} \cup \widehat{\Gamma_K})$ is a product of a graph and a field automorphism, then all involutions in the coset $\alpha\mathrm{Diag}_K$ are Diag_K-conjugate to α. Furthermore, $O^{2'}(C_K(\alpha)) \cong {}^2\Sigma(q^{\frac{1}{2}})$.*

PROOF. Remember that our notation is not exactly the same as that in [**27**]. Part (i)(a), (iii) and (iv) are taken from [**27**, Propositiona 4.9.1 and 4.9.2]. Parts (i)(b), (i)(c), (ii)(a) and (ii)(b) are found in [**27**, Proposition 4.9.2] and [**5**, (19.8)]. Part (ii)(c) is given in [**5**, (19.5)]. $\square$

LEMMA A.17. *Let p be a prime and M be a group with $O_p(M) = 1$. If K is a group of Lie type in characteristic p such that $M \leq K$ and p does not divide $|K : M|$, then $K = M$.*

PROOF. Suppose first that K is a genuine group of Lie type. Let $U \in \mathrm{Syl}_p(M)$ and $B = N_K(U)$ be a Borel subgroup of K. Then [**27**, Theorem 2.6.7] yields MB is a parabolic subgroup in K. As $O_p(M) = 1$, we have $MB = K$. Furthermore by [**27**, Theorem 2.6.7], B normalizes M and so $M = K$ as $K = O^{p'}(K)$. This proves the result when K is genuine.

It remains to treat the groups $K/Z(K) \cong {}^2\mathrm{G}_2(3)'$, $\mathrm{Sp}_4(2)'$, $\mathrm{G}_2(2)'$ and ${}^2\mathrm{F}_4(2)'$. As ${}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8)$ and $\mathrm{Sp}_4(2)' \cong \mathrm{PSL}_2(9)$, we may apply Dickson's Theorem [**33**, Satz 8.27] to obtain the result. We use [**14**, Tables 8.5 and 8.6] to obtain the result for $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$. This leaves us with $K \cong {}^2\mathrm{F}_4(2)'$. So $|K| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. Thus, if $O_t(M) \neq 1$ for some odd prime t, then $|O_t(M)| \leq 3^3$, 5^2 or equal 13. Hence in any case $C_U(O_t(M)) \neq 1$ and so $O_t(M)$ commutes with a 2-central involution r, contrary to $C_K(r)$ having characteristic 2. Thus $F^*(M)$ is semisimple and as the centralizer of an involution is soluble, $F^*(M)$ is a simple group. Since M is a $7'$-group and $|S| = 2^{11}$, $F^*(M)$ is not an alternating group. Using [**27**, Table 5.3] for the orders of the sporadic simple groups, we see that M is not sporadic. If $F^*(M)$ is a group of Lie type in odd characteristic r, then, using Lemma A.2, $F^*(M)$ is either $\mathrm{PSL}_3(3)$, $\mathrm{PSU}_3(3)$, ${}^2\mathrm{G}_2(3)'$ or $\mathrm{PSL}_2(r^a)$ where $r^a \in \{3, 3^2, 3^3, 5, 5^2, 13\}$. Thus
$$2^{11} = |M|_2 \leq |\mathrm{Aut}(M)|_2 \leq 2^6,$$
a contradiction.

So consider the case that $F^*(M)$ is a group of Lie type in characteristic 2. Suppose that $F^*(M)$ has Lie rank at least 2 and let P be a maximal parabolic subgroup of $F^*(M)$. Then by the Borel-Tits Theorem [**27**, Theorem 3.1.3], P is contained in a parabolic subgroup of the genuine group of Lie type $\mathrm{Aut}(K)$. In particular, $P/O_2(P)$ is either $\mathrm{SL}_2(2)$ or a subgroup of ${}^2\mathrm{B}_2(2)$. Since ${}^2\mathrm{F}_4(2)$ is the only group of Lie type in characteristic two, which possesses a parabolic subgroup with Levi complement a Suzuki group, we deduce that $P/O_2(P) \cong \mathrm{Sym}(3)$. It follows that $F^*(M) \cong \mathrm{PSL}_3(2)$, $\mathrm{PSp}_4(2)'$ or $\mathrm{G}_2(2)'$, which yields $|\mathrm{Aut}(F^*(M))|_2 < 2^{11}$, a contradiction. Hence $F^*(M) \cong \mathrm{SL}_2(2^f)$, $\mathrm{PSU}_3(2^f)$ or ${}^2\mathrm{B}_2(2^f)$ with $f > 1$ and in the last case odd. Since there is always an element of order $2^f - 1$ in $N_{F^*(M)}(S \cap F^*(M))$, we have $2^f - 1 = |N_{F^*(M)}(S \cap F^*(M))/(S \cap F^*(M))| \in \{3, 5\}$ which means that $f = 2$. But then $|S \cap F^*(M)| \leq 2^6$ and $|\mathrm{Aut}(F^*(M))|_2 < 2^{11}$, a contradiction. □

Lemma A.18. *Let X be a p-minimal group such that $K = F^*(X)$ is a simple group of Lie type in characteristic p. Let $U \in \mathrm{Syl}_p(K)$. Then $N_X(U)$ is maximal in X and one of the following holds:*

(i) *$K \cong \mathrm{PSL}_2(p^e)$;*
(ii) *$K \cong \mathrm{PSU}_3(p^e)$;*
(iii) *$p = 2$ and $K \cong {}^2\mathrm{B}_2(2^{2e+1})$, $e \geq 1$;*
(iv) *$p = 3$ and $K \cong {}^2\mathrm{G}_2(3^{2e+1})'$, $e \geq 0$;*
(v) *$p = 2$ and $K \cong \mathrm{PSL}_3(2^e)$ and $X > K$ and $N_X(U)$ interchanges the two minimal parabolic subgroups of K containing $N_K(U)$;*
(vi) *$p = 2$ and $K \cong \mathrm{PSp}_4(2^e)'$ and $X > K$ and $N_X(U)$ interchanges the two minimal parabolic subgroups of K containing $N_K(U)$.*

Proof. Let $U \in \mathrm{Syl}_p(K)$ and $U_0 \geq U$ be a Sylow p-subgroup of X. Let M be the unique maximal subgroup of X containing U_0 and let $F = N_X(U)$. Then $X = KF$, F permutes the minimal parabolic subgroups of K which contain $N_K(U)$ and $F \leq M$. If Θ is an orbit of F on the minimal parabolic subgroups of K, then $\langle\Theta\rangle F = X$ or $\langle\Theta\rangle \leq M$. Since $K \not\leq M$ and K is generated by the minimal parabolic subgroups containing $N_K(U)$, there must exist an orbit Ψ such that $K = \langle\Psi\rangle$. That is F is transitive on the minimal parabolic subgroups of K containing $N_K(U)$. If K has rank 1, then we have cases (i) to (iv). So the rank of K is at least 2. By Theorem A.10 and Lemmas A.13 and A.14, K is not a twisted group and $|\Psi| = 2$ or 3. Since $\mathrm{P}\Omega_8^+(p^e)$ has rank 4, we have $|\Psi| = 2$ and K has rank 2. Now X has a normal subgroup Y of index 2. If p is odd, then $X = YF \leq M$, a contradiction. Hence $p = 2$. Now Theorem A.11 (v) gives the result. □

Lemma A.19. *Suppose that X is a simple group of Lie type defined in characteristic p and of rank one, $S \in \mathrm{Syl}_p(X)$ and $B = N_X(S)$.*

(i) *If* $X \cong \mathrm{PSL}_2(p^e)$, *then* $B = SH$ *where* H *is cyclic of order* $(p^e-1)/\gcd(p^e-1,2)$ *and* H *acts irreducibly on* S.

(ii) *If* $X \cong \mathrm{PSU}_3(p^e)$, *then* $B = SH$ *where* H *is cyclic of order* $(p^{2e}-1)/\gcd(p^e+1,3)$ *and* S *is special of order* q^3. *The subgroup* H *acts faithfully and irreducibly on* $S/Z(S)$ *which has order* q^2 *and* $C_H(Z(S))$ *has order* $(p^e+1)/\gcd(p^e+1,3)$. *If* p *is odd, then* S *has exponent* p *while, when* $p=2$, $\Omega_1(S) = Z(S)$.

(iii) *If* $X \cong {}^2\mathrm{B}_2(2^{2e+1})$, *then* $B = SH$ *where* H *is cyclic of order* $(2^{2e+1}-1)$, S *is special with* $\Omega_1(S) = Z(S)$, SH *is a Frobenius group and* H *acts transitively on* $\Omega_1(S)$.

Proof. In case (i), we obtain the result by an elementary calculation in $\mathrm{SL}_2(p^e)$ noting that B is the image of the subgroup of lower triangular matrices.

For (ii) we refer to [**33**, Satz II.10.12] where most of the required calculations are performed. Enough information is also provided to calculate the remaining points.

For (iii), we refer to [**73**, page 113 and Theorem 9]. $\square$

B. Properties alternating groups

In this short appendix we present the basic structural results that we require about the alternating and symmetric groups.

Lemma B.1. *Assume that* $n \geq 8$, $X = \mathrm{Sym}(n)$, $H \leq X$ *and* $F^*(H)$ *is quasisimple. If* H *contains a Sylow 2-subgroup of* $F^*(X)$, *then* $F^*(H) \cong \mathrm{Alt}(m)$ *for* $m \in \{n-3, n-2, n-1, n\}$. *In particular if* H *is transitive then* $m = n$.

Proof. If $n = 8$, then every proper over-group of the Sylow 2-subgroup has characteristic 2 by Lemma A.17 and thus we may assume $n \geq 9$.

Assume first that H is primitive. Then, as H contains as fours group transitive on 4 points, Marggraf's Theorem [**80**, Theorem 13.5] implies $H \geq \mathrm{Alt}(n)$ and so we are done.

Assume next that H is transitive, but not primitive. Then H is contained in $\mathrm{Sym}(c) \wr \mathrm{Sym}(b)$ where $n = cb$ with $c \geq 2$ and $b \geq 4$. Since $F^*(H)$ is quasisimple and H contains a Sylow 2-subgroup of $F^*(X)$ this is impossible.

Thus H is not transitive and so H is isomorphic to a subgroup of $\mathrm{Sym}(a) \times \mathrm{Sym}(b)$ where $a+b = n$ and a is the length of a maximal orbit of H. Further, as $n \geq 9$, we have that H contains a Sylow 2-subgroup of $\mathrm{Alt}(9)$, and so $a \geq 8$. Since H contains a Sylow 2-subgroup of $F^*(X)$,

$H \cap \mathrm{Sym}(a) \neq 1$, which gives $F^*(H) \leq \mathrm{Sym}(a)$ and so $F^*(H) = \mathrm{Alt}(a)$ by induction. Furthermore, $\mathrm{Sym}(b) \cap \mathrm{Alt}(n)$ has odd order for otherwise $1 \neq H \cap \mathrm{Sym}(b) \leq C_H(F^*(H)) \leq \mathrm{Sym}(a)$. Therefore $b \leq 3$ and the result follows. □

LEMMA B.2. *Assume that $n \geq 5$, $X = \mathrm{Sym}(n)$, $H \leq X$ and $F^*(H)$ is quasisimple. If H contains a Sylow 3-subgroup of X, then $F^*(H) \cong \mathrm{Alt}(m)$ for $m \in \{n-2, n-1, n\}$. In particular if H is transitive $n = m$.*

PROOF. Suppose first that H is primitive. Then, as H contains a 3-cycle, Jordan's Theorem [**80**, Theorem 13.3] implies $F^*(H) = \mathrm{Alt}(n)$.

If H is transitive, but not primitive, then $H \leq \mathrm{Sym}(c) \wr \mathrm{Sym}(d)$ with $n = dc$ where d and c both greater than 1. If $c \neq 2$, the $F^*(H) \leq \mathrm{Sym}(c) \cap \mathrm{Sym}(d)$ which is absurd. Hence $c = 2$ but this is impossible as the Sylow 3-subgroup of $\mathrm{Alt}(n)$ does not preserve blocks of size 2. Therefore we have:

(B.2.1) If H acts transitively, then $F^*(H) \cong \mathrm{Alt}(n)$.

Assume now that H is intransitive. Write $H \leq \mathrm{Sym}(a) \times \mathrm{Sym}(b)$, where a is the length of a maximal orbit of $F^*(H)$. As $a \geq 5$, we get that $F^*(H) \cap \mathrm{Sym}(a) \neq 1$ and so $F^*(H) \leq \mathrm{Sym}(a)$ and $|\mathrm{Sym}(b)|$ is not divisible by 3. This gives $b \leq 2$. As $H \cap \mathrm{Sym}(a)$ contains a Sylow 3-subgroup of $\mathrm{Alt}(a)$ and H acts transitively on the orbit of length a, we have that $F^*(H) \cong \mathrm{Alt}(a)$ by (B.2.1). Thus, as $b \leq 2$, we get $m = a \in \{n-2, n-1\}$ and we are done. □

LEMMA B.3. *Assume that X is 2-minimal and $F^*(X) \cong \mathrm{Alt}(n)$ for some $n \geq 5$. Then either $F^*(X) = \mathrm{Alt}(2^a + 1)$ for some $a \geq 2$ or $F^*(X) \cong \mathrm{Alt}(6) \cong \mathrm{Sp}_4(2)'$ and X involves a graph or a graph-field automorphism of X.*

PROOF. See [**40**, Lemma 2.2] for the cases where X is contained in $\mathrm{Sym}(n)$. For $F^*(X) \cong \mathrm{Alt}(6)$ and X contained in $\mathrm{Sym}(6)$, the intersection with X of the parabolic subgroups $2 \wr \mathrm{Sym}(3)$ and $2 \times \mathrm{Sym}(4)$ of $\mathrm{Sp}_4(2) \cong \mathrm{Sym}(6)$, show that X is not 2-minimal. Suppose that X contains a graph or a graph-field automorphism of $F^*(X)$. Let $S \in \mathrm{Syl}_2(X)$ and assume that S is not a maximal subgroup of X. Let M be a maximal subgroup of X containing S. Then, as $X/F^*(X)$ is a 2-group, $M \cap F^*(X)$ is proper over-group of $T = S \cap F^*(X)$ which is normalized by S. The only proper over-groups of T in $F^*(X)$ are the parabolic subgroups of $F^*(X)$ and these are not normalized by S. Hence S is a maximal subgroup of X and it follows that X is 2-minimal. □

LEMMA B.4. *Assume that X is 3-minimal and $F^*(X) \cong \mathrm{Alt}(n)$ for some $n \geq 5$. Then $X = F^*(X)$ and $n \in \{6, 9, 3^a + 1 \mid a \geq 2\}$.*

PROOF. Let Ω be a set of size n and $T \in \mathrm{Syl}_3(X)$. Since $T \leq F^*(X)$, and $X = F^*(X)N_X(T)$, we conclude that $X = F^*(X)$. We will repeatedly apply Jordan's Theorem [**80**, Theorem 13.3] to see that certain pairs of subgroups generate X.

Assume that $n = a_t 3^t + \cdots + a_1 3^1 + a_0$ where $a_i = 0, 1, 2$ be the 3-adic decomposition of n. Assume that $\sum_{i=1}^{t} a_i \geq 3$. Then T has at least 3-orbits on Ω. In particular, we may find distinct non-empty T-invariant subsets α, β, γ and δ of Ω such that $\Omega = \alpha \cup \beta = \gamma \cup \delta$ are disjoint decompositions. But then T is a subgroup of $H = \mathrm{Sym}(\alpha) \times \mathrm{Sym}(\beta)$ and of $K = \mathrm{Sym}(\gamma) \times \mathrm{Sym}(\delta)$. Since $X = \langle K \cap X, H \cap X \rangle$, we have that X is not 3-minimal in this case. Thus $1 \leq \sum_{i=1}^{t} a_i \leq 2$. If T fixes a point, we then have $X \cong \mathrm{Alt}(3^a + 1)$. Continuing the considerations in this case, if $H \geq T$ is transitive on Ω, then it is 2-transitive and as T contains a 3-cycle we conclude that $H \geq \mathrm{Alt}(\Omega)$. Hence every proper over-group of T in X fixes the same point as T and so $\mathrm{Alt}(3^a + 1)$ is 3-minimal.

Suppose that $n = 3^a + 3^b$ with $a \geq b \geq 1$. Then T is contained in subgroups $H = \mathrm{Sym}(3) \wr \mathrm{Sym}(n/3)$ and $K = \mathrm{Sym}(3^a) \times \mathrm{Sym}(3^b)$. If $n \neq 6$, then $X = \langle H \cap X, K \cap X \rangle$, and so X is not 3-minimal. Therefore $n = 6$ and $a = b = 1$, then $K \leq H$ and indeed we obtain $X = \mathrm{Alt}(6)$ is 3-minimal.

Finally assume that $n = 3^a$. If $a > 2$, then T is contained in $H = \mathrm{Sym}(3) \wr \mathrm{Sym}(3^{a-1})$ and in $K = \mathrm{Sym}(3^{a-1}) \wr \mathrm{Sym}(3)$ and $X = \langle H \cap X, K \cap X \rangle$. Hence, as $n \geq 5$, $n = 9$. In this case, T acts transitively on Ω and any proper subgroup of X containing T must be imprimitive. But then setting $H = \mathrm{Sym}(3) \times \mathrm{Sym}(3)$, $H \cap X$ is the unique maximal subgroup of X containing T and so $X \cong \mathrm{Alt}(9)$ is 3-minimal. □

C. Small modules for finite simple groups

This appendix focuses on small representations of simple groups. The results have been applied throughout the proof of Main Theorem 1. In particular, Section 5 requires most of the results presented here. For studying irreducible modules for groups of Lie type the following lemma is fundamental.

LEMMA C.1. *Let X be a genuine group of Lie type in characteristic p and V be an absolutely irreducible $\mathrm{GF}(p^e)X$-module. If P is a parabolic subgroup of X and $U = O_p(P)$. Then $C_V(U)$ is an irreducible $\mathrm{GF}(p^e)P$-module. In particular, for S a Sylow p-subgroup of X and $B = N_X(S)$, we have that $C_V(S)$ is 1-dimensional as a $\mathrm{GF}(p^e)B$-module.*

PROOF. See [**70**] or [**27**, Theorem 2.8.11]. □

We fix the following notation. If X is a simple group then we denote by $R_t(X)$ the minimal dimension of a faithful projective representation of X in characteristic t over a splitting field and by

$$R_{p'}(X) = \min_{t \neq p} R_t(X).$$

LEMMA C.2. *Let p be a prime and X be a sporadic simple group. Then the following table presents lower bounds for $R_p(X)$.*

X	Mat(11)	Mat(12)	Mat(22)	Mat(23)	Mat(24)	J_1	J_2	J_3	J_4
$R_p(X)$	5	6	6	11	11	7	6	9	110
X	HiS	McL	He	Ru	Suz	O'N	Co_1	Co_2	Co_3
$R_p(H)$	20	21	18	28	12	31	24	22	22

X	M(22)	M(23)	M(24)$'$	F_5	LyS	F_3	F_2	F_1
$R_p(X)$	27	234	702	56	110	48	234	729

PROOF. See [**37**, page 187] for these approximations. □

For the groups Mat(11) and Mat(22) and $p = 3$ we will need more precise information.

LEMMA C.3. *Let X be a quasisimple group with $X/Z(X) \cong$ Mat(11) or Mat(22).*

(i) *If $X \cong$ Mat(11), then X has no 6-dimensional irreducible* GF(3)*-module.*
(ii) *If $X/Z(X) \cong$ Mat(22), then $R_3(X) \geq 7$.*

PROOF. By [**34**, Theorem 7.1], Mat(11) has no 6-dimensional irreducible GF(3)-module, which is (i).

Let $X/Z(X) \cong$ Mat(22). By [**27**, Table 5.3c], $X/Z(X)$ has a subgroup of shape $2^3 : \mathrm{SL}_3(2) = 2^3{:}\mathrm{PSL}_2(7)$ and this subgroup has a preimage in X which is an elementary abelian group of order at most 2^4 extended by $\mathrm{PSL}_2(7)$. Since the minimal faithful permutation representation of $\mathrm{PSL}_2(7)$ is of degree 7 as 7 does not divide $|\mathrm{Alt}(m)|$, $m \leq 6$, then Lemma 2.23 implies that $R_3(X) \geq 7$, which is (ii). □

The next result is due to Wagner.

LEMMA C.4. *Let X be an alternating group of degree n with $n \geq 9$. Then, for all primes p, $R_p(X) = n - 1 - \delta_{n,p} \geq n - 2$.*

PROOF. See [**74, 75, 76**]. □

LEMMA C.5. *Let X be a simple group of Lie type defined over* GF(q)*, $q = p^e$. Then lower bounds for $R_{p'}(X)$ and $R_p(X)$ are presented in the following table.*

X	*lower bounds for* $R_{p'}(X)$	$R_p(X)$	*exceptions*
$\mathrm{PSL}_2(q), q$ *odd*	$(q-1)/2$	2	$R_{p'}(\mathrm{PSL}_2(9)) = 3$
$\mathrm{PSL}_2(q), q$ *even*	$(q-1)$	2	$R_{p'}(\mathrm{PSL}_2(4)) = 2$
$\mathrm{PSL}_m(q), m \geq 3$	$(q^m-1)/(q-1)-m$	m	$R_{p'}(\mathrm{PSL}_3(2)) = 2$ $R_{p'}(\mathrm{PSL}_3(4)) = 4$ $R_{p'}(\mathrm{PSL}_4(2)) = 7$ $R_{p'}(\mathrm{PSL}_4(3)) = 26$
$\mathrm{PSU}_m(q), m \geq 3$ *odd*	$q(q^{m-1}-1)/(q+1)$	m	
$\mathrm{PSU}_m(q), m \geq 4$ *even*	$(q^m-1)/(q+1)$	m	$R_{p'}(\mathrm{PSU}_4(3)) = 6$ $R_{p'}(\mathrm{PSU}_4(2)) = 4$
$\mathrm{PSp}_{2m}(q), m \geq 2, q$ *even*	$q(q^m-1)(q^{m-1}-1)/2(q+1)$	$2m$	
$\mathrm{PSp}_{2m}(q), m \geq 2, q$ *odd*	$(q^m-1)/2$	$2m$	
$\mathrm{P\Omega}^+_{2m}(q), m \geq 4, q = 2,3$	$q(q^{2m-2}-1)/(q^2-1)$ $-(q^{m-1}-1)/(q-1)-7\delta_{2,p}$	$2m$	$R_{p'}(\mathrm{P\Omega}^+_8(2)) = 8$
$\mathrm{P\Omega}^+_{2m}(q), m \geq 4, q \neq 2,3$	$q(q^{2m-2}-1)/(q^2-1)$ $+q^{m-1}-m$	$2m$	
$\mathrm{P\Omega}^-_{2m}(q), m \geq 4$	$q(q^{2m-2}-1)/(q^2-1)$ $-q^{m-1}-m+2$	$2m$	
$\mathrm{P\Omega}_{2m+1}(q), m \geq 3, q \neq 3$	$(q^{2m}-1)/(q^2-1)-m$	$2m+1$	
$\mathrm{P\Omega}_{2m+1}(3), m \geq 3$	$(3^{2m}-1)/(3^2-1)-(3^m-1)/2$	$2m+1$	$R_{p'}(\mathrm{P\Omega}_7(3)) = 27$
$\mathrm{E}_6(q)$	$q^9(q^2-1)$	27	
${}^2\mathrm{E}_6(q)$	$q^9(q^2-1)$	27	
$\mathrm{E}_7(q)$	$q^{15}(q^2-1)$	56	
$\mathrm{E}_8(q)$	$q^{27}(q^2-1)$	248	
$\mathrm{F}_4(q), q$ *even*	$\frac{1}{2}q^7(q^3-1)(q-1)$	26	$R_{p'}(\mathrm{F}_4(2)) \geq 44$
$\mathrm{F}_4(q), q$ *odd*	$q^6(q^2-1)$	$26-\delta_{p,3}$	
$\mathrm{G}_2(q)$	$q(q^2-1)$	$7-\delta_{p,2}$	$R_{p'}(\mathrm{G}_2(3)) = 14$ $R_{p'}(\mathrm{G}_2(4)) = 12$
${}^3\mathrm{D}_4(q)$	$q^3(q^2-1)$	8	
${}^2\mathrm{F}_4(q), q = 2^{e+1}, e > 1$	$q^4(q-1)\sqrt{q/2}$	26	
${}^2\mathrm{B}_2(q), q = 2^{e+1}, e > 1$	$(q-1)\sqrt{q/2}$	4	$R_{p'}({}^2\mathrm{B}_2(8)) = 8$
${}^2\mathrm{G}_2(q), q = 3^{e+1}, e > 1$	$q(q-1)$	7	
${}^2\mathrm{G}_2(3)'$	2	7	
$\mathrm{Sp}_4(2)'$	2	3	
$\mathrm{G}_2(2)'$	3	6	
${}^2\mathrm{F}_4(2)'$	16	26	

PROOF. For the genuine groups of Lie type the bounds for $R_{p'}(X)$ are found in [**68**]. For $R_p(X)$ we refer to [**37**, Theorem 5.4.13]. For the remaining four simple groups, as ${}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8)$, $\mathrm{Sp}_4(2)' \cong \mathrm{PSL}_2(9)$ and $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$ we obtain the bounds using the results we already have. For $R_{2'}({}^2\mathrm{F}_4(2)')$ the proof is the same as in [**38**, Lemma 4.9] and for $R_2({}^2\mathrm{F}_4(2)')$ we refer to [**37**, Proposition 5.4.13]. □

LEMMA C.6. *Let X be a simple group of Lie type defined over* $\mathrm{GF}(q)$, $q = p^e$.

(i) *If $R_{p'}(X) \leq 4R_p(X)$, then X is one of the following groups.*
- $\mathrm{PSL}_2(q)$, $q \leq 17$ *with q odd,* $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_4(2)$.
- $\mathrm{PSU}_3(3), \mathrm{PSU}_3(4), \mathrm{PSU}_4(2), \mathrm{PSU}_4(3), \mathrm{PSU}_5(2)$, $\mathrm{PSU}_6(2)$.
- $\mathrm{PSp}_4(2)'$, $\mathrm{PSp}_4(3)$, $\mathrm{PSp}_4(5)$, $\mathrm{PSp}_6(2)$, $\mathrm{PSp}_6(3)$, $\mathrm{P\Omega}_7(3)$, $\mathrm{P\Omega}^+_8(2)$, $\mathrm{P\Omega}^-_8(2)$.

- $\mathrm{F}_4(2)$, $\mathrm{G}_2(2)'$, $\mathrm{G}_2(3)$, $\mathrm{G}_2(4)$, ${}^3\mathrm{D}_4(2)$, ${}^2\mathrm{F}_4(2)'$, ${}^2\mathrm{B}_2(8)$, ${}^2\mathrm{G}_2(3)'$.

(ii) *If $R_{p'}(X) \leq R_p(X)$, then $X \cong \mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(5)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSp}_4(2)'$, $\mathrm{PSU}_4(2)$, ${}^2\mathrm{G}_2(3)'$, $\mathrm{G}_2(2)'$, ${}^2\mathrm{F}_4(2)'$ or $\mathrm{P}\Omega_8^+(2)$, where we must have equality besides when $X \cong \mathrm{PSL}_3(2)$, $\mathrm{PSp}_4(2)'$, $\mathrm{G}_2(2)'$, ${}^2\mathrm{F}_4(2)'$ or ${}^2\mathrm{G}_2(3)'$.*

(iii) *If $R_{p'}(X) \leq 8$, then X is one of the following groups*
- $\mathrm{PSL}_2(q)$, $q \leq 17$ *with q odd,* $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_4(2)$.
- $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(3)$.
- $\mathrm{PSp}_4(2)'$, $\mathrm{PSp}_4(3)$, $\mathrm{PSp}_6(2)$, $\mathrm{P}\Omega_8^+(2)$.
- $\mathrm{G}_2(2)'$, ${}^2\mathrm{B}_2(8)$, ${}^2\mathrm{G}_2(3)'$.

PROOF. This result is obtained from the data presented in the table from Lemma C.5. □

LEMMA C.7. *Suppose that p and r are primes with $p \neq r$ and X is a simple group of Lie type defined in characteristic r. Then $R_r(X) \geq m_p(X)$.*

PROOF. Let $k = R_r(X)$. Then by the definition of $R_r(X)$, we have that X is isomorphic to a subgroup of $\mathrm{PGL}_k(r^e)$ for some suitable e. Therefore $m_p(X) \leq m_p(\mathrm{PGL}_k(r^e)) \leq k$ by [**37**, 5.5.2] and consequently $m_p(X) \leq R_r(X)$. □

A special role is played by the so-called quadratic and cubic representations of quasisimple groups.

DEFINITION C.8. Suppose that p is a prime, A is a group and V is a non-trivial $\mathrm{GF}(p)A$-module. Then

(i) A acts *quadratically* on V provided $[V, A, A] = 0$; and
(ii) A acts *cubically* on V provided $[V, A, A, A] = 0$.

If A acts cubically on V but not quadratically on V, then we say that A acts *strictly cubically* on V.

We remark that

LEMMA C.9. *If A acts quadratically and faithfully on a vector space V defined over $\mathrm{GF}(p)$, then A is an elementary abelian p-group.*

PROOF. This is well-known. □

We will now study quadratic modules more closely. The first result is independent of the classification of the finite simple groups.

LEMMA C.10. *Suppose that p is an odd prime, V is a faithful $\mathrm{GF}(p)X$-module and $x \in \mathrm{GL}(V)$ normalizes X. If x acts quadratically on V and $|X|$ is coprime to p, then $[X,x] \le O_2(X)$. In particular $[E(X),x]=1$.*

PROOF. By coprime action, $X = C_X(x)[X,x]$. Let X be a minimal counterexample, then $X=[X,x]$. Set $Y = X\langle x\rangle$. Let r be a prime which divides $|X|$ and $R \in \mathrm{Syl}_r(X)$. Then $N_Y(R)$ contains a conjugate of x by the Frattini Argument. Thus $R\langle x\rangle$ is a subgroup of Y. By [**22**, Chap. 3, Theorem 8.4], if r is odd, $O_p(R\langle x\rangle) \neq 1$ and so $[R,x]=1$. In particular, letting $T \in \mathrm{Syl}_2(X)$ be x-invariant, we have $X = TC_X(x)$. Therefore $X=[X,x]=[T,x] \le T$, which is a contradiction as surely X is not a 2-group.

For the second assertion we now have that $[E(X),x] \le O_2(E(X)) \le Z(E(X))$. So $[\langle x\rangle, E(X), E(X)] = 1$. The Three Subgroup Lemma gives $[\langle x\rangle, E(X)] = 1$, the assertion. $\square$

LEMMA C.11. *Let X be a finite group, p an odd prime and V be a faithful, irreducible* $\mathrm{GF}(p)X$*-module. Assume the following conditions.*

(a) *There is a non-trivial subgroup A of X which acts quadratically on V and $X = \langle A^X\rangle$; and*
(b) $C_X(F(X)) \le F(X)$.

Then we have the following:

(i) $|A| = p = 3$;
(ii) *$F(X) = O_2(X) = Z(X)E$, where E is an extraspecial 2-group of order 2^{1+2n}, $Z(X)$ is cyclic of order 2 or 4; and*
(iii) *$X/O_2(X)$ is isomorphic to* $\mathrm{Alt}(2n+1)$, $\mathrm{Alt}(2n+2)$, $\mathrm{GU}_n(2)$, $\Omega^{\pm}_{2n}(2)$ *or* $\mathrm{Sp}_{2n}(2)$*. Furthermore $F(X)/Z(X)$ is a natural module for $X/F(X)$.*

PROOF. See [**16**, Theorem A]. $\square$

LEMMA C.12. *Let X be a finite group, p be an odd prime and V be a faithful, irreducible* $\mathrm{GF}(p)X$*-module. Assume that*

(a) *$A \le X$ with $\langle A^X\rangle = X$ and A acts quadratically on V; and*
(b) *K is a normal quasisimple subgroup of X and $C_X(K) = Z(X)$.*

Then either $Z(X) \le K$ and $X = K$ is a group of Lie type in characteristic p, or $|A| = p = 3$ and one of the following holds:

(i) *$X \cong \mathrm{PGU}_n(2)$, $n \ge 5$;*
(ii) *$|Z(X)| = 2$, $X/Z(X) \cong \mathrm{Alt}(n)$, $n \ge 5$ and $n \neq 6$;*
(iii) *$|Z(X)| = 2$, $X/Z(X) \cong$ $\mathrm{P}\Omega^+_8(2)$, $\mathrm{G}_2(4)$, $\mathrm{PSp}_6(2)$, Co_1, Suz or J_2.*

PROOF. See [**17**, Theorem A]. $\square$

LEMMA C.13. *Suppose that X is a group with $F^*(X)$ quasisimple and V is an irreducible faithful* $\mathrm{GF}(2)X$*-module. Assume that $A \le X$ is a 2-subgroup of order at least 4 and that A acts quadratically on V.*

(i) *If $F^*(X)/Z(F^*(X))$ is a sporadic simple group, then one of the following hold:*
 (ia) *$F^*(X) \cong \mathrm{Mat}(12)$ and V is 10-dimensional.*
 (ib) *$F^*(X) \cong \mathrm{Mat}(22)$ and V is 10-dimensional.*
 (ic) *$F^*(X) \cong \mathrm{Mat}(24)$ and V is 11-dimensional.*
 (id) *$F^*(X) \cong \mathrm{J}_2$ and V is 12-dimensional.*
 (ie) *$F^*(X) \cong \mathrm{Co}_2$ and V is 22-dimensional.*
 (if) *$F^*(X) \cong \mathrm{Co}_1$ and V is 24-dimensional.*
 (ig) *$F^*(X) \cong 3{\cdot}\mathrm{Suz}$ and V is 24-dimensional.*
 (ih) *$F^*(X) \cong 3{\cdot}\mathrm{Mat}(22)$ and V is 12-dimensional.*
 Furthermore, if $|A| \ge 8$, then $F^*(X) \cong 3{\cdot}\mathrm{Mat}(22)$ and, in this case, if $|A| = 16$, then $N_{F^*(X)}(A)/A \cong 3{\cdot}\mathrm{Alt}(6)$.

(ii) *If $F^*(X)/Z(F^*(X))$ is a group of Lie type defined in odd characteristic which is not isomorphic to a group of Lie type defined in characteristic 2, then $F^*(X) \cong 3{\cdot}\mathrm{PSU}_4(3)$. Furthermore $\dim V = 12$ and $|A| \le 2^5$.*

(iii) *If $F^*(X)/Z(F^*(X))$ is an alternating group, then either V is the natural module or a spin module or $F^*(X) \cong 3{\cdot}\mathrm{Alt}(6)$ and V is a 6-dimensional module, or $F^*(X) \cong 3{\cdot}\mathrm{Alt}(7)$ and V is 12-dimensional. Furthermore,*
 (iiia) *if $|A| > 8$, then V is a natural module or $X \cong \mathrm{Alt}(8)$ and $|V| = 16$.*
 (iiib) *If V is a spin module and $X \not\cong \mathrm{Alt}(6)$ or $\mathrm{Alt}(8)$, then either $|A| = 4$ and A is conjugate to $\langle (12)(34), (13)(24)\rangle$, or $|A| = 8$, $n = 9$ and A is conjugate in $\mathrm{Sym}(9)$ to $\langle (12)(34)(56)(78), (13)(24)(57)(68), (14)(26)(37)(48)\rangle$.*

PROOF. (i) This is [**49**, Theorems 1, 2 and 3].

(ii) This is [**48**, Theorem and Proposition 3.2]. To see that $|A| \le 2^5$ we argue as follows: by [**27**, Proposition 6.2.2] we have $\mathrm{Out}(\mathrm{PSU}_4(3)) \cong \mathrm{Dih}(8)$ acts faithfully on the 3-part of the Schur multiplier of $\mathrm{PSU}_4(3)$. This shows $Z(F^*(X))$ is normalized by a fours group in $\mathrm{Out}(\mathrm{PSU}_4(3))$ and just centralized by a group of order 2. If $A \not\le F^*(X)$, then by quadratic action we have that A has to centralize $Z(F^*(X))$. Hence $|A : A \cap F^*(X)| \le 2$. Since $m_2(\mathrm{PSU}_4(3)) = 4$ by [**27**, Theorem 4.10.5], we see that $|A| \le 2^5$.

(iii) If $Z(F^*(X)) \ne 1$ this is [**45**, Lemma 7.4]. So assume that $F^*(X) \cong \mathrm{Alt}(n)$. Then we get that V is the natural module or a spin module from [**49**, Theorem 4]. The final statements are presented in [**45**, Lemma 7.5]. □

LEMMA C.14. *Let X be a classical group defined over* $\mathrm{GF}(q)$*, V a natural module and $A \leq X$ be a quadratic subgroup of X of maximal order. Then the following hold:*

(i) *if $X \cong \mathrm{SL}_n(q)$ with $n \geq 2$, then $|A| \geq q^{n^2/4}$ if n is even and $|A| \geq q^{(n+1)(n-1)/4}$ if n is odd;*
(ii) *if $X \cong \mathrm{SU}_n(q)$ with $n \geq 3$, then $|A| \geq q^{n^2/4}$ if n is even and $|A| \geq q^{(n-1)^2/4}$ if n is odd;*
(iii) *if $X \cong \mathrm{Sp}_{2n}(q)$ with $n \geq 2$, then $|A| \geq q^{(n+1)n/2}$;*
(iv) *if $X \cong \Omega^+_{2n}(q)$, then $|A| \geq q^{n(n-1)/2}$;*
(v) *if $X \cong \Omega^-_{2n}(q)$, then $|A| \geq q^{(n-1)(n-2)/2}$; and*
(vi) *if $X \cong \Omega_{2n+1}(q)$, then $|A| \geq q^{n(n-1)/2}$.*

PROOF. This result is taken from [**45**, Lemma 3.4]. □

The next lemma is about transvection subgroups of certain classical groups.

LEMMA C.15. *Let $X \cong \mathrm{Sp}_{2n}(p^e)$, $\mathrm{O}^{\pm}_n(p^e)$ or $\mathrm{GU}_n(p^e)$ and V be the corresponding natural module. Assume that $Y \leq X$ acts quadratically on V and $\dim[V,Y] = 1$. If X is either symplectic or unitary, then $|Y| \leq p^e$ and, if X is orthogonal, then $p = 2$ and $|Y| = 2$.*

PROOF. Since Y acts quadratically on V, we have $C_V(Y) = [V,Y]^{\perp}$ by Lemma 2.20 and Y is an elementary abelian p-group by Lemma C.9. Since $\dim[V,Y] = 1$, we deduce that $C_V(Y)$ is a hyperplane of V. Let $U \leq C_V(Y)$ be a non-degenerate space of dimension $n-2$. Then Y centralizes U and leaves $U^{\perp}$ invariant. Now $U^{\perp}$ is a 2-dimensional symplectic, orthogonal or unitary space. Thus Y embeds into $\mathrm{Sp}_2(p^e)$, $\mathrm{O}^{\pm}_2(p^e) \cong \mathrm{Dih}(p^e \pm 1)$ or $\mathrm{GU}_2(p^e)$. In the first and the last case we see that Y has order at most p^e. In the second case we see that $\mathrm{Dih}(p^e \pm 1)$ has order coprime to p unless $p = 2$ and then we have that $|Y| = 2$. □

LEMMA C.16. *Assume $X \cong \Omega^-_6(2)$ and V is the natural* $\mathrm{GF}(2)X$*-module. Then there is a fours subgroup of X which operates quadratically on V.*

PROOF. The group $\mathrm{SO}^-_6(2)$ contains a subgroup $D = \mathrm{SO}^-_2(2) \times \mathrm{SO}^-_2(2) \times \mathrm{SO}^-_2(2) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. The Sylow 2-subgroup of D acts quadratically on V. Thus $\Omega^-_6(2)$ contains a quadratic fours subgroup. □

LEMMA C.17. *Let $X = \mathrm{Alt}(5)$ and V be a $\mathrm{GF}(2)X$-module. Assume that there is a submodule V_1 of V such that both V_1 and V/V_1 are natural $\mathrm{SL}_2(4)$-modules. Let U be a Sylow 2-subgroup of X. Then*

(i) *Any X-orbit in V of length 15 generates a proper submodule; and*

(ii) *If U acts quadratically on V then V over V_1 splits.*

PROOF. (i) Choose $v \in V$ with $|v^X| = 15$. We may assume $v \in C_V(U)$. If $C_V(U) \leq V_1$, then $v^X \subseteq V_1$ and we are done. Hence we may assume that $C_V(U) \not\leq V_1$. Let $A = N_X(U) = \langle U, \rho\rangle \cong \mathrm{Alt}(4)$ with ρ of order 3. Then, as ρ acts fixed-point-freely on V and A acts on $C_V(U)$, we get $|C_V(U)| = 16$ and $C_V(U) = C_V(u) = [V, u]$ for all $u \in U^{\#}$. Let $t \in X$ be an involution with $X = \langle t, U\rangle$. Then $v^t \notin C_V(U)$ since otherwise X centralizes $\langle v, v^t\rangle$. Since $C_V(\rho) = 0$ and $C_V(U) = C_V(u)$ for all $u \in U^{\#}$, $|(v^t)^A| = 12$ and, as $C_V(U) = [V, U]$,

$$\langle (v^t)^A\rangle + C_V(U)/C_V(U) = \langle (v^t)^{\langle\rho\rangle}\rangle + C_V(U)/C_V(U)$$

has dimension at most 3. Since v^A has size 3,

$$v^X = (v^t)^A \cup v^A \subseteq \langle (v^t)^A\rangle + C_V(U) < V$$

and this proves the first claim.

For part (ii) we have that $C_V(U) \not\leq V_1$. In particular there is some $v \in V \setminus V_1$ such that $|v^X| = 15$. By (i) we have that v^X is contained in a proper submodule and so $V_2 = \langle v^X\rangle$ is a natural $\mathrm{SL}_2(4)$-module and $V = V_1 \oplus V_2$. □

DEFINITION C.18. Suppose that X is a group and V is a $\mathrm{GF}(p)X$-module. Then, for natural numbers m, V is an *mF-module* with *m-offender* $A \leq X$ if $A/C_A(V)$ is an elementary abelian p-group and

$$|V/C_V(A)| \leq |A/C_A(V)|^m.$$

We call an mF-module *sharp* if for any m-offender A we have that

$$|V/C_V(A)| = |A/C_A(V)|^m.$$

We call V a *dual mF-module* with *dual m-offender* $A \leq X$ if $A/C_A(V)$ is an elementary abelian p-group and

$$|[V, A]| \leq |A/C_A(V)|^m.$$

If $m = 1$, then $1F$-modules are called F-modules and dual $1F$-modules are called dual F-modules the corresponding subgroup A is called an offender or a dual offender respectively.

LEMMA C.19. *Let V be a faithful $\mathrm{GF}(p)X$-module and A be an elementary abelian p-subgroup of X. Let V^* be the dual module of V.*

(i) *If A acts quadratically on V, then A acts quadratically on V^*.*
(ii) *If A acts (strictly) cubically on V, then A acts (strictly) cubically on V^*.*
(iii) *If A is an m-offender on V, then A is a dual mF-offender on V^*.*
(iv) *If A is a dual m-offender on V, then A is an m-offender on V^*.*

PROOF. Parts (i) and (ii) are an easy calculation using the definition of the dual module. We prove (iii) and (iv).

Suppose that A is an m-offender on V. Then, as V is a faithful $\mathrm{GF}(p)X$-module, $|V/C_V(A)| \leq |A|^m$. By Lemma 2.19, $C_V(A)^\dagger = [V^*, A]$ and $V^*/C_V(A)^\dagger \cong C_V(A)^*$. Thus, as duality preserves dimension, we have

$$|A|^m \geq |V/C_V(A)| = |V^*|/|C_V(A)^*| = |C_V(A)^\dagger| = |[V^*, A]|.$$

Hence A is a dual m-offender on V^*. Similarly, if $|[V, A]| \leq |A|^m$, then, as $V^*/[V, A]^\dagger \cong [V, A]^*$ and $[V, A]^\dagger = C_{V^*}(A)$, we obtain

$$|A|^m \geq |[V, A]| = |[V, A]^*| = |V^*/[V, A]^\dagger| = |V^*/C_{V^*}(A)|.$$

Thus A is an m-offender on V^*. This proves (iii) and (iv). □

Quadratic action and F-modules play a pivotal role in many sophisticated group theoretical problems such as problems involving factorisations or pushing-up. The two concepts are linked as follows: suppose that V is an F-module with offender A. Then we may apply the Thompson replacement theorem [**22**, Chap. 8, Theorem 2.5] to the semidirect product VA to see that VA contains a subgroup B which is also an offender on V and which operates quadratically on V.

In the next lemma we identify certain modules as "natural modules" and "spin" or "half spin" modules. The formal definitions of these modules is given in [**47**, Section A.2]. For example, if $X \cong \mathrm{Sym}(n)$ or $\mathrm{Alt}(n)$, then the non-trivial irreducible section of the n-dimensional permutation is called the natural X-module.

We have taken the following lemma from [**15**].

LEMMA C.20. *Suppose that G is p-minimal, $S \in \mathrm{Syl}_p(G)$ and M be the unique maximal subgroup of G which contains S. Let V be a faithful $\mathrm{GF}(p)P$-module. Assume that there exists an elementary abelian subgroup $A \leq T$ of order p^n and*

(i) *$|V/C_V(A)| \leq |A|$ and $|A_0||C_V(A_0)| < |A||C_V(A)|$ for every $1 \neq A_0 < A$,*
(ii) *$[C_V(T), P] \neq 1$, and*

(iii) *$P = \langle A, A^x \rangle$ for every $x \in G \setminus M$.*

Then $P \cong \mathrm{SL}_2(p^n)$, $C_V(A) = [V, A]C_V(P)$, and $V/C_V(P)$ is a natural $\mathrm{SL}_2(p^n)$-module for P.

Proof. See [**15**, Lemma 3.5]. □

Lemma C.21. *Let X be a group such that $F^*(X)$ is quasisimple and let V be a faithful, irreducible $\mathrm{GF}(2)F^*(X)$-module which is an F-module for X. Then $F^*(X)$ is either a classical group defined in characteristic 2, $\mathrm{G}_2(2^e)'$ ($e \geq 1$), $\mathrm{Alt}(n)$, ($n \geq 5$), or $3{\cdot}\mathrm{Alt}(6)$. Furthermore, one of the following holds:*

(i) *$F^*(X)$ is a classical group in characteristic 2 and V is a natural module.*
(ii) *$F^*(X) \cong \mathrm{Alt}(n)$, $n \geq 5$ and V is a natural module.*
(iii) *$F^*(X) \cong \mathrm{SL}_n(2^e)$, $e \geq 1$, and V is the exterior square of a natural module. Furthermore, in this case, V is sharp.*
(iv) *$F^*(X) \cong \mathrm{Sp}_6(2^e)$ or $\Omega^+_{10}(2^e)$, $e \geq 1$, and V is a spin module or half-spin module, respectively. If $F^*(X) \cong \Omega^+_{10}(2^e)$, then V is sharp.*
(v) *$F^*(X) \cong \mathrm{G}_2(2^e)$ and V is a natural module. In this case V is sharp.*
(vi) *$F^*(X) \cong 3{\cdot}\mathrm{Alt}(6)$ and $|V| = 2^6$ and V is sharp.*
(vii) *$X \cong \mathrm{Alt}(7)$ and $|V| = 2^4$ and V is sharp.*

Proof. This can be obtained by combining [**45**, Theorems 2 and 3]. □

Lemma C.22. *Suppose that X is a group with $F^*(X)$ quasisimple. Let V be a faithful, irreducible $\mathrm{GF}(p)X$-module. Assume that $X = \langle A^X \rangle$, A is a dual offender on V and $[v, A] = [V, A]$ for all $v \in V \setminus C_V(A)$. Then one of the following holds:*

(i) *$X \cong \mathrm{SL}_n(p^e)$ or $\mathrm{Sp}_{2n}(p^e)$ with $n \geq 2$ and V is a natural module;*
(ii) *$X \cong \mathrm{Alt}(6)$ or $\mathrm{Alt}(7)$, $\dim V = 4$ and $|A| = 4$; or*
(iii) *$p = 2$ and $X = \mathrm{O}^{\pm}_{2n}(2)$ with $n \geq 3$ or $\mathrm{Sym}(n)$ with $n = 5$ or $n \geq 7$, V is the corresponding natural module and $|A| = 2$.*

Proof. This is [**46**, 3.1]. □

Lemma C.23. *Let $X \cong \mathrm{PSU}_3(p^e)$ or $\mathrm{SU}_3(p^e)$ and V be an irreducible $\mathrm{GF}(p)$-module for X. Let S be a Sylow p-subgroup of X and $A = Z(S)$. Then A does not induces an F-module offender on V.*

Proof. We have $|A| = p^e$. If A induces an F-module offender, then $|V : C_V(A)| \leq p^e$. By Lemma A.6 X is generated by three conjugates of A. This implies $|V| \leq p^{3e}$. Hence X is a subgroup of $\mathrm{GL}_{3e}(p)$. We have that $p^{3e}+1$ divides the order of X. Let r be a primitive prime divisor of $p^{6e}-1$ according to Theorem 2.28. Then r divides $|X|$ but r does not divide $|\mathrm{GL}_{3e}(p)|$, a contradiction. □

Next we study the class of $2F$-modules which will come up when studying p-minimal subgroups. We do not need the full strength of the classification given in [**29**] and [**30**]. In particular we do not require the classification of the $2F$-modules for groups of Lie type in defining characteristic given in [**30**].

Theorem C.24. *Suppose that p is a prime and X is p-minimal, $Y = F^*(X)$ is quasisimple but not isomorphic to a group of Lie type in characteristic p, and that V is a faithful $\mathrm{GF}(p)G$-module which is a cubic $2F$-module or dual $2F$-module. Then one of the following holds:*

(i) *$p = 2$ and $Y \cong \mathrm{Alt}(2^a+1)$ with $a \geq 3$(two possible actions for $\mathrm{Alt}(9)$ both with $|V| = 2^8$);*
(ii) *$p = 3$ and $Y \cong \mathrm{Alt}(9)$ or $\mathrm{Alt}(3^a+1)$ with $a \geq 2$;*
(iii) *$p = 3$ and $Y \cong 2{\cdot}\mathrm{Alt}(9)$;*
(iv) *$p = 3$ and $Y \cong \mathrm{Sp}_6(2)$; or*
(v) *$p = 3$ and $Y \cong 2{\cdot}\mathrm{Sp}_6(2)$.*

Proof. By Lemma C.19 it is enough to prove the theorem for cubic $2F$-modules. Suppose $Y/Z(Y)$ is an alternating group which is not a simple group of Lie type. Then [**29**, Theorem 6.2 and Table 6.3] yields that $p \in \{2,3\}$. Then, as X p-minimal, Lemmas B.3 and B.4 imply that (i), (ii) or (iii) holds.

If $Y/Z(Y)$ is a simple group of Lie type defined in characteristic r with $r \neq p$ (which cannot be identified with a simple group of Lie type in characteristic p), then we apply [**29**, Theorem 6.4 and Table 6.5]. This yields $p = 2$ and $Y/Z(Y) \cong \mathrm{PSU}_4(3)$ or $p = 3$ and $Y \cong 2{\cdot}\mathrm{PSL}_3(4)$, $\mathrm{Sp}_6(2)$, $2{\cdot}\mathrm{Sp}_6(2)$ or $2{\cdot}\Omega_8^+(2)$.

In the first case, [**14**, Table 8.10] shows that the centralizer of a 2-central involution is a maximal subgroup of X and the subgroup $4^2 : \mathrm{Sym}(4)$ is normalized by a Sylow 2-subgroup but does not centralize an involution. Thus $Y/Z(Y) \cong \mathrm{PSU}_4(3)$ is not 2-minimal.

Suppose that $p = 3$. If $Y/Z(Y) \cong \mathrm{Sp}_6(2)$ we have (iv) or (v). Thus we have to deal with $Y \cong 2{\cdot}\mathrm{PSL}_3(4)$ or $2{\cdot}\Omega_8^+(2)$. By [**27**, Proposition 6.2.2] we have that $\mathrm{Out}(\mathrm{PSL}_3(4))$ acts faithfully on the 2-part of the Schur multiplier of $Y/Z(Y)$. Hence, as X is 3-minimal, $X = O^{3'}(X)$ and so we now see that $X = Y$. Using [**14**, Tables 8.3 and 8.4], we see that $\mathrm{PSL}_3(4)$ has maximal subgroups $\mathrm{PSU}_3(2)$ and $\mathrm{PSL}_2(9)$ both containing a Sylow 3-subgroup. Hence this group is not 3-minimal. So we may assume that $X \cong 2{\cdot}\Omega_8^+(2)$. In this case [**14**, Table 8.50] shows that the subgroup $(3\times\Omega_6^-(2))$:2 is a maximal subgroup of $X/Z(X)$ and so as this subgroup does not contain $\mathrm{SO}_2^-(3)\wr\mathrm{Sym}(4)\cap X$ we have that X is not 3-minimal.

If $Y/Z(Y)$ is a sporadic simple group, then the appropriate reference is [**29**, Theorem 6.6 and Table 6.7] which gives $p = 3$ and $Y/Z(Y) \cong \mathrm{Mat}(11)$ or $\mathrm{Mat}(12)$ or $p = 2$ and $Y/Z(Y) \cong \mathrm{Mat}(22), \mathrm{Mat}(23), \mathrm{Mat}(24)$ or J_2. All maximal subgroups of these groups are given in [**27**, Tables 5.3a ... 5.3e] and these lists reveal that the groups listed are not p-minimal. □

Lemma C.25. *Suppose $X \cong {}^2\mathrm{G}_2(3^e)$ and V is a faithful* GF(3)X-*module. Then V is not a $2F$-module.*

Proof. This is taken from [**29**, Lemma 8.5]. □

Lemma C.26. *Suppose that $X \cong \mathrm{PSU}_3(2^e)$ and V is a faithful* GF(2)X-*module. If V is a $2F$-module with 2-offender A, then A acts quadratically on V.*

Proof. Let $S \in \mathrm{Syl}_2(X)$ and assume that $A \le S$. Since A is elementary abelian we have $A \le Z(S)$ and $|A| \le 2^e$ by Lemma A.19 (iii). Noticing that all the involutions in $Z(S)$ are conjugate, we have, for $z \in Z(S)^\#$, $|V : C_V(z)| \le |V : C_V(A)| \le 2^{2e}$. If $|V : C_V(z)| = 2^{2e}$, then $|A| = 2^e$ and $A = Z(S)$ and, furthermore, as $C_V(z) \ge [V, z]$, $Z(S)$ acts quadratically on V.

Notice that by Lemma A.19, for $z \in Z(S)^\#$, we have $C_{N_X(S)}(z) = C_{N_X(S)}(Z(S)) = SH$ where H is cyclic of order $2^e + 1$.

Suppose that $|V : C_V(z)| < 2^{2e}$. Then $C_X(z)$ acts on $V/C_V(z)$. As $|V : C_V(z)| < 2^{2e}$, H does not act faithfully on $V/C_V(z)$ and we see that $\langle C_H(V/C_V(z))^S\rangle = SC_H(V/C_V(z))$ centralizes $V/C_V(z)$ and so $[V, S] \le C_V(z)$ for all $z \in Z(S)$. Hence $[V, S, Z(S)] = 0$ and therefor A acts quadratically. □

Lemma C.27. *Suppose $X \cong {}^2\mathrm{B}_2(2^e)$ and V is a faithful* GF(2)X-*module. If V is a $2F$-module with 2-offender A, then $A = Z(S)$ acts quadratically on V.*

Proof. We start as in the previous lemma, let $S \in \mathrm{Syl}_2(X)$ and assume that $A \le S$. Then, by Lemma A.19 (iv), $A \le Z(S)$ and $|A| \le 2^e$. As all the involutions in Z are conjugate, we have, for $z \in Z(S)^\#$, $|V : C_V(z)| \le |V : C_V(A)| \le 2^{2e}$. Now z inverts an element of order $2^e \pm 2^{(e+1)/2} + 1$ one of which contains a primitive prime divisor of $2^{4e} - 1$. It follows that $|V : C_V(z)| = 2^{2e}$ and so $A = Z(S)$ and $C_V(Z(S)) = C_V(z)$ for all $z \in Z(S)$. Hence $Z(S)$ acts quadratically on V. □

LEMMA C.28. *Let* $X \cong \mathrm{SL}_2(p^e)$ *and* V *be an irreducible* $\mathrm{GF}(p)X$*-module. Assume that* V *is a* $2F$*-module with* 2*-offender a Sylow* p*-subgroup* S *of* X*. Then* V *is either the natural module for* X*, the* 4*-dimensional module for* $\mathrm{SL}_2(p^e) \cong \Omega_4^-(p^{e/2})$ *or,* p *is odd and* V *is the* 3*-dimensional* $\mathrm{PSL}_2(p^e) \cong \Omega_3(p^e)$*-module. The same also holds if* V *is a dual* $2F$*-module with dual* 2*-offender a Sylow* p*-subgroup* S *of* X*.*

PROOF. Let first V be a $2F$-module. By Definition C.18 we have $|V/C_V(S)| \leq |S|^2$. Assume that the field of definition of V is $\mathrm{GF}(p^f)$. Then, setting $e/f = x$, we have $\dim_{\mathrm{GF}(p^f)} V \leq 2x+1$ by Lemma C.1. Let $\langle\sigma\rangle = \mathrm{Gal}(\mathrm{GF}(p^e)/\mathrm{GF}(p^f))$. By the Steinberg Tensor Product Theorem [**27**, Corollary 2.8.6] we have that

$$V \otimes_{\mathrm{GF}(p^f)} \mathrm{GF}(p^e) = V_1^{\sigma_1} \otimes \cdots \otimes V_r^{\sigma_r}$$

of algebraic conjugates of basic modules. Then, as V is defined over $\mathrm{GF}(p^f)$, $V^\sigma \cong V$ by [**2**, 26.3]. In particular, there are at least x (the order of σ) tensor factors in the above expression. Since $\dim_{\mathrm{GF}(p^e)} V_1 \geq 2$, we have $\dim V \geq 2^x$. Hence we require that $2x+1 \geq 2^x$. Hence $x \leq 2$. If $x = 2$, we must have that V_1 is 2-dimensional, so $V \otimes_{\mathrm{GF}(p^f)} \mathrm{GF}(p^e) = V_1 \otimes V_1^\sigma$, which is the orthogonal module. If $x = 1$, V is defined over $\mathrm{GF}(p^e)$ and so $\dim V \leq 3$ and is a basic module. Application of [**13**] or [**27**, Example 2.8.10] now gives that V is the natural module or the 3-dimensional module for p odd.

As all modules are self-dual we see with Lemma C.19 that the assertion also holds if V is a dual $2F$-module. □

We need the following rather explicit result about the 8-dimensional $\mathrm{GF}(2)\mathrm{Alt}(9)$-modules.

LEMMA C.29. *Suppose that* $G \cong \mathrm{Sym}(9)$*,* $H = G'$ *and* Q *is an elementary abelian subgroup of* G *normalized by a Sylow* 2*-subgroup* S *of* H*. Let* V *be an irreducible* 8*-dimensional* $\mathrm{GF}(2)HQ$*-module and assume*

(a) $C_V(Q) = C_V(S)$ *has dimension* 1*; and*

(b) $[V,Q,Q] = C_V(Q)$*.*

Then $|[V,Q]| > 2|Q|$*.*

PROOF. Aiming for a contradiction assume that $|[V,Q]| \leq 2|Q|$. Then, as $m_2(G) = 4$, $|[V,Q]| \leq 2^5$. Recall that HQ is a 2-minimal group. The maximal subgroup of H containing S is $\mathrm{Alt}(8)$.

Suppose that $w \in Q$ is a product of at most two transpositions. Since $[V,Q]$ is normalized by S and $HQ = \langle O^2(C_{HQ}(w)), S\rangle$, $[V,w] < [V,Q]$.

Hence $|[V,w]| \le 2^4$. Since $w \in Q$ and Q is abelian, by (a), $C_V(Q) \le [V,w]$ and so $C_{HQ}(w)$ does not centralize $[V,w]$ for otherwise HQ normalizes $C_V(Q)$. If w is a transposition then $C_{HQ}(w) \cong \langle w\rangle \times \mathrm{Sym}(7)$. But $\mathrm{Sym}(7)$ does not act on a 4-dimensional space. It follows that w is a product of two transpositions. As $C_{HQ}(w)$ contains A_5, we get $|[V,w]| = 16$. Now we have $C_V(w) = [V,w] < [V,Q]$ and so $|Q| = 2^4$. Thus $|Q \cap O^2(C_{HQ}(w))| = 2^2$ and $O^2(C_{HQ}(w)) \cong \mathrm{Alt}(5)$. But $[V,w,Q] \le [V,Q,Q] = C_V(Q)$ by (b) and has order 2 by (a) this contradicts the fact that $\mathrm{Alt}(5)$ has no transvections on its non-trivial irreducible GF(2)-modules. Hence Q contains no elements which are transpositions or products of two transpositions.

Since a Sylow 2-subgroup of G is isomorphic to $\mathrm{Dih}(8) \wr 2$ and any elementary abelian subgroup of order 2^4 is contained in the base group of this wreath product, we obtain

$$|Q| \le 2^3$$

and consequently

$$|[V,Q]| \le 2^4.$$

Let $Q_0 = Q \cap H$. Baring in mind that Q is normalized by S, $Q_0 \ne 1$. The non-trivial elements of Q_0 are of cycle type 2^4. Suppose $|Q| \le 4$. For $w \in Q_0^{\#}$ then $[V,w] \le [V,Q]$ has order at most 2^3 and so $|C_V(w)| \ge 2^5$. We will draw the same conclusion if $|Q| = 8$. In this case $|Q_0| \ge 2^2$. Choose $w \in Q_0^{\#}$. If $[V,w] = [V,Q]$, then as all involution in Q_0 are conjugate we have that $[V,w_1] = [V,Q]$ for all $w_1 \in Q_0$. As Q does not act quadratically by (b), we have $|Q_0| = 4$ and $G = QH$. Now $[V,Q]$ is invariant under $K = \langle C_{HQ}(z) \mid z \in Q_0^{\#}\rangle$. We have that $C_H(w)$ is a minimal parabolic subgroup of $H \cong \mathrm{SL}_4(2)$. Hence $K \cap H$ is a parabolic subgroup of H with Levi factor $\mathrm{SL}_3(2)$. As $Q \not\le H$, some element of Q induces an outer automorphism on $H \cong \mathrm{Alt}(8)$ and so $K = G$. But then $[V,Q]$ is G-invariant, a contradiction as V is irreducible. This contradiction shows $[V,w] < [V,Q]$ and so again $|C_V(w)| = 2^5$. That is

$$|C_V(w)| \ge 2^5 \text{ for } w \text{ a product of four transpositions.}$$

Since w inverts an element f of order 5, we have $|C_V(f)| = 2^4$. Now select $k = (1,2,3,4,5)$ and $w = (2,6)(3,7)(4,8)(5,9)$ and we see that $\langle k,w\rangle$ centralizes a non-zero subspace of V. On the other hand we have $a = [w,k]^3 = (1,7,3)(2,5,8)(4,9,6)$ and $b = kaw = (1,9,2)(3,5)(6,8)$. Hence $\langle k,w\rangle$ contains a 3-cycle. As $\langle w,k\rangle$ is primitive, Jordan's Theorem [**80**, Theorem 13.3] implies that $H = \langle w,k\rangle$. But then $\mathrm{Alt}(9)$ has a fixed point on V and this is our final contradiction. □

Lemma C.30. *Suppose that* $X \cong \mathrm{Alt}(9)$ *and* W *is a* $\mathrm{GF}(2)X$*-module of dimension* 9 *with* U *a submodule of* W *of codimension* 1. *If* U *is a spin module for* X*, then* $C_W(X) \neq 0$.

Proof. Let $A = C_X((1,2,3))$ and $B = C_X((4,5,6))$. Then $X = \langle A, B\rangle$, $C_W((1,2,3))$ is normalized by A and $C_W((4,5,6))$ is normalized by B. As 3-cycles act fixed point freely on the spin module, we have that $C_W((1,2,3))$ and $C_W((4,5,6))$ are 1-dimensional. Thus $C_W((4,5,6)) = C_W((1,2,3))$. Then $C_W((1,2,3))$ is invariant under $X = \langle A, B\rangle$, the assertion. □

D. p-local properties of groups of Lie type in characteristic p

In this section we will compile some facts about the p-local subgroups of the simple groups of Lie type in characteristic p. If $G \neq \mathrm{F}_4(2^n)$, $\mathrm{PSp}_{2m}(2^n)$ or $\mathrm{G}_2(3^n)$, then by Lemma A.3 the centre of a Sylow p-subgroup of G is a long root group. The structure of the normalizer of a long root group in these cases and in the cases of $\mathrm{F}_4(2^n)$, $\mathrm{PSp}_{2m}(2^n)$, $\mathrm{G}_2(q)$ and ${}^2\mathrm{F}_4(q)$, will be given in Lemmas D.1, D.5, D.11, D.10 and D.13. In the next lemma we use the notation V_n to denote a natural module for a classical group defined in dimension n. Thus, if X is a classical group defined over $\mathrm{GF}(p^e)$, then $|V_n| = p^{ne}$.

Lemma D.1. *Let* p *be a prime,* X *be a simple group of Lie type defined in characteristic* p *and* R *be a long root subgroup of* X*. Set* $Q = O_p(N_X(R))$ *and* $L = O^{p'}(N_X(R)/Q)$*. Then for specified* X*, the following table displays the Levi section* $L/Z(L)$*, the* p*-rank of* Q/R *and, for the classical groups* X*, describes the action of* L *on* Q/R*.*

X	$L/Z(L)$	$m_p(Q/R)$	Q/R
$\mathrm{PSL}_m(p^e), m \geq 5$	$\mathrm{PSL}_{m-2}(p^e)$	$2(m-2)e$	$V_{m-2} \oplus V_{m-2}^*$
$\mathrm{PSU}_m(p^e), m \geq 5$	$\mathrm{PSU}_{m-2}(p^e)$	$(m-2)2e$	V_{m-2}
$\mathrm{PSp}_{2m}(p^e), m \geq 2, p$ *odd*	$\mathrm{PSp}_{2(m-1)}(p^e)$	$2(m-1)e$	V_{2m-2}
$\mathrm{P\Omega}_{2m+1}(p^e), m \geq 3, p$ *odd*	$\mathrm{PSL}_2(p^e) \times \mathrm{P\Omega}_{2(m-2)+1}(p^e)$	$2(2(m-2)+1)e$	$V_2 \otimes V_{2m-3}$
$\mathrm{P\Omega}^{\pm}_{2m}(p^e), m \geq 4$	$\mathrm{PSL}_2(p^e) \times \mathrm{P\Omega}^{\pm}_{2(m-2)}(p^e)$	$4(m-2)e$	$V_2 \otimes V_{2m-4}$
$\mathrm{P\Omega}^{\pm}_6(p^e)$	$\mathrm{PSL}_2(p^e)$	$4e$	$V_2 \oplus V_2$
$\mathrm{E}_6(p^e)$	$\mathrm{PSL}_6(p^e)$	$20e$	
${}^2\mathrm{E}_6(p^e)$	$\mathrm{PSU}_6(p^e)$	$20e$	
$\mathrm{E}_7(p^e)$	$\mathrm{P\Omega}^+_{12}(p^e)$	$32e$	
$\mathrm{E}_8(p^e)$	$\mathrm{E}_7(p^e)$	$56e$	
$\mathrm{F}_4(p^e), p$ *odd*	$\mathrm{PSp}_6(p^e)$	$14e$	
${}^3\mathrm{D}_4(p^e)$	$\mathrm{PSL}_2(p^{3e})$	$8e$	

Furthermore, other than for $X \cong \mathrm{PSL}_m(p^e)$ *and* $\mathrm{P\Omega}^{\pm}_6(p^e)$*,* Q/R *is an irreducible* L*-module and, for the exceptional groups, it is defined over* $\mathrm{GF}(p^e)$*. If* $X \cong \mathrm{P\Omega}^-_6(p^e)$*, then* $C_X(R)$ *acts irreducibly on* Q/R *unless*

$p^e = 3$. *If* $X \cong \mathrm{PSL}_2(p^e)$, $\mathrm{PSL}_3(p^e)$ *or* $\mathrm{PSU}_3(p^e)$, *we have that* Q *is a Sylow* p*-subgroup of* X*.*

Proof. This can be checked using the Chevalley commutator formula (see [**27**, Chapter 3.2]). But we will sketch some arguments. Compare also [**27**, Example 3.2.3].

We begin with the classical groups. For simplicity we consider quasisimple variants $\mathrm{SL}_n(p^e)$, $\mathrm{SU}_n(p^e)$, $\mathrm{Sp}_{2n}(p^e)$ and $\Omega_n^{\pm}(p^e)$. Let V be the corresponding natural module.

We start with $X \cong \mathrm{SL}_n(p^e)$, $n \geq 4$. Then $r \in R^{\#}$ induces a transvection with center $\langle v\rangle$ on V. Set $W = C_V(r)$. Let X_1 be the stabiliser of v in X, then by [**27**, Example 3.2.3] $O^{p'}(X_1) = Q_1L_1$, where $L_1 \cong \mathrm{SL}_{n-1}(p^e)$ and Q_1 may be considered as the natural module for L_1. We have $[W, Q_1] = \langle v\rangle$. Let X_{n-1} be the stabiliser of W, then also, by [**27**, Example 3.2.3], we have that $O^{p'}(X_{n-1}) = Q_{n-1}L_{n-1}$, where $L_{n-1} \cong \mathrm{SL}_{n-1}(p^e)$ and Q_{n-1} is the natural module. We have $R = Q_1 \cap Q_{n-1}$ and $[V, Q_{n-1}] = W$. Now we see that $C_X(R) = Q_1Q_{n-1}L_{1,n-1}$, where $L_{1,n-1} \cong \mathrm{SL}_{n-2}(p^e)$. Furthermore $L_{1,n-1}$ induces the natural module on Q_1/R and the dual module on Q_{n-1}/R. An easy calculation shows $[Q_1, Q_{n-1}] = R = Z(Q)$. This proves all the claims in this case.

Next consider $X \cong \Omega_n^{\epsilon}(p^e)$, $\epsilon = \pm$ and $n \geq 7$. Let v be an isotropic vector in V and X_v be the stabiliser of v in X. Then the structure of $O^{p'}(X_v)$ is given in [**18**, Proposition 3.1]. We have $O^{p'}(X_v) = Q_vL_v$, where $L_v \cong \Omega_{n-2}^{\epsilon}(p^e)$ and Q_v is the natural module for L_v. We may assume that $R \leq Q_v$ and that $[V, R] = \langle v, w\rangle$, which is of dimension 2. Furthermore we have that $[v^{\perp}, Q_v] = \langle v\rangle$. Let $L_{v,w}$ be the stabiliser of w in L_v, then $L_{v,w} \cong \Omega_{n-4}^{\epsilon}(p^e)$. We see that for the normalizer $X_{v,w}$ of $[V, R]$ we have $X_{v,w} \geq \langle Q_v, Q_w, L_{v,w}\rangle$. Set $Q_1 = C_{Q_v}(w)$ and $Q_2 = C_{Q_w}(v)$. Then Q_1Q_2 is normal in $X_{v,w}$ and Q_v/Q_1 induces the full transvection group with center $\langle v\rangle$ and Q_w/Q_2 the one with center $\langle w\rangle$ on $[V, R]$. As V is a $\mathrm{GF}(p^e)$-module, we see that $\langle Q_v, Q_w\rangle/Q_1Q_2 \cong \mathrm{SL}_2(p^e)$. Hence $O_p(X_{v,w}) = Q_1Q_2$ and

$$\langle Q_v, Q_w, L_{v,w}\rangle/O_p(X_{v,w}) \cong \Omega_{n-4}^{\epsilon}(p^e) \times \mathrm{SL}_2(p^e).$$

As this group is invariant under $N_X(S)$ we see $O^{p'}(X_{v,w}) = \langle Q_v, Q_w, L_{v,w}\rangle$. Recall, as $n \geq 7$, we have that $\Omega_{n-4}^{\epsilon}(p^e) \neq 1$. We have that $Q_1Q_2/R \cong \mathrm{Hom}_{\mathrm{GF}(p^e)}(\langle v, w\rangle^{\perp}/\langle v, w\rangle, \langle v, w\rangle) \cong V_{n-4} \otimes V_2$. This proves the result for the orthogonal groups in dimension at least 7. In dimension 6, the first part of the proof is just the same. Only now $O^{p'}(X_{v,w})/Q_1Q_2 \cong \mathrm{SL}_2(p^e)$

and so Q_1Q_2/R is a direct sum of two natural $\mathrm{SL}_2(p^e)$-modules.

The result for $X = \mathrm{Sp}_{2n}(p^e)$ follows from [**18**, Proposition 3.2] and the one for $\mathrm{SU}_n(p^e)$ comes from [**18**, Proposition 3.3]. The only thing which remains to prove is that L acts irreducibly on Q/R as a $\mathrm{GF}(p)$-module. In the case of $X \cong \mathrm{Sp}_{2n}(p^e)$ this is obvious as L acts transitively on the on the non-trivial elements of Q/R. Consider $X \cong \mathrm{SU}_n(p^e)$. In this case $L \cong \mathrm{SU}_{n-2}(p^e)$. As Q/R is a vector space over $\mathrm{GF}(p^{2e})$ it is enough to show that the stabiliser of an isotropic 1-space U over $\mathrm{GF}(p^{2e})$ in L acts irreducibly on this space as considered over $\mathrm{GF}(p)$. Now any such 1-space is a subspace of a non-degenerate unitary 2-space. On this 2-space $\mathrm{GU}_2(p^e)$ acts irreducibly. In particular the stabiliser of a 1-space acts irreducibly considered as a $\mathrm{GF}(p)$-space. As $n-2 \geq 3$, we are in a position to adjust determinants to obtain $\mathrm{GU}_2(p^e)$ is contained in L and the result follows.

Next consider $X \cong \mathrm{E}_6(p^e)$, $\mathrm{E}_7(p^e)$ or $\mathrm{E}_8(p^e)$. The facts aside from the irreducibility of L on Q/R can be found in [**18**, Proposition 4.4]. For $X \cong \mathrm{F}_4(p^e)$ we cite [**18**, Proposition 4.5]. For the fact that the action on Q/R is irreducible and defined over $\mathrm{GF}(p^e)$ we use [**6**, Theorem 2]. Suppose that $X \cong {}^2\mathrm{E}_6(p^e)$. Then again everything apart from the irreducibility of L on Q/R can be found in [**18**, Proposition 4.6]. The irreducible action of L on Q/R and field of definition comes from [**6**, Theorem 3]. In [**27**, Example 3.2.5] the reader will find the calculation for ${}^3\mathrm{D}_4(p^e)$, and we remark ${}^2\mathrm{E}_6(p^e)$ is also discussed in the same example. □

The groups $\mathrm{PSL}_3(2^e)$ and $\mathrm{PSp}_4(2^e)$ play a special role in the proof of the theorems. Hence we have to have a very detailed knowledge of their 2-local structure.

Lemma D.2. *Let $X \cong \mathrm{PSL}_3(q)$, $q = 2^e$, and S be a Sylow 2-subgroup of X. Then X possesses two parabolic subgroups P_1, P_2 which contain S, such that $E_i = O_2(P_i)$ is elementary abelian of order q^2 and $O^{2'}(P_i/E_i) \cong \mathrm{SL}_2(q)$, for $i = 1, 2$. Furthermore P_i induces the natural module on E_i, $i = 1, 2$, $S = E_1E_2$ and any involution of S is contained in $E_1 \cup E_2$. Finally there is an automorphism α of X, which normalizes S with $P_1^\alpha = P_2$.*

Proof. [**42**, Lemma 2.40]. □

Lemma D.3. *Let $X \cong \mathrm{PSp}_4(q)$, $q = 2^e > 2$, and S be a Sylow 2-subgroup of X. Then X has exactly two parabolic subgroups P_1, P_2 which contain S. For $i = 1, 2$, $E_i = O_2(P_i)$ is elementary abelian of* *order q^3 and $P_i/E_i \cong \mathrm{GL}_2(q)$. We have that E_i is an indecomposable module for P_i and $Z(O^{2'}(P_i)) = R_i$ is a root group. Furthermore $Z(S) = R_1R_2 = S'$, $S = E_1E_2$ and any involution in S is contained in $E_1 \cup E_2$. There is an automorphism α of X with $R_1^\alpha = R_2$ and $P_1^\alpha = P_2$.*

Proof. This is [**42**, Lemma 2.48]. □

Lemma D.4. *Suppose that X is a group such that $F^*(X) \cong \mathrm{PSL}_3(2^e)$ with $e \geq 1$ or $F^*(X) \cong \mathrm{Sp}_4(2^e)$ with $e \geq 2$. Let $T \in \mathrm{Syl}_2(X)$ and $S = T \cap F^*(X)$. Then*

(i) *every elementary abelian normal subgroup of T is contained in S;*
(ii) *$J(T) = J(S)$.*

Proof. We adopt the notation from Lemmas D.2 and D.3. Let Q be an elementary abelian normal subgroup of T and assume that $w \in Q \setminus F^*(X)$. If $E_1^w = E_2$, then for $e \in E_1 \setminus E_2$, we have $[w,e] \in S \setminus (E_1 \cup E_2)$. Since $[e,w] \in Q \cap S$ and $E_1 \cap E_2 = Z(S)$, we have a contradiction as $[e,w]$ has order 2. Hence Q normalizes E_1 and E_2. Therefore, by Lemma A.16, w induces a field automorphism on $F^*(X)$. It follows that w induces a field automorphism on P_1/E_1 and then on $S/E_1 \cong E_2E_1/E_1$. Hence $[E_2,w] \not\leq E_1 \cap E_2$ and similarly $[E_1,w] \notin E_1 \cap E_2$. Since $[S,Q] \geq [E_1,w][E_2,w]$, we see that $Q \cap S$ has elements of order 4, a contradiction. This proves (i).

Now consider (ii). Let A be an elementary abelian subgroup of T of maximal rank and assume that $A \not\leq S$. Then, by Lemmas D.2 and D.3,

$$m_2(A) \geq m_2(S) = \begin{cases} 2e & F^*(X) \cong \mathrm{PSL}_3(2^e) \\ 3e & F^*(X) \cong \mathrm{Sp}_4(2^e). \end{cases}$$

From Theorem A.11, $|AS/S| \leq 4$ if $F^*(X) \cong \mathrm{PSL}_3(2^e)$ and $|AS/S| \leq 2$ if $F^*(X) \cong \mathrm{PSp}_4(2^e)$. In particular, $m_2(A \cap S) \leq m_2(S) - 1$. Let $w \in A \setminus S$. We use Lemma A.16 without further reference. Assume first that w induces a graph-field automorphism on $F^*(X)$. Then $F^*(X) \cong \mathrm{PSL}_3(2^e)$ and $O^{2'}(C_{F^*(X)}(w)) \cong \mathrm{PSU}_3(2^{e/2})$ and $2e-2 \leq m_2(A \cap S) \leq m_2(C_S(w)) \leq e/2$. This is impossible and so conclude that in both cases we now have $|AS/S| \leq 2$. In particular, $m_2(A \cap S) \leq m_2(S) - 1$. If w induces a field automorphism on $F^*(X)$, then

$$m_2(A \cap S) \leq m_2(C_S(w)) \leq \begin{cases} e & F^*(X) \cong \mathrm{PSL}_3(2^e) \\ 3e/2 & F^*(X) \cong \mathrm{Sp}_4(2^e). \end{cases}$$

which is impossible. Suppose that $w \in A$ is conjugate to a graph automorphism. If $F^*(X) \cong \mathrm{PSL}_3(2^e)$, then $C_{F^*(X)}(w) \cong \mathrm{PSp}_2(2^e)$

and, if $F^*(X) \cong \mathrm{PSp}_4(2^e)$, then $C_{F^*(X)}(w) \cong {}^2\mathrm{B}_2(2^e)$. In both cases, $m_2(C_S(w)) \le e$ which is impossible. Thus $A \le S$ and $J(S) = J(T)$ as claimed. $\square$

LEMMA D.5. *Suppose that $X \cong \mathrm{Sp}_{2n}(q)$ with $q = 2^e$ and $n \ge 3$, and let R_1 be a long root subgroup and R_2 be a short root subgroup of X. For $i = 1, 2$, set $Q_i = O_2(N_X(R_i))$ and*

$$L_i = O^{2'}(N_X(R_i)/Q_i).$$

Then

(i) *$L_1 \cong \mathrm{Sp}_{2n-2}(q)$, Q_1 is elementary abelian and Q_1/R_1 is a natural $\mathrm{Sp}_{2n-2}(q)$-module; and*
(ii) *$L_2 \cong \mathrm{Sp}_{2n-4}(q) \times \mathrm{SL}_2(q)$, $\Phi(Q_2) = Q_2' = R_2$, $Z(Q_2)/R_2$ is a natural $\mathrm{SL}_2(q)$-module and $Q_2/Z(Q_2)$ is the tensor product of natural modules of the direct factors of L_2. In addition, if $q > 2$, then $Z(Q_2)$ does not split over R_2 as an L_2-module.*

PROOF. Let V be the natural module for X. The structure of $N_X(R_1)$ in part (i) is taken from [**18**, Proposition 3.2].

So we consider $N_X(R_2)$. Let $V = V_1 \perp V_2$ where $\dim V_1 = 4$. For $i = 1, 2$, set $Y_i = \mathrm{Sp}(V_i)$ and let $Y = Y_1 \times Y_2 \le X$. We may suppose that $R_1R_2 \le Y_1$ and the parabolic subgroups $P_1 = N_{Y_1}(R_1)$ and $P_2 = N_{Y_1}(R_2)$ preserve totally isotropic subspaces of V_1 or dimensions 1 and 2 respectively. In particular, we see that $P_2 \times Y_2$ normalizes $W = [V, R_2] = [V_1, R_2]$ which is totally isotropic of dimension 2 admitting P_2 irreducibly and being centralized by Y_2. Furthermore, setting $E_2 = O_2(P_2)$, we have that E_2 is elementary abelian of order q^3 by Lemma D.3. Furthermore, $[V, E_2] = W$. Now $[V, E_2, Q_2] = [W, Q_2] < W$ is normalized by P_2 and so $[V, E_2, Q_2] = 0$ and as $W^\perp = W \perp V_2$, a similar argument give $[V, Q_2, E_2] = 0$. The Three Subgroup Lemma now yields $R_2 \le Z(Q_2)$. Since $N_X(R_2)$ normalizes the chain $V > W^\perp > W > 0$ we also see that $N_X(R_2)/Q_2 = (P_2 \times Y_2)Q_2/Q_2$. Notice that $E_2 = C_{P_2}(W)$ and so $E_2 = C_X(W^\perp)$ and Q_2/E_2 embeds into $\mathrm{Hom}_{\mathrm{GF}(q)}(W^\perp/W, W)$. Since $|Q_2/E_2| = q^{4n-8}$, we deduce that Q_2/E_2 is isomorphic to the tensor product of the natural P_2/E_2-module with a natural Y_2-module. Since this module is irreducible, we also deduce that either $E_2 = Z(Q_2)$ or Q_2 is abelian. Since Q_2Y_2 centralizes E_2, Q_2Y_2 centralizes R_1 and hence normalizes Q_1. From the structure of $N_X(R_1)/Q_1$, we see that $|Q_2Q_1/Q_1|$ has order at most q^{2n-3} Since $|Q_2| = q^{4n-5}$ and $Q_1 \not\le Q_2$, we deduce that $|Q_1Q_2/Q_1| = q^{2n-3}$ and $|Q_1 : Q_1 \cap Q_2| = q$. In particular, Q_2 does not centralizes $Q_1 \cap Q_2$ and $\Phi(Q_2) \le Q_1$ and so we have $Z(Q_2) = E_2$ and $\Phi(Q_2) \leq Q_1 \cap E_2 = R_1R_2$. Since Q_2 is non-abelian and $N_X(R_2)$ acts irreducibly on R_2 by Lemma A.4, we have $\Phi(Q_2) = Q_2' = R_2$. □

Lemma D.6. *Suppose that $X \cong \mathrm{Sp}_{2n}(2^e)$ with $n \geq 3$. Let V be the natural symplectic module, P be the stabiliser of a maximal isotropic subspace of V and $S \in \mathrm{Syl}_2(P)$. Then $J(S) = O_2(P)$ is elementary abelian.*

Proof. By [**30**, Lemmas 3.12 and 3.13] the 2-rank of X is $(n+1)n/2$ and if A is an elementary abelian subgroup of X of maximal 2-rank, then A is conjugate in X to $O_2(P)$. Hence if $J(S) \neq O_2(P)$, then $O_2(P)$ is not weakly closed in S with respect to X contrary to [**28**, Lemma 4.2]. □

Lemma D.7. *Suppose that $X \cong \mathrm{F}_4(q)$ with $q = 2^e$ and let R_1 be a long root subgroup and R_2 be a short root subgroup of X. For $i = 1, 2$, set $Q_i = O_2(N_X(R_i))$ and*

$$L_i = O^{2'}(N_X(R_i)/Q_i).$$

Then, for $i = 1, 2$, we have $L_i \cong \mathrm{Sp}_6(q)$ and

$$\Phi(Q_i) = R_i.$$

Furthermore, as L_i-modules, $Z(Q_i)/R_i$ is a natural module of dimension 6, $Q_i/Z(Q_i)$ is a spin module of dimension 8, the modules $Z(Q_i)$ and Q_i/R_i are indecomposable.

Proof. This can be taken from [**18**, Proposition 4.5] or [**27**, Example 3.2.4, page 100]. □

Lemma D.8. *Suppose that $X \cong \mathrm{F}_4(q)$ with $q = 2^e$, $S \in \mathrm{Syl}_2(X)$ and $\Omega_1(Z(S)) = R_1R_2$ with R_1 a long root subgroup of X and R_2 a short root subgroup of X. We use the notation introduced in Lemma* D.7 *and additionally set $I_{12} = C_X(R_1R_2)$, $Q_{12} = O_2(I_{12})$ and $L_{12} = I_{12}/Q_{12}$. For $i = 1, 2$, define*

$$V_i = [Z(Q_i), Q_{12}]R_1R_2,$$

put $V_{12} = V_1V_2$ and $W_{12} = Z(Q_1)Z(Q_2)$.

Then the following hold:

(i) *$L_{12} \cong \mathrm{Sp}_4(q)$ and $Q_{12} = Q_1Q_2$.*
(ii) *V_{12} and W_{12} are normal in I_{12} and*

$$1 < R_1R_2 < V_{12} < W_{12} < Q_{12}.$$

In addition, we have $Z(Q_1) \cap Z(Q_2) = R_1R_2$, $Q_1 \cap Q_2 = V_{12}$ is elementary abelian and, setting $\overline{V_{12}} = V_{12}/R_1R_2$,

$$\overline{V_{12}} = \overline{V_1} \oplus \overline{V_2},$$

where $\overline{V_1}$ and $\overline{V_2}$ are irreducible L_{12}-modules of $\mathrm{GF}(q)$*-dimension* 4 *which are not isomorphic as* $\mathrm{GF}(2)L_{12}$*-modules. Furthermore, if $q > 2$, $W_{12}' = R_1R_2$ whereas, if $q = 2$, $W_{12}' = \langle r_1r_2\rangle$ where $r_i \in R_i^{\#}$.*

(iii) *$[V_{12}, W_{12}] = 1$ and W_{12}/V_{12} has order q^2 and is centralized by L_{12}.*

(iv) *We have*

$$Q_{12}/W_{12} \cong Q_1W_{12}/W_{12} \oplus Q_2W_{12}/W_{12},$$

Q_1W_{12}/W_{12} and Q_2W_{12}/W_{12} are irreducible, non-isomorphic L_{12}-modules of $\mathrm{GF}(q)$*-dimension* 4*. Furthermore, as L_{12}-modules, for $i = 1, 2$,*

$$Q_iW_{12}/W_{12} \cong V_{3-i}/R_1R_2.$$

(v) *We have*

$$Q_{12}/V_{12} = Q_1/V_{12} \oplus Q_2/V_{12}$$

is a direct sum of two indecomposable L_{12}-modules of $\mathrm{GF}(q)$*-dimension* 5*.*

(vi) *The group* $\mathrm{Aut}(Q_{12})$ *has a subgroup of index* 2 *which normalizes all of R_1, R_2, Q_1, Q_2, $Z(Q_1)$, $Z(Q_2)$, V_{12} and W_{12}.*

Proof. By Definition A.9 there is an automorphism α of X such that $R_1^\alpha = R_2$. So α exchanges $C_X(R_1)$ and $C_X(R_2)$ and so normalize I_{12} and exchanges the parabolic subgroups $C_{I_{12}}(R_1)$ and $C_{I_{12}}(R_2)$. In particular, this allows us to apply symmetric arguments for $i = 1, 2$.

We use the structure of $C_X(R_i)$, $i = 1, 2$, as given in Lemma D.7. Thus

$$\Phi(Q_1 \cap Q_2)' \le \Phi(Q_1) \cap \Phi(Q_2) = R_1 \cap R_2 = 1$$

and so $Q_1 \cap Q_2$ is elementary abelian. By [**27**, Table 3.3.1], $|Q_1 \cap Q_2| \le q^{11}$ and so $|Q_2Q_1/Q_1| \ge q^4$. Now $N_{L_1}(I_{12}/Q_1)$ is a parabolic subgroup in $L_1 \cong \mathrm{Sp}_6(q)$ which normalizes Q_{12}/Q_1 and Q_{12}/Q_1 is an indecomposable $\mathrm{GF}(q)L_{12}$-module of order q^5 with $L_{12} \cong \mathrm{Sp}_4(q)$, see Lemma D.5. Since I_{12} normalizes Q_1Q_2, we deduce that $Q_2Q_1 = Q_{12}$. This proves (i).

Using (i), we have $Q_1/(Q_1 \cap Q_2) \cong Q_{12}/Q_2$ as L_{12}-modules and so $Q_1/(Q_1 \cap Q_2)$ is an indecomposable L_{12}-module of $\mathrm{GF}(q)$-dimension 5. By symmetry, the same is true for $Q_2/(Q_1 \cap Q_2)$ and so we have proved

$$Q_{12}/(Q_1 \cap Q_2) = Q_1/(Q_1 \cap Q_2) \oplus Q_2/(Q_1 \cap Q_2)$$

is a direct sum of two indecomposable L_{12}-modules of $\mathrm{GF}(q)$-dimension 5. Since this module is invariant under the action of α, then non-trivial modules involved are not isomorphic. Notice that $Z(Q_1)(Q_1 \cap Q_2)/(Q_1 \cap$

Q_2) is normalized by $N_X(S)$ and so has order q. Hence $W_{12}/(Q_1 \cap Q_2)$ has order q^2 and is centralized by L_{12}.

By construction, $V_1 \leq Q_2$ and $V_2 \leq Q_1$ and so $V_{12} \leq Q_{12}$ and as $V_1 \neq V_2$ are both I_{12}-invariant, $|V_{12}| = q^{10}$ and so $V_{12} = Q_1 \cap Q_2$ and $V_1 \cap V_2 = R_1R_2$. Thus $\overline{V_{12}} = \overline{V_1} \oplus \overline{V_2}$. Since $V_1^{\alpha} = V_2$, we have $\overline{V_1}$ is not isomorphic to $\overline{V_2}$ as L_{12}-modules.

As, by definition, $[W_{12}, Q_1]R_1R_2 = V_2$ and $[W_{12}, Q_2]R_1R_2 = V_1$, we have that V_1/R_1R_2 is isomorphic to Q_2W_{12}/W_{12} and V_2/R_1R_2 is isomorphic to Q_1W_{12}/W_{12} as a $GF(2)$ L_{12}-module. we have now proved all parts (i) to (v) other than the statement about W'_{12} given in part(ii).

Using the fact that $Z(Q_i)$ is an indecomposable L_{12}-module, we have

$$[W_{12}, Z(Q_i)]R_{3-i} = R_1R_2.$$

We have that $N_X(S)$ acts on W'_{12}. If $q > 2$ $N_X(S)$ induces a homocyclic group of shape $(q-1) \times (q-1)$ on R_1R_2 and so the only non-trivial invariant subgroup under this group and α is R_1R_2. Hence $W'_{12} = R_1R_2$ when $q > 2$. If $q = 2$, then, as $|Z(Q_1)/V_1| = |Z(Q_2)/V_2| = 2$, we have W'_{12} has order 2 and then as W'_{12} is invariant under α we get $W'_{12} = \langle r_1r_2 \rangle$.

Finally we come to part (vi). Since α conjugates Q_1 to Q_2, to prove (vi) it suffices to show that the set $\{Q_1, Q_2\}$ is permuted by $\mathrm{Aut}(Q_{12})$.

We know $W_{12} = Z(Q_1)Z(Q_2)$ and $V_{12} = Q_1 \cap Q_2$. For $i = 1, 2$, let $F_i = V_{12}Z(Q_i)$ and assume that $x = f_1f_2 \in W_{12}$ which is not contained in F_1 or F_2. We claim that x has order 4. Suppose false. Then $1 = x^2 = f_1f_2f_1f_2 = [f_1, f_2]$ which means that $f_1 \in C_{F_1}(f_2)$. As $f_2 \notin F_1$, f_2Q_1 induces a GF(q) transvection on $Z(Q_1)$. Now $f_1 = z_1v_1$ where $z_1 \in Z(Q_1)$ and $v_1 \in V_{12} \leq F_2$. Since f_2 centralizes f_1 and v_1 it must also centralize z_1. Hence $z_1 \in C_{Z(Q_1)}(f_2) = Z(Q_1) \cap V_{12}$. But then $x = f_1f_2 = z_1v_1f_2 \leq F_2$ which is a contradiction. Since $V_{12} = Q'_{12}$, and $W_{12} = C_{Q_{12}}(V_{12})$, V_{12} and W_{12} are both invariant under the action of $\mathrm{Aut}(Q_{12})$. Hence $\mathrm{Aut}(Q_{12})$ permutes the set of involutions in W_{12} and therefore $\{F_1, F_2\}$ is permuted by $\mathrm{Aut}(Q_{12})$. Now we see that $\{C_{Q_{12}}([F_1, Q_{12}]), C_{Q_{12}}([F_1, Q_{12}])\}$ is permuted by $\mathrm{Aut}(Q_{12})$. Now $C_{Q_{12}}([F_1, Q_{12}]) = W_{12}Q_1$ and Q_1 has index q in this group. Now $Q_1/Z(Q_{12})$ is the unique elementary abelian subgroup of order q^{13} in $W_{12}Q_1/Z(Q_{12})$. Therefore $\{Q_1, Q_2\}$ is permuted by $\mathrm{Aut}(Q_{12})$ as claimed. $\square$

Lemma D.9. *Let $X \cong \mathrm{F}_4(q)$, $q = 2^e$, and S be a Sylow 2-subgroup of X. Set $Z_2 = Z_2(S)$ the second centre of S. Then $P = N_X(Z_2(S))$ is a parabolic subgroup of X,*

$$O^{2'}(P/O_2(P)) = F_1/O_2(P) \times F_2/O_2(P),$$

$F_1/O_2(P) \cong F_2/O_2(P) \cong \mathrm{SL}_2(q)$ and $Z_2 = U_1 \oplus U_2$, with $U_1 = \langle R_2^{F_1} \rangle$ a natural $F_1/O_2(P)$-module and $U_2 = \langle R_1^{F_2} \rangle$ a natural $F_2/O_2(P)$-module. Moreover, for $i = 1, 2$, $[U_i, F_{3-i}] = 1$.

Proof. We employ the notation from Lemmas D.7 and D.8. In particular, we select S so that $Z(S) = R_1R_2$. First of all we have that, for $i = 1, 2$, $[Z_2, Q_i] \le Z(S) \le Z(Q_i)$. Hence, as $Q_i/Z(Q_i)$ is a non-trivial L_i-module, $Z_2(S) \le Q_1 \cap Q_2 = V_{12}$. By Lemma D.8 we have that V_{12}/R_1R_2 is a direct sum of two irreducible $\mathrm{Sp}_4(q)$-modules and so by Lemma C.1 we have that $|Z_2(S)| = q^4$.

Let P be the parabolic subgroup of X containing $N_X(S)$ such that $O^{2'}(P/O_2(P)) = F_1/O_2(P) \times F_2/O_2(P)$, $F_1/O_2(P) \cong F_2/O_2(P) \cong \mathrm{SL}_2(q)$ with notation chosen so that $[F_1, R_1] = 1 = [F_2, R_2]$. Then $L_1 = F_1I_{12}$ and $L_2 = F_2I_{12}$. In particular, $[R_1, F_2] \ne 1 \ne [R_2, F_1]$. We set $U_1 = \langle R_2^{F_1} \rangle$ and $U_2 = \langle R_1^{F_2} \rangle$. Then U_1U_2 is normalized by F_1F_2 and is centralized by $O_2(P)$. Since $U_1 \le Z(Q_1)$ and is F_1-invariant, we see that U_1/R_1 is a natural $F_1/O_2(P)$-module. Furthermore, by construction, $[U_2, F_1] = 1 = [U_1, F_2]$ and so $U_1 \ne U_2$ and $Z_2(S) = U_1U_2$. Furthermore, $U_1 \cap U_2$ is centralized by F_1F_2 and, as $C_{R_1R_2}(F_1F_2) = 1$, we deduce that $Z_2 = U_1 \oplus U_2$. Finally we observe that $P = N_X(Z_2(S))$. $\square$

The groups $\mathrm{G}_2(q)$ and ${}^2\mathrm{F}_4(q)$ play a special role in this paper. Hence we have a closer look at their parabolic subgroups.

Lemma D.10. *Suppose that $X \cong \mathrm{G}_2(p^e)$, $p \ne 3$, $p^e \ne 2$, $S \in \mathrm{Syl}_p(X)$, $P_1 = N_X(R)$ where R a long root subgroup contained in $Z(S)$, and $P_2 = N_X(Z_2(S))$. For $i = 1, 2$, put $Q_i = O_p(P_i)$ and $L_i = O^{p'}(P_i/Q_i)$. Then*

(i) *P_1 and P_2 are maximal parabolic subgroups of X;*
(ii) *$L_1 \cong L_2 \cong \mathrm{SL}_2(p^e)$;*
(iii) *$Q_1' = \Phi(Q_1) = Z(Q_1) = R$;*
(iv) *If $p^e \ne 4$, then L acts irreducibly on Q_1/R;*
(v) *If $p^e = 4$, then P acts irreducibly on Q_1/R while $L \cong \mathrm{SL}_2(4) \cong$ $\mathrm{Alt}(5)$ induces a direct sum of two natural $\mathrm{Alt}(5)$-modules on Q_1/R. Furthermore, in the latter case, if $R < E \le Q_1$ is normalized by L_1, then E is not abelian.*
(vi) *We have $Z_2(S) \le Q_1$ and $Z_2(S)$ is a natural L_2-module. Furthermore, setting $W = \bigcap_{x \in P_2} Q^x$, we have W is elementary abelian of order p^{3e}, $W/Z_2(S)$ is centralized by L_2 and Q_2/W is a natural L_2-module.*

Proof. Up to the statement concerning the structure of E when $X \cong \mathrm{G}_2(4)$ everything can be extracted from [**19**, 10.10 and page 238] or [**27**, Example 3.2.4 page 99].

So suppose that $X \cong \mathrm{G}_2(4)$ and E is such that $R < E \le Q_1$ is normalized by L_1. Suppose that E is abelian. Then $E \ne Q$. Let W be such that $W/R = C_{E/R}(S)$. As Q/R is a direct sum of two natural Alt(5)-modules, E/R is a natural Alt(5)-module for L_1. Therefore $|W| = 8$, W is normalized by $N_L(S)$ and $N_L(S)/Q \cong \mathrm{Alt}(4)$. Obviously, $C_Q(W)$ is also normalized by $N_L(S)$. Because E is abelian, $C_Q(W) \ge E$, Q/E is a natural Alt(5)-module for L and $|Q : C_Q(W)| \le 4$, we deduce that $|Q : C_Q(W)| = 2$. Set $U = [W, Q]$. Then $|U| = 2$ and $W/U \le Z(Q/U)$. As L centralizes U, we get that $E/U \le Z(Q/U)$. Now choose $g \in N_X(R)$ with $E^g \ne E$. Then we have that $EE^g = Q$. As E^g is abelian and $[E, E^g] = U$, we see that $Q' = U$. But $N_G(R)$ acts irreducibly on R by Lemma A.4 and so $Q' = R$. This provides a contradiction. □

LEMMA D.11. *Suppose that $X \cong \mathrm{G}_2(3^e)$, $S \in \mathrm{Syl}_3(X)$, P_1 and P_2 are the maximal parabolic subgroups of X containing S, and $Q_i = O_3(P_i)$ for $i = 1, 2$. Then*

(i) $P_1 \cong P_2$;
(ii) $O^{3'}(P_1/Q_1) \cong \mathrm{SL}_2(3^e)$, $S = Q_1Q_2$ *and* $Z(S) = R_1R_2$ *where* R_i *is a root subgroup centralized by* $O^{3'}(P_i)$;
(iii) $Q_1' = \Phi(Q_1) = R_1$;
(iv) $|Z(Q_1)/R_1| = 3^{2e}$, $Z(Q_1) = [Z(Q_1), O^{3'}(P_1)] \times R_1$ *and in addition* $R_2 \le [Z(Q_1), O^{3'}(P_1)]$;
(v) Q_1/R_1 *is an indecomposable extension of two natural modules for* $O^{3'}(P_1/Q_1)$;
(vi) *all the elements of* $Z(Q_1)^\#$ *and* $Z(Q_2)^\#$ *are* 3*-central;*
(vii) $Q_1 \cap Q_2 = Z(Q_1)Z(Q_2)$ *is elementary abelian of order* 3^{4e} *and every element of order* 3 *in* X *is conjugate into* $Q_1 \cap Q_2$;
(viii) *if* $x \in (Q_1 \cap Q_2) \setminus (Z(Q_1) \cup Z(Q_2))$, *then* x *is not* 3*-central and* $Q_1 \cap Q_2 \in \mathrm{Syl}_3(C_X(x))$; *and*
(ix) Q_1 *and* Q_2 *have exponent* 3 *and every element of order* 3 *in* S *is contained in* $Q_1 \cup Q_2$.

In particular, we have

$$O^{3'}(P_1) \sim ((3^e)^2 \times (3^e)^{1+2}){:}\mathrm{SL}_2(3^e).$$

PROOF. Part (i) follows from the existence of the graph automorphism of $F^*(H)$ (see Definition A.9). Parts (ii), (iii) and (v) as well as the first statement in (iv) can be extracted from [**27**, Example 3.2.4, page 99]. We take part (ix) from [**53**, Lemma 6.5].

Since $S = Q_1Q_2$ by (ii), $Q_1 \cap Q_2$ has order 3^{4e} and (iii) shows that $Q_1 \cap Q_2$ is elementary abelian. If $Z(Q_1) \not\le Q_2$, then, as $Z(Q_1)$ centralizes $Q_1 \cap Q_2$ which has index 3^e in Q_2 and $O^{3'}(P_2/Q_2) \cong \mathrm{SL}_2(3^e)$, we have that Q_2 has only one non-central P_2-chief factor in Q_2 and this contradicts (v). Thus $Z(Q_1)Z(Q_2) \leq Q_1 \cap Q_2$. Since $Z(Q_1) \cap Z(Q_2) \leq C_{Z(Q_1)}(S) = C_{Z(Q_1)}(S)$, part (v) implies that $Z(Q_1) \cap Z(Q_2)$ has index at least 3^e in $Z(Q_1)$. As $|Z(Q_i)| = 3^{3e}$ by (iv), we have $Z(Q_1)Z(Q_2)$ has order 3^{4e}. We conclude that $Q_1 \cap Q_2 = Z(Q_1)Z(Q_2)$ and $Z(Q_1) \cap Z(Q_2) = R_1R_2 = Z(S)$. Since every element of order 3 is contained in Q_1 or Q_2 and, for $i = 1, 2$, $O^{3'}(P_i/Q_i)$ acts transitively on the non-trivial elements of $Q_i/Z(Q_i)$, we see that every element of order 3 in X is conjugate to an element of $Q_1 \cap Q_2$. This proves (vii).

Because Q_1 has exponent 3, $Z(Q_1)$ is a $O^{3'}(P_1/Q_1)$-module and the centre of $O^{3'}(P_1/Q_1)$ inverts $Z(Q_1)/R_1$ and centralizes R_1. Thus

$$Z(Q_1) = [Z(Q_1), O^{3'}(P_1/Q_1)] \times R_1.$$

This proves a further statement of part (iv). Using

$$[Z(Q_1), O^{3'}(P_1/Q_1)] \leq Z(Q_1) \leq Q_2$$

we deduce

$$[[Z(Q_1), O^{3'}(P_1/Q_1)], Q_2] \leq Q_2' = R_2.$$

Since $|[[Z(Q_1), O^{3'}(P_1/Q_1)], Q_2]| = 3^e$, we have equality. Thus (iv) holds.

Suppose that $x \in (Q_1 \cap Q_2) \setminus (Z(Q_1) \cup Z(Q_2))$ and $y \in S$ centralizes x. Then, as $S = Q_1Q_2$, $y = q_1q_2$ for some $q_i \in Q_i$, $i = 1, 2$. Hence

$$1 = [x, y] = [x, q_1q_2] = [x, q_2][x, q_1]^{q_2}.$$

Since $Q_1' = R_1$ and $Q_2' = R_2$, $R_1 \cap R_2 = 1$ and $R_1R_2 = Z(S)$, we deduce that $[x, q_1] = [x, q_2]^{-1} \in R_1 \cap R_2 = 1$. So we may as well assume that $y = q_1 \notin Q_1 \cap Q_2$. Now, as $Q_1 \cap Q_2 = Z(Q_1)Z(Q_2)$, we can write $x = z_1z_2$ with $z_1 \in Z(Q_1)$ and $z_2 \in Z(Q_2)$. Hence $1 = [y, x] = [q_1, z_1z_2] = [q_1, z_2]$. Thus $z_2 \in C_{Z(Q_2)}(q_1)$. As $Z(Q_2)/R_2$ is a natural P_2/Q_2-module, we see that $z_2 \in Z(S)$. But then $x \in Z(Q_1)$, a contradiction. Hence $C_S(x) = Q_1 \cap Q_2$. Notice that $N_X(Q_1 \cap Q_2) \geq N_X(S)$ and so, as the proper over-groups of $N_X(S)$ in X are P_1 and P_2 and these subgroups do not normalize $Q_1 \cap Q_2$, we have $N_X(Q_1 \cap Q_2) = N_X(S)$. If $W \in \mathrm{Syl}_2(C_G(x))$ is such that $W \geq Q_1 \cap Q_2$, then

$$Q_1 \cap Q_2 \leq N_W(Q_1 \cap Q_2) \leq C_S(x) = Q_1 \cap Q_2$$

and so $W = Q_1 \cap Q_2$. Hence $Q_1 \cap Q_2 \in \mathrm{Syl}_3(C_G(x))$ and, in particular, x is not 3-central. This proves (viii). □

We now finish with the group $\mathrm{G}_2(2)$.

LEMMA D.12. *Suppose that $X \cong \mathrm{G}_2(2)$, $S \in \mathrm{Syl}_2(X)$, $R = Z(S)$ is a long root subgroup and $z \in R^{\#}$. Let P_1 and P_2 be the maximal* *parabolic subgroups of X which contain S and choose notation so that $P_1 \geq C_X(R)$. For $i = 1, 2$, set $Q_i = O_2(P_i)$. Then the following statements hold.*

(i) $X' \cong \mathrm{SU}_3(3)$ *and every involution of* X' *is conjugate to* z. *In particular,* $P_1 = C_X(z)$,
$$C_{X'}(z) = P_1 \cap X' \cong \mathrm{GU}_2(3)$$
and $Q_1 \cong 4 \circ \mathrm{Q}_8$. *Moreover,* $m_2(X') = 2$.

(ii) *If* $i \in S \setminus X'$, *then* $C_X(i) \cong \langle i \rangle \times \mathrm{Sym}(4)$.

(iii) $Q_2 \cap X' \cong 4 \times 4$ *and* $(P_2 \cap X')/(Q_2 \cap X') \cong \mathrm{Sym}(3)$ *and there exists an involution in* Q_2 *which inverts* $Q_2 \cap X'$.

(iv) Q_1 *is extraspecial of order* 32 *and* $+$*-type,* $P_1/Q_1 \cong \mathrm{Sym}(3)$ *and* $O^2(P_1) \cong \mathrm{SL}_2(3)$.

Proof. By [**1**], $\mathrm{G}_2(2)' \cong \mathrm{SU}_3(3)$ and consequently that $\mathrm{G}_2(2) \cong \mathrm{Aut}(\mathrm{SU}_3(3))$. Therefore, the number of conjugacy classes of involutions and the centralizers of a representative can be taken from [**27**, Table 4.5.1, page 172]. That the 2-rank of X' is 2 can be read from [**27**, Theorem 4.10.5 (c)]. The group of monomial matrices in $\mathrm{SU}_3(3)$ has shape (4×4):Sym(3) and so this gives the structure of $P_2 \cap X'$. Since the outer automorphism of X' can be chosen to be inverse transpose map, we can also deduce the structure of P_2 as described.

This leave part (iv). It is clear that $|Q_1| = 32$. Let $H = \mathrm{G}_2(4)$. Then X is the centralizer in H of a field automorphism α. Now z corresponds to a root element and so $Q_1 = C_{O_2(C_H(z))}(\alpha)$. The structure of $C_H(z)$ is given in Lemma D.10. From this it follows that Q_1 is extraspecial. Finally, we have $O^2(P_1) = O^2(P_1 \cap X') \cong \mathrm{SL}_2(3)$.

□

Lemma D.13. *Let $X \cong {}^2\mathrm{F}_4(q)$ with $q = 2^{2e+1}$, $S \in \mathrm{Syl}_2(X)$, R be a long root subgroup in $Z(S)$, $P = C_X(R)$ and $Q = O_2(P)$. Then*

(i) $P/Q \cong {}^2\mathrm{B}_2(q)$.

(ii) $R = Z(Q)$, $Z_2(Q)$ *is elementary abelian and* $Z_2(Q)/R$ *is an irreducible* 4*-dimensional module for* P/Q.

(iii) $C_Q(Z_2(Q))$ *is of order* q^6, $\Phi(C_Q(Z_2(Q))) = R$ *and* $Q/C_Q(Z_2(Q))$ *is the natural* P/Q*-module.*

(iv) *If* $q > 2$, *then* $Q/Z_2(Q)$ *is an indecomposable module.*

(v) *If* $q = 2$, *then* $F^*(X) = \mathrm{F}_4(2)'$ *has index* 2 *in* X. *We have that* $R = Z(O_2(P \cap F^*(X)))$, $Z_2(Q) = Z_2(Q \cap F^*(X))$ *and* $|(Q \cap F^*(X))/Z_2(Q)| = 16$. *Furthermore,* $(Q \cap F^*(X))/Z_2(Q)$ *and* $Z_2(Q)/R$ *admit* $P \cap F^*(X)$ *irreducibly.*

(vi) *Let* $P_1 = N_X(Z_2(S))$. *Then* P_1 *is a maximal parabolic subgroup of* X, $P_1 \neq P$, P_1 *normalizes* $Z_3(S)$ *which has order* q^3 *and* P_1 *induces* $\mathrm{GL}_2(q)$ *on* $Z(O_2(P_1)) = Z_2(S)$.

PROOF. For the structure of P see [**27**, Example 3.2.5 page 101] or [**19**, 12.9]. For part (vi) we refer to [**19**, 12.9]. □

Additionally we require a special fact about ${}^2\mathrm{F}_4(2)$.

LEMMA D.14. *Suppose that* $X \cong {}^2\mathrm{F}_4(2)$, *let* $S \in \mathrm{Syl}_2(X)$ *and* $R = Z(S)$. *Set* $Q = O_2(C_X(R))$ *and* $Q^* = Q \cap F^*(X)$. *Then* Q^* *is generated by involutions.*

PROOF. We use the results and notation from [**27**] especially Corollary 2.4.6 and the passages on pages 101 and 102. Thus we have root groups X_1 to X_{16} with X_i of order 2 if i is even and cyclic of order 4 if i is odd. For odd i we define $Y_i = \Omega_1(X_i)$. The opposite root group of X_i is X_{i+8} for $1 \leq i \leq 8$. We have $S = \prod_{i=1}^{8} X_i$. By [**27**, Theorem 3.3.2 (d)], $S \cap F^*(H)$ contains the subgroups X_i, i even. In particular, note that, as $Q = \prod_{i=2}^{8} X_i$, $X_2 \leq Q^*$ (see [**27**, page 102]). Furthermore, by Lemma D.13 (v), $Z_2(Q) = Z_2(Q^*)$ and $Z_2(Q) = Y_5Y_3X_4X_6X_7$ by [**27**, Example 3.2.5, page 102]. Since $Q^*/Z_2(Q^*)$ is an irreducible $N_{X'}(R)$-module by Lemma D.13 (v) and $X_2 \leq Q^*$, we have proved the result. □

For the proof of Theorem 3 we need to know those groups of Lie type which have the centralizer of some p-central element which is soluble.

LEMMA D.15. *Suppose that* X *is a simple group of Lie type defined in characteristic* p *of rank at least* 2. *Assume that* $C_X(z)$ *is soluble for some* p*-central element of* X. *Then one of the following holds.*

(i) $X \cong \mathrm{PSL}_3(p^e)$ *for some* $f \geq 1$;
(ii) $p = 2$ *and* $X \cong \mathrm{PSp}_6(2)$, $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, $\mathrm{PSU}_5(2)$, $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$, ${}^2\mathrm{F}_4(2)'$, $\mathrm{P}\Omega_6^+(2) \cong \mathrm{PSL}_4(2)$, $\mathrm{P}\Omega_8^+(2)$ *or* $\mathrm{PSp}_4(2^e)'$ *for some* $f \geq 1$; *or*
(iii) $p = 3$ *and* $X \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSU}_4(3)$, $\mathrm{P}\Omega_7(3)$, $\mathrm{P}\Omega_8^+(3)$ *or* $\mathrm{G}_2(3^e)$ *for some* $f \geq 1$.

PROOF. Let $S \in \mathrm{Syl}_p(X)$ and n represent the rank of X. Then by Lemma A.3 either $Z(S)$ is a long root group or $X \cong \mathrm{PSp}_{2n}(2^e)'$, $\mathrm{F}_4(2^e)$ or $\mathrm{G}_2(3^e)$ for $e \geq 1$ and $Z(S)$ is the product of the root groups corresponding to the highest long root and the highest short root. As $\mathrm{G}_2(3^e)$ is one of the groups in the statement of the lemma, we may assume that $X \not\cong G_2(3^e)$. Furthermore we also may assume that $X \not\cong \mathrm{PSL}_3(p^e)$ or $\mathrm{PSp}_4(2^e)$. Using Lemmas D.1, D.5, D.10 and D.13 it is easy to see that if $z \in Z(S)$ is a long root element and if $p^e > 3$ and $n \geq 3$, then

$C_X(z)$ is non-soluble.

So assume first that $Z(S)$ is a root group. Suppose further that $n = 2$. Let $z \in Z(S)$ be a long root element. If $X \cong \mathrm{PSp}_4(p^e)$ with $p^e > 3$ and p odd, then by Lemma D.5 $C_X(z)$ is non-soluble besides $X \cong \mathrm{PSp}_4(3)$ which is listed in (iii). If $G \cong \mathrm{PSU}_4(p^e)$ or $\mathrm{PSU}_5(p^e)$, then by Lemma D.1 $C_X(z)$ contains a section isomorphic to $\mathrm{PSL}_2(p^e)$ or $\mathrm{PSU}_3(p^e)$ respectively. Hence $C_X(z)$ is non-soluble if $f \geq 2$ or $p^e = 3$ and $X \cong \mathrm{PSU}_5(3)$. Thus $\mathrm{PSU}_4(2)$ and $\mathrm{PSU}_5(2)$ are included in (ii) and $\mathrm{PSU}_4(3)$ is listed in (iii). If $G \cong \mathrm{G}_2(p^e)'$, then by Lemma D.10 $C_X(z)$ contains a section isomorphic to $\mathrm{PSL}_2(p^e)$ and so is non-soluble unless $p^e = 2$, which is listed in (ii). If $X \cong {}^2\mathrm{F}_4(2^{2e+1})$, then by Lemma D.13 $C_X(z)$ contains a section isomorphic to ${}^2\mathrm{B}_2(2^{2e+1})$ and is thus non-soluble if $e > 1$ and ${}^2\mathrm{F}_4(2)'$ is itemized in (ii). This completes the analysis when $n = 2$.

So assume that $n \geq 3$ and $p^e \in \{2, 3\}$. If $n \geq 4$, then $C_X(z)$ is non-soluble (containing a section of Lie rank at least 2) or $X \cong \mathrm{P}\Omega_8^+(p)$ and these groups are included in (ii) and (iii). We now may assume that the rank of X is 3 and that $p^e = p \in \{2, 3\}$. Thus

$$X \cong \mathrm{PSL}_4(2), \mathrm{P}\Omega_8^-(2), \mathrm{PSU}_6(2), \mathrm{PSU}_7(2) \text{ or}$$
$$X \cong \mathrm{PSp}_6(3) \text{ or } \Omega_7(3).$$

Application of Lemma D.1 shows that for $X \cong \mathrm{P}\Omega_8^-(2)$, $C_X(z)$ contains a section isomorphic to $\mathrm{PSL}_2(4)$, for $X \cong \mathrm{PSU}_6(2)$ or $\mathrm{PSU}_7(2)$, $C_X(z)$ has a section isomorphic to $\mathrm{PSU}_4(2)$ or $\mathrm{PSU}_5(2)$ and for $X \cong \mathrm{PSp}_6(3)$, $C_X(z)$ contains a section isomorphic to $\mathrm{PSp}_4(3)$. So all these cases are eliminated. The remaining groups are $\mathrm{PSL}_4(2)$ listed in (ii) and $X \cong \mathrm{PSL}_4(3)$, $\mathrm{P}\Omega_7(3)$ presented in (iii).

So we are left with the case where $Z(S)$ is not a root subgroup of K. If $X \cong \mathrm{F}_4(2^e)$, then by Lemma D.7 $C_X(Z(S))$ contains a section isomorphic to $\mathrm{PSp}_4(2^e)'$ and so this group is not listed. Suppose that $X \cong \mathrm{PSp}_{2n}(2^e)$. Then also by Lemma D.5 $C_X(Z(S))$ contains a section isomorphic to $\mathrm{PSp}_{2(n-2)}(2^e)'$. This group is not soluble if $2^e > 2$ or $n > 3$. Hence $\mathrm{PSp}_6(2)$ is listed in (ii). $\square$

We will now study the structure of $O_p(N_X(R))$ for those groups X of Lie type in which $R = Z(S)$ is a long root subgroup where, as usual, S is a Sylow p-subgroup of X. Recall the definition of a semi-extraspecial group from Definition 2.9.

Lemma D.16. *Suppose that X is a group of Lie type in characteristic p listed in the table of Lemma* D.1 *or is* $\mathrm{G}_2(p^e)$*,* $p \neq 3$ *and* $p^e \neq 2$*. Assume that R is a long root subgroup of X and $Q = O_p(N_X(R))$. Then Q is semi-extraspecial. Moreover, if $x \in Q \setminus R$, then $|Q : C_Q(x)| = |R|$ and $[x, Q] = R$.*

Proof. Suppose U is a maximal subgroup of R and assume that $Q' \leq U$. Then, as $N_X(R)$ by Lemma A.4 acts irreducibly on R, we deduce that Q is abelian which is not the case. Hence Q/U is not abelian. In all the cases other than $X \cong \mathrm{PSL}_n(p^e)$, $\mathrm{P}\Omega_6^-(p^e)$ or $X \cong \mathrm{G}_2(4)$, $L = O^{p'}(N_X(Q)/Q)$ acts irreducibly on Q/R and centralizes R by Lemmas D.1 and D.10. In the irreducible cases, then Q/U admits L, so, if $W = Z(Q/U)$, we have $W = Q/U$ or $W = R/U$. As Q/U is not abelian, the latter is the case and so Q is semi-extraspecial. Similarly, if $X \cong \mathrm{P}\Omega_6^-(p^e)$, then, as Q is not extraspecial, $p^e \neq 3$ and Lemma D.1 shows that $C_X(R)$ acts irreducibly on Q/R. Thus we are done in this case as well.

Suppose $X \cong \mathrm{PSL}_n(q)$ and $Z(Q/U) > R/U$. Then, the action of L on Q described in Lemma D.1, implies that Q/U has centre W/U of order pq^{n-2} and W has order q^{n-1}. Let E_1 and E_2 be the elementary abelian normal subgroups of Q of order q^{n-1} which are normalized by a maximal parabolic subgroup of X. Then, without loss of generality $E_1 \cap W = R$. Now $[E_1, W] \leq U$ and so, as E_1 is a natural module for $N_X(E_1)/E_1 \cong \mathrm{SL}_{n-2}(q)$ and $|U| < q$, we conclude that $[E_1, W] = 1$, which is a contradiction.

So suppose $X \cong \mathrm{G}_2(4)$. Then, according to Lemma D.10, $L \cong \mathrm{SL}_2(4) \cong \mathrm{Alt}(5)$ acts on Q/R preserving a decomposition into a direct sum of two 4-dimensional natural $\mathrm{Alt}(5)$-modules. Let $P_1 = N_X(Q)$ and P_2 be the parabolic subgroup of X containing S with $P_2 \neq P_1$. Then Lemma D.10 yields $W = Z(O_2(P_2)) = Z_2(S)$ has order 16 and is contained in Q. Furthermore, W is a natural module for $O^{2'}(P_2/O_2(P_2)) \cong \mathrm{SL}_2(4)$ and $S = QO_2(P_2)$. It follows that for $w \in W \setminus R$, we have $[w, Q] = R$.

Again let U be a maximal subgroup of R. Then we may either apply the previous argument or Q/U could potentially have a centre of order 2^5. Assume the latter case and let X be the preimage of the centre. Then $X \cap Z(O_2(P_2)) \not\leq R$ and so we get that $[X, Q] = R$ a contradiction. The final statements follow from Lemma 2.10. $\square$

To exploit the results about quadratic action given in Appendix C we frequently use the following lemma.

Lemma D.17. *Suppose that X is isomorphic to one of the following $\mathrm{PSL}_n(p^e), n \geq 4$, $\mathrm{PSU}_n(p^e), n \geq 3$, $\mathrm{PSp}_{2n}(p^e), n \geq 2, p$ odd, $\mathrm{E}_6(p^e)$, ${}^2\mathrm{E}_6(p^e)$, $\mathrm{E}_7(p^e)$, $\mathrm{E}_8(p^e)$ or $\mathrm{F}_4(p^e), p$ odd. Let R be a long root subgroup of X, $Q = O_p(N_X(R))$ and $L = O^{p'}(N_X(R))/Q$. Then we have the following statements.*

(i) *There is an X-conjugate T of R in $N_X(Q)$ such that $T \cap Q = 1$ and T acts quadratically on Q/R. Moreover, for $t \in T^{\#}$, $C_{Q/R}(t) = C_{Q/R}(T)$. In particular if $e > 1$, there is a group of order p^2 in L, which acts quadratically on Q/R.*
(ii) *If $p^e = p$, then either there is an elementary abelian group of order p^2 in L which acts quadratically on Q/R or $X \cong \mathrm{PSL}_4(p)$, $\mathrm{PSp}_4(p)$, $\mathrm{PSU}_4(p)$, $\mathrm{PSU}_5(p)$, or p is odd and X is possibly one of ${}^2\mathrm{E}_6(p)$, $\mathrm{E}_8(p)$ or $\mathrm{F}_4(p)$ and the $(L/Z(L), |Q/R|)$ is one of $(\mathrm{PSU}_6(p), p^{20})$, $(\mathrm{E}_7(p), p^{56})$ or $(\mathrm{PSp}_6(p), p^{14})$.*

Proof. According to [**27**, Eq. (3.2.5), page 104] if we have a conjugate R^h of R such that $\langle R, R^h\rangle \cong \mathrm{SL}_2(p^e)$, then $N_X(\langle R, R^h\rangle) = \langle R, R^h\rangle\widetilde{L}$, where $\widetilde{L}$ is the Levi complement in $N_X(R)$. As the rank of X is at least three, we have that $\widetilde{L}$ is a genuine group of Lie type and as $N_X(\langle R, R^h\rangle)$ contains a torus, we have that $S \cap \widetilde{L}$ contains a conjugate T of R. This proves the first part of (i).

Suppose that $y \in C_{Q/R}(t)$. Then $t^y \in \langle t, R\rangle$ and so $t^y \in RT$. Since $[\langle R, R^h\rangle, T] = 1$, Lemma A.5 implies that $t^y \in T$ as it cannot be in R. Thus $T^y = T$ by Lemma A.4 (iii). But then $[y, T] \leq Q \cap T = 1$. Hence $y \in C_{Q/R}(T)$. This proves the final part of (i).

Let P be a maximal parabolic of X with $N_X(Q) \not\leq P$ and set $Y_P = \langle R^P\rangle$. Suppose $Y_P \not\leq Q$. As Y_P is normalized by $N_X(S)$, we see first of all that $Z(S/Q) \cap Y_PQ/Q \neq 1$ and further by Lemma A.4 that $TQ/Q \leq Y_PQ/Q$. Hence (i) holds as $[Q/R, Y_P, Y_P] = 1$. If $|Y_P : Y_P \cap Q| \geq p^2$, then also (ii) holds. By Lemma D.16 in all cases Q is semi-extraspecial.

Suppose that X is a classical group. If X is linear, let P be the stabiliser of a 2-space and otherwise let P be the stabiliser of a maximal isotropic subspace. In all cases, the subgroup V described in Lemma D.22 is contained in Y_P. Thus $Y_P \not\leq Q$ by Lemma D.22 (ii). This proves (i) in all these classical cases. Suppose that X is an exceptional group and that P is a maximal parabolic subgroup of X containing $N_X(S)$ chosen as in the following table. Here M denotes $O^{p'}(P/O_p(P))/Z(O^{p'}(P/O_p(P)))$ and similarly N is the central quotient of $O^{p'}(N_X(S) \cap P)/O_p(N_X(S) \cap P)$.

X	M	N
${}^2\mathrm{E}_6(p^e)$	$\Omega_8^-(p^e)$	$\Omega_6^-(p^e)$
$\mathrm{E}_6(p^e)$	$\Omega_{10}^+(p^e)$	$\mathrm{SL}_5(p^e)$
$\mathrm{E}_7(p^e)$	$\mathrm{E}_6(p^e)$	$\Omega_{10}^+(p^e)$
$\mathrm{E}_8(p^e)$	$\Omega_{14}^+(p^e)$	$\Omega_{12}^+(p^e)$
$\mathrm{F}_4(p^e)$	$\Omega_7(p^e)$	$\mathrm{Sp}_4(p^e)$

where these structures have been obtained from [**27**, Examples 3.2.4 and 3.2.5 pages 99 to 101] except for $\mathrm{E}_8(p^e)$ where $P/O_p(P)$ is taken as $M_{\{\alpha_8\}}$ and calculated as described in the previous citation.

Using the descriptions of the Levi sections given in Lemma D.1 and comparing this information with the details presented in the above table, we see that

(D.17.1) $QO_p(P)/O_p(P)$ is not normalized by the normalizer of a root subgroup of $Z(S/O_p(P))$.

If $Y_P \le Q$, then $[Q, Y_P] = R$. Let $y \in P\backslash N_X(R)$, then $[Y_P, QO_p(P) \cap Q^y O_p(P)] \le R \cap R^y = 1$ by Lemma A.4. Hence $QO_p(P)/O_p(P)$ is a trivial intersection set in $P/O_p(P)$. Since $QO_p(P)/O_p(P)$ contains a root subgroup R_1, and $N_{P/O_p(P)}(R_1)$ does not normalize $QO_p(P)/O_p(P)$ by (D.17.1), we have a contradiction. This completes the proof of (i).

Suppose now that $p^e = p$. If we have one of $\mathrm{PSL}_4(p)$, $\mathrm{PSp}_4(p)$, $\mathrm{PSU}_4(p)$, $\mathrm{PSU}_5(p)$ then there is no elementary abelian group of order p^2 in $N_X(Q)/Q$, hence these groups are listed as exceptions in (ii). Thus we may assume that the untwisted Lie rank of X is at least 4. We consider $Y_P Q/Q$, which is normalized by $P \cap N_X(R)$. If $|Y_P Q/Q| > p$, we have nothing more to prove. Thus $Y_P Q/Q$ is a root subgroup in $Z(S/Q)$ and $P \cap N_X(R)$ is the normalizer of a root subgroup in $N_X(R)/Q$. The Levi sections of $P \cap N_X(R)$ are given in the table above in the case of the exceptional group and the Levi sections of $N_{N_X(R)}(Y_P Q)$ are given in Lemma D.1. The groups $\mathrm{E}_6(p)$ and $\mathrm{E}_7(p)$ have incompatible structures and so are eliminated while the other cases for p odd are listed in (ii). We will return to the case $p = 2$ later.

Suppose that X is a classical group and take P to be as described earlier. Then if X is a linear group, $P \cap N_X(R)$ is maximal in $N_X(R)$ and so cannot be the normalizer of a root subgroup in $N_X(R)/Q$, recall that the Lie rank is at least 4. If X is unitary or symplectic, then the Levi section of $N_X(R) \cap P$ is a linear group defined over $\mathrm{GF}(p^2)$ respectively $\mathrm{GF}(p)$, whereas the normalizer of a root subgroup involves a unitary group over $\mathrm{GF}(p)$ or a symplectic group. By [**37**, Proposition 2.9.1] we conclude that $n = 6$ and $X \cong \mathrm{PSp}_6(p)$ is a symplectic group (as in this case p is odd). However in $\mathrm{PSp}_6(p)$ we have Y_P has order p^6 and Q is extraspecial of order p^5. Thus $|Y_PQ/Q| = p^3$, a contradiction.

Suppose that $p^e = 2$. Then as just seen we have only to deal with $X \cong {}^2\mathrm{E}_6(2)$ and $X \cong \mathrm{E}_8(2)$. Suppose that $QY_P = QO_2(P)$. Then $\Phi(O_2(P)) \leq \Phi(Q \cap O_2(P)) = R$ as $Y_P \leq Z(O_2(P))$. Since P does not normalize R, we have $O_2(P)$ is elementary abelian. Hence, as Q is extraspecial, $|O_2(P)| \leq 2^{12}$ in the first case and 2^{30} in the second case. On the other hand, using Lemma A.2 we contradict $|S|$. Hence $Y_PQ < O_2(P)Q$. Now applying Lemma D.1 shows that $O_2(P)Q/Q$ is extraspecial of order greater than 8. In particular, there is an involution $x \in O_2(P)Q/Q$ which is not contained in Y_PQ/Q. Now $[Q/R, Y_P, x] = 1 = [Q/R, x, Y_P]$ and so $[Q/R, \langle x, Y_P\rangle] \leq C_{Q/R}(Y_P) \cap C_{Q/R}(x)$ as involutions act quadratically on Q/R. As $\langle x, Y_PQ/Q\rangle$ is a fours group, this proves (ii). $\square$

Lemma D.18. *Let $X \cong \Omega^+_{2n}(2)$ with $n \geq 3$ and i an involution in the centre of a Sylow 2-subgroup of X. Then X is generated by $2n$ conjugates of i.*

Proof. Suppose that $n = 3$. Then by [**37**, Proposition 2.9.1], $X \cong \mathrm{Alt}(8)$. Furthermore, X has a maximal subgroup isomorphic to $Y = \mathrm{Sym}(6)$ and i corresponds to an element of cycle type 2^4 in X. These elements act on the 6 set preserved by Y with cycle type 2^3 and, as Y is generated by 5 transpositions it is also generated by 5 elements of cycle type 2^3 and so by 5 conjugates of i. Hence X is generated by 6 conjugates of i. Now suppose that $n > 3$ and the result is true for $3 \leq m < n$. Let P be the stabiliser of a non-zero singular vector in the natural module for X. Then P has shape $2^{2n-2}{:}\Omega^+_{2n-2}(2)$. By induction this group is generated by $2n-1$ conjugates of i. As P is maximal, we therefore have X is generated by $2n$ conjugates of i. $\square$

Lemma D.19. *Let $X \cong \mathrm{PSL}_n(2)$, $n \geq 5$, $\mathrm{PSU}_n(2)$, $n \geq 5$, $\mathrm{P}\Omega^\pm_{2n}(2)$, $n \geq 4$, $\mathrm{E}_n(2)$, $n = 6, 7, 8$, ${}^2\mathrm{E}_6(2)$ or ${}^3\mathrm{D}_4(2)$. Let S be a Sylow 2-subgroup of X and $Q = O_2(C_X(r))$, $\langle r\rangle = Z(S)$. Suppose that there is some involution $i \in Z(S/Q)$ such that $|[Q/\langle r\rangle, i]| = 4$, then $X = \mathrm{PSL}_n(2)$, or $\mathrm{PSU}_n(2)$.*

Proof. By Lemma D.1 and Lemma D.16 in these cases we have that Q is an extraspecial group. Furthermore in case of $\mathrm{PSL}_n(2)$ and $\mathrm{PSU}_n(2)$ we have that i is a transvection and so the lemma holds for these groups. Assume now that $X \cong \mathrm{P}\Omega^\pm_{2n}(2)$. Then according to Lemma D.1 we have that $N_X(Q)/Q \cong \Omega^\pm_{2n-4}(2) \times \mathrm{Sym}(3)$ and $Q/\langle r\rangle$ is a direct sum of two orthogonal modules. As $\Omega^\pm_{2n-4}(2)$ contains no transvections on the natural module (see Lemma C.22), we see that

$|[Q/\langle r\rangle, i]| \geq 16$. So assume now $X \cong \mathrm{E}_n(2)$ or ${}^2\mathrm{E}_6(2)$. Then we have by Lemma D.1 that $N_X(Q)/Q \cong \mathrm{PSL}_6(2)$, $\mathrm{P}\Omega_{12}^+(2)$, $\mathrm{E}_7(2)$ or $\mathrm{PSU}_6(2)$. The case $\mathrm{P}\Omega_{12}^+(2)$ is impossible by Lemma D.18 as $N_X(Q)/Q$ is generated by 12 conjugates of i, $i_1, \ldots, i_{12}$ and $Q/R = \langle [Q/R, i_j] \mid 1 \leq j \leq 12\rangle$ has order at most 2^{24}. In the remaining case Lemma D.1 gives $C_{N_X(Q)/Q}(i) \cong 2_+^{1+8} : \mathrm{SL}_4(2)$, $2_+^{1+32} : \Omega_{12}^+(2)$ or $2_+^{1+8} : \mathrm{SU}_4(2)$. Hence $[C_{N_X(Q)/Q}(i), [Q/\langle r\rangle, i]] = 1$. We therefore get a contradiction to Lemma C.1.

So we are left with $X \cong {}^3\mathrm{D}_4(2)$. By Lemma D.1 we have that $N_X(Q)/Q \cong \mathrm{SL}_2(8)$. But then i inverts some element ω of order 9 in $N_X(Q)$. As $|[V, i]| = 4$, this shows $[V, \omega^3] = 1$, a contradiction. □

Lemma D.20. *Suppose that X is a group and $F^*(X)$ is a group of Lie type in characteristic p. Let S be a Sylow p-subgroup of $F^*(X)$ and assume that $R = \Omega_1(Z(S))$ is a long root subgroup of order p^e. Set $Q = O_p(C_X(R)))$ and let A be a subgroup of $N_{F^*(X)}(R)/Q$ of order at least p^e. If $|[Q/R, A]| = p^e$, then either p is odd and $F^*(X) \cong \mathrm{PSp}_{2n}(p^e)$ or $F^*(X) \cong \mathrm{G}_2(2)'$.*

Proof. Lemma D.13 shows that $F^*(X)$ is not ${}^2\mathrm{F}_4(2^{2e+1})'$ and Lemma A.3 shows that X is not $\mathrm{Sp}_{2n}(2^e)$, $\mathrm{F}_4(2^e)$ or $\mathrm{G}_2(3^e)$. Thus, by Lemmas D.1 and D.10, Q/R is a module over $\mathrm{GF}(p^e)$ or $\mathrm{GF}(p^{2e})$ for $L = O^{p'}(N_X(R)/Q)$. As $|[Q/R, A]| = p^e$, we get that Q/R is a $\mathrm{GF}(p^e)L$-module. In particular, $X \not\cong \mathrm{PSU}_n(p^e)$. As Q/R is defined over $\mathrm{GF}(p^e)$, $[y, A] = [Q/R, A]$ for all $y \in Q \setminus C_{Q/R}(A)$. Furthermore, as $|A| \geq p^e$, A is a dual offender on Q/R and every element of A operates as a $\mathrm{GF}(p^e)$-transvection on Q/R. Thus Lemma D.1 implies that X is not a linear group. If X is an orthogonal group defined of dimension m at least 7, then Q/R is a tensor product module for $L = L_1L_2$ where $L_1 \cong \mathrm{SL}_2(p^e)$ and $L_2 \cong \Omega_{m-4}^{\pm}(p^e)$. Since Q/R is a direct sum of more than one non-trivial L_i-module, we see that A cannot act non-trivially on either L_1 or L_2. Hence X is not an orthogonal group. If L is soluble, then by D.1 and D.10 we are left with $X \cong \mathrm{PSp}_4(3)$ or $\mathrm{G}_2(2)'$ and these groups are listed. The remaining groups all have L is quasisimple and Q/R is an irreducible module. Thus by comparing the possibilities for Q/R and L given in Lemmas D.1 and D.10 with the possibilities given by Lemma C.22 yields $L \cong \mathrm{Sp}_{2n-2}(p^e)$ with Q/R the natural module and so $X \cong \mathrm{PSp}_{2n}(p^e)$. □

In our proof of Theorem 4 the structure of p-minimal subgroups P of G such that $Z_2(S) \leq Z(O_p(P))$, $S \in \mathrm{Syl}_p(G)$, play a very important role. To handle this type of situation we require the structure of the corresponding groups in simple groups of Lie type.

Lemma D.21. *Suppose that X is a simple group of Lie type defined over* $\mathrm{GF}(p^e)$ *and of rank at least* 2, *let S be a Sylow p-subgroup of X, R be a long root subgroup contained in $Z(S)$ and set $Q = O_p(N_X(R))$. Assume that $X \not\cong \mathrm{PSL}_3(p^e)$, ${}^2\mathrm{F}_4(2^{2e+1})'$, $\mathrm{F}_4(2^e)$, $\mathrm{PSp}_{2n}(2^e)'$, $\mathrm{G}_2(2)'$ or $\mathrm{G}_2(3^e)$. Then the following hold*

(i) *$Z(S) = Z(Q) = R$ is a long root subgroup and $Z_2(S) \le Q$.*
(ii) *Either $|Z_2(S)| = p^{2e}$, or $X \cong \mathrm{PSU}_n(p^e)$ or $\mathrm{PSL}_n(p^e)$ in which case $|Z_2(S)| = p^{3e}$.*

Proof. The first statement of (i) is already recorded in Lemma A.3. Since $Q = O_p(N_G(R))$, the Borel-Tits Theorem [**27**, Theorem 3.1.3] implies that $Q = C_{N_X(R)}(Q/R)$. Hence, as $R = Z(S) \ge [Z_2(S), Q]$, we have $Z_2(S) \le Q$.

Denote by $L = O^{p'}(N_X(Q))$. Assume first that $X \cong \mathrm{P}\Omega^\pm_m(p^e)$, $m \ge 7$. Then $L = L_1L_2$, $L_1 \cong \mathrm{SL}_2(p^e$, $L_2 \cong \Omega^\pm_{m-4}(p^e)$ and Q/R is the tensor product module (see Lemma D.1). Now Q/R is a direct sum of two natural L_2-modules and then by Lemma C.1 $|C_{Q/R}(S \cap L_2)| = p^{2e}$. As L_1 induces the natural module on $C_{Q/R}(S \cap L_2)$, we get $|C_{Q/R}(S)| = p^e$. Therefore $|Z_2(S)| = p^{2e}$. Thus from now on $X \not\cong \mathrm{P}\Omega^\pm_m(p^e)$, $m \ge 7$.

Assume next that L acts irreducibly on Q/R. Then, by Lemma C.1, $|C_{Q/R}(S)| = |k|$ where k is the field of definition of the L-module Q/R. By Lemma D.1 this is $\mathrm{GF}(p^e)$ as long as $X \not\cong \mathrm{PSU}_n(p^e)$, in which case Q/R is defined over $\mathrm{GF}(p^{2e})$. In the first cases we therefore have $|Z_2(S)| = p^{2e}$ and in the exceptional case $|Z_2(S)| = p^{3e}$.

Using Lemmas D.1 and D.10 it remains to consider $X \cong \mathrm{PSL}_n(p^e)$, $\mathrm{P}\Omega^\pm_6(p^e)$, $\mathrm{G}_2(4)$. If we have $X \cong \mathrm{PSL}_n(p^e)$ there are two irreducible modules in Q/R and so $|Z_2(S)| = p^{3e}$. If we have $X \cong \mathrm{G}_2(4)$, then we have two natural $\mathrm{Alt}(5)$-modules and so $|Z_2(S)| = 16 = 4^2$. When $X \cong \mathrm{P}\Omega^\pm_6(p^e)$, Q/R has order p^{4e} and $|C_{Q/R}(S)| = p^{2e}$. $\square$

Lemma D.22. *Let X, S and R be as in Lemma* D.21. *Assume that $P > N_X(S)$ is a parabolic subgroup of X chosen to be maximal subject to $P \cap N_X(R) = N_X(S)$. Set $V = \Omega_1(Z(O_p(P)))$ and $K = O^{p'}(P/O_p(P))$. Then $Z_2(S) = V \cap Q$ and the following hold.*

(i) *If $V = Z_2(S)$, then $K \cong \mathrm{SL}_2(p^e)$ and V is a natural K-module. In particular, all elements in $Z_2(S)^\#$ are conjugate in P.*
(ii) *If $V > Z_2(S)$, then $V = Z_2(S)R^g$ for suitable $g \in P$ and we have*
  (a) *$X \cong \mathrm{PSp}_{2n}(p^e)$, $n \ge 2$, $|V| = p^{3e}$, $K \cong \Omega_3(p^e)$ and V is a natural K-module;*

(b) $X \cong \mathrm{PSU}_n(q)$, $n \geq 4$, $|V| = p^{4e}$, $K \cong \Omega_4^-(p^e)$ *and* V *is a natural* K*-module; or*

(c) $X \cong \mathrm{PSL}_n(q)$, $n \geq 4$, $|V| = p^{4e}$, $K \cong \Omega_4^+(p^e)$ *and* V *is a natural* K*-module.*

Proof. We have that P is a minimal parabolic subgroup or in case of $X \cong \mathrm{PSL}_n(q)$ we have $P = P_1P_2$, where the P_i are minimal parabolic subgroups. By Lemma D.21 (i), $Z_2(S) \leq Q$.

Let first $X \cong PSL_n(q)$. Let X_1 be the stabiliser of a point and X_2 be the stabiliser of a hyperplane in the natural representation of $\mathrm{GL}_n(q)$. Then $O_p(X_i)$ is the natural module for $X_i/O_p(X_i) \cong \mathrm{GL}_{n-1}(q)$. Furthermore we have that $P_i \leq X_i$, $i = 1, 2$. This shows $|\langle R^{P_i}\rangle| = q^2$ and so $\langle R^{P_1}, R^{P_2}\rangle = Z_2(S)$. Hence we have that $Z_2(S) \geq V$.

Assume now that P is a minimal parabolic. We have that P contains the Borel subgroup B. If Q/R is an irreducible module for $O^{p'}(N_X(R))$, we receive by Lemma C.1 that B acts irreducibly on $Z_2(S)/Z(S)$ and so $Z_2(S) \leq V$. If $X \cong \mathrm{P}\Omega_n(q)$, then we have a tensor product module and so the same holds. In case of $X \cong \mathrm{G}_2(4)$, we get the result with Lemma D.10. Hence in any case we have

(D.22.1) $Z_2(S) \leq V$.

Suppose first that $V = Z_2(S)$. If $|Z_2(S)| = q^2$, then $V = RR^g$ for a suitable $g \in P$. But as Q is semi-extraspecial by Lemma D.16, all elements in the coset Rr^g, $r \in R$ are conjugate by Q. This implies that all elements in V are conjugate under $\langle Q, Q^g\rangle$.

Assume that $|Z_2(S)| \neq q^2$. Then by Lemma D.21 we have $|Z_2(S)| = q^3$. Assume first that $X \cong \mathrm{PSU}_n(q)$. Then we have that $O^{p'}(P/O_p(P)) \cong \mathrm{PSL}_2(q^2)$. But $\mathrm{PSL}_2(q^2)$ cannot act non-trivially on a group of order at most q^3. So we may consider $X \cong \mathrm{PSL}_n(q)$. Now $P = P_1P_2$. But again this group cannot act faithfully on a group of order q^3. This can be seen as follows: we have $[V, P_1]$ is of order at most q^3 and so it must be an irreducible P_1-module. But then as P_2 is non-abelian we get that $[[V, P_1], P_2] = 1$. As $|[V/[V, P_1]| \leq q$, we now get $[V, P_2] = 1$. So we have shown that

(D.22.2) If $V = Z_2(S)$, then all elements in $Z_2(S)^\#$ are conjugate.

If $X = \mathrm{PSL}_n(q)$, we just have seen that $S = QO_p(P)$. If $X \neq \mathrm{PSL}_n(q)$, then P is a minimal parabolic of type PSL_2 and so B acts irreducibly on $S/O_p(P)$ hence also

$$S = O_p(P)Q.$$

Further we see that

$$C_S(Z_2(S)) = O_p(P).$$

Assume now $V > Z_2(S)$. We first will show that $V \cap Q = Z_2(S)$. Otherwise by Lemma D.16 we have that $|Q : Q \cap O_p(P)| \geq |V \cap Q/R| > |Z_2(S)/R|$. But then $C_Q(Z_2(S)) \not\leq O_p(P)$, a contradiction. So we have

$$V \cap Q = Z_2(S).$$

Then $[V, Q] \leq Z_2(Q)$. Assume first that $|Z_2(S)| = q^2$. Then $[Q/R, V] = Z_2(S)/R$ has order q. As $VQ/Q \cap Z(S/Q)$ is normalized by $N_X(S)$ and is non-trivial, we have $|VQ/Q| \geq q$ by Lemma A.4. Thus Lemma D.20 implies that $X \cong \mathrm{PSp}_{2n}(q)$. Then application of Lemma C.15 gives $|V : V \cap Q| \leq q$. Now $VQ/Q \leq Z(S/Q)$. We have that the Borel subgroup B acts irreducibly on $Z(S/Q)$, so $VQ/Q = Z(S/Q)$. In particular $|V| = q^3$ and $V = Z_2(S)R^g$ for suitable $g \in P$. Furthermore $|[V, Q]| = q^2 = |Q/C_Q(V)|^2$. So V is a dual 2F-module, which by Lemma C.28 gives the statement for $\mathrm{PSp}_{2n}(q)$.

So assume that $|Z_2(S)| = q^3$. If $X \cong \mathrm{PSU}_n(q)$, then V induces transvections on Q/R regarded as a $\mathrm{GF}(q^2)$-module and so again by Lemma C.15 $|V : V \cap Q| = q$. Now $|V| = q^4$, $V = Z_2(S)R^g$ and $|[Q, V]| = q^3 < q^4 = |Q/C_Q(V)|$. So again V is a dual 2F-module and the statement for $\mathrm{PSU}_n(q)$ follows from Lemma C.28.

If we have $X \cong \mathrm{PSL}_n(q)$, then by Lemma D.1 we have that $Q/R = E_1E_2$, where both E_1 and E_2 are modules for $N_X(R)$ and $|Z_2(S)/R \cap E_i| = q$. Suppose that $VQ/Q \not\leq Z(S/Q)$ As V induces transvections on the natural module to a point, we now get that VQ/Q is the full transvection group to $Z_2(S)/R \cap E_1$. On the other hand the same is true for E_2. But these transvection groups generate a non-abelian group. This shows that $VQ/Q = Z(S/Q)$ again. So we have $|V| = q^4$.

We had that $P = P_1P_2$, where P_i normalizes the intersection Z_i of $Z_2(S)$ with the preimage of E_i, $i = 1, 2$. But if $[V, O^{p'}(P_1)] = [Z_1, O^{p'}(P_1)]$, then this group is centralized by $O^{p'}(P_2)$ and so $P_2 \leq N_X(R)$, a contradiction. So we have that V involves two natural P_1-modules and the same applies for P_2, which shows that we have a tensor product module, which is the statement for $\mathrm{PSL}_n(q)$. □

LEMMA D.23. *Let X, S, V be as in Lemma* D.22. *Set*

$$V(Q, S) = Z(C_S(C_Q(Z_2(S)))).$$

Then $V = V(Q, S)$.

PROOF. Assume first that $V = Z_2(S)$. Then $|Z_2(S)| = p^{2e}$. Let $x \in S \setminus Q$, such that x centralizes $C_Q(Z_2(S))$. Then x induces a

GF(p^e)-transvection on Q/R. Thus $V(Q,S)$ acts as a group of GF(p^e)-transvections on Q/R. As $V(Q,S) \trianglelefteq S$, we have $V(Q,S)Q/Q \cap Z(S/Q) \neq 1$. As $N_X(S)$ normalizes Q and $Z_2(S)$ it also normalizes $V(Q,S)$ and so $|V(Q,S)Q/Q| \geq p^e$. We have

$$[Q, C_Q(Z_2(S)), V(Q,S)] = 1 = [C_Q(Z_2(S)), V(Q,S), Q] = 1.$$

The Three Subgroup Lemma gives $[Q, V(Q,S), C_Q(Z_2(S))] = 1$. That is $[Q, V(Q,S)] \leq C_Q(C_Q(Z_2(S))) = Z_2(S)$. Hence $[Q, V(Q,S)] \leq Z_2(S)$. Application of Lemma D.20 shows $F^*(H) \cong \mathrm{PSp}_{2n}(p^e)$, p odd. But then V is the orthogonal 3-dimensional module for $P/C_P(V)$, a contradiction. Hence we have $V(Q,S) \leq Q$ and so $V(Q,S) = Z_2(S)$.

We may assume that $V \neq Z_2(S)$. As seen in Lemma D.22 we have $|V : V \cap Q| = p^e$ and $V \cap Q = Z_2(S)$. Let P be as in Lemma D.22, then P induces an orthogonal group on V and so Q does not induces GF(p^e)-transvections on V. In particular $C_Q(Z_2(S)) \leq O_p(P)$ and then $[V, C_Q(Z_2(S))] = 1$. This shows that $V \leq C_S(C_Q(Z_2(S)))$. As $S = O_p(P)Q$, we get that

$$V \leq Z(C_S(C_Q(Z_2(S)))) = V(Q,S).$$

Now suppose that $x \in V(Q,S)$. Then x induces a GF(p^e)-transvection on Q/R in case of $X \cong \mathrm{PSp}_{2n}(p^e)$, a GF($p^{2e}$)-transvection on Q/R in case of $X \cong \mathrm{PSU}_n(p^e)$ and a GF(p^e)-transvection to a point on both natural $\mathrm{SL}_{n-2}(p^e)$-modules in Q/R. As $|V : V \cap Q| = p^e$, we get in the first two cases by Lemma C.15 that $x \in QV$. For $\mathrm{SL}_{n-2}(p^e)$ the only group in S/Q, which induces transvections to a point in both modules is a root group, hence again we have that $x \in QV$. So $x = yv$, $v \in V$, $y \in Q$. As $[x, Z_2(S)] = 1$ we also have $[y, Z_2(S)] = 1$. We now get that $y \in Z(C_Q(Z_2(S))) = Z_2(S)$, by the structure of Q. Hence $y \in V$ and so $x \in V$. This shows $V = V(Q,S)$, the assertion. $\square$

Lemma D.24. *Let $X \cong {}^3\mathrm{D}_4(q)$, $q = p^e$, and S be a Sylow p-subgroup of X. Then we have*

(i) *$|Z(S)| = q$;*
(ii) *$|Z_2(S)| = q^2$;*
(iii) *$|Z_3(S)| \geq q^5$;*
(iv) *if $Q = O_p(C_X(Z(S)))$, then $Z_3(S) \leq Q$ and $|C_Q(t)| \geq q^3$ for any element $t \in S$, $o(t) = p$.*

Proof. By Lemma D.1 we have that $R = Z(S)$ is of order q, $|S| = q^{12}$ and $|Q| = q^9$. Furthermore by Lemma D.16 Q is semi-extraspecial. By Lemma D.21 we have that $|Z_2(S)| = q^2$ and $Z_2(S) = RR^g$ for suitable $g \in X$. So (i) and (ii) hold. In particular $Z_2(S) \leq Q^g$. Set $P = \langle Q, Q^g \rangle$, then $Z_2(S)$ is normal in P and by Lemma D.22

we have that $P/O_p(P) \cong SL_2(q)$ and acts naturally on $Z_2(S)$. As $Q \cap Q^g$ is elementary abelian, we get that $|Q \cap Q^g| \leq q^5$. Furthermore $U = (Q \cap O_p(P))(Q \cap O_p(P))^g$ is a normal subgroup of P and $|U : Q \cap Q^g| \geq q^6$. As $|O_p(P)| \leq q^{11}$ we get equality everywhere, i.e. $U = O_p(P)$ and $|Q \cap Q^g| = q^5$. As $Q' = R \leq Z_2(S)$, we see that $[\langle Q, Q^g \rangle, Q \cap Q^g] \leq Z_2(S)$. In particular $Q \cap Q^g \leq Z_3(S)$. So (iii) holds.

As $[Q, O_p(P)] \leq Q \cap O_p(P)$ we have $Z_3(S) \leq Q$. Finally if $t \in S$, $o(t) = p$, then we have $[Q \cap Q^g, t] \leq Z_2(S)$. Hence $|[Q \cap Q^g, t]| \leq q^2$ and so $|C_{Q \cap Q^g}(t)| \geq q^3$. This proves (iv). □

LEMMA D.25. *Let X be a genuine group of Lie type over a field of characteristic p and α be an automorphism of X with α^p inner. If α centralizes a Sylow p-subgroup S of X, then α is inner.*

PROOF. Suppose false. Then, by Theorem A.11(ii), α is in the coset of either a graph, graph-field or a field automorphism of X. As α centralizes S, we see that α has to normalize every parabolic subgroup of X which contains S and so α is not in the coset of a graph automorphism. Hence, by Lemma A.14, α has to induce a field automorphism on X. If X is not a twisted group, then α acts non-trivially on $Z(S)$ which is impossible. Hence X is a twisted group. If $p = 3$, then $X \cong {}^3\mathrm{D}_4(3^e)$. Let $L = O^{3'}(N_X(Z(S))/O_3(N_X(Z(S))))$. By Lemma D.1, $L \cong \mathrm{PSL}_2(3^{3e})$ and α induces a field automorphism on L, which certainly does not centralize a Sylow 3-subgroup of L. Hence $p = 2$.

Then $X \cong {}^2\mathrm{F}_4(2^{2e+1})$, $\mathrm{PSU}_n(2^e)$, $\Omega^-_{2n}(2^e)$, $n \geq 3$, or ${}^2\mathrm{E}_6(2^e)$. By Lemma A.13, the group ${}^2\mathrm{F}_4(2^{2e+1})$ has no outer automorphisms of order two. By [**37**, (4.2.3)] $\mathrm{PSU}_n(2^e)$ possesses a parabolic subgroup P with Levi section L such that $L \cong \mathrm{PSL}_{\lfloor \frac{n}{2} \rfloor}(2^{2e})$. In $\Omega^-_{2n}(2^e)$ the point stabiliser P has Levi section $\Omega^-_{2n-2}(2^e)$ and in ${}^2\mathrm{E}_6(2^e)$ by [**27**, Example 3.2.5, page 101] there is a parabolic P subgroup with Levi section $\Omega^-_8(2^e)$. In all cases α induces a field automorphism on the Levi section, in particular α acts non-trivially. But α centralizes $O_2(P)$ and as $C_P(O_2(P)) \leq O_2(P)$, we see that α must centralize the Levi section, a contradiction. □

We end this appendix with some results about specific groups.

LEMMA D.26. *Let $X \cong \mathrm{P}\Omega^+_8(3)$. Then the following hold.*

(i) *There is an involution $i \in X$ such that $E(C_X(i)) \cong \Omega^-_6(3) \cong 2{\cdot}\mathrm{PSU}_4(3)$.*
(ii) *If R is a root subgroup of X, then*

$$N_X(R)/O_3(N_X(R)) \sim (\mathrm{SL}_2(3) \circ \mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)) : 2 \sim 2^{1+6}_-.3^3.2.$$

Furthermore $O_3(N_X(R))$ is extraspecial of order 3^9.
(iii) *We have* $\mathrm{Out}(X) \cong \mathrm{Sym}(4)$.

PROOF. Part (i) is either [**63**, Lemma 3.8] or [**14**, Table 8.50] and part (ii) follows immediately from Lemmas D.1 and D.16. Part (iii) can be read of from [**27**, Lemma 2.5.12(b) and (j)]. □

LEMMA D.27. *Suppose that $X \cong \mathrm{P}\Omega_7(3)$. Then there is a parabolic subgroup P of X such that $P/O_3(P) \cong \mathrm{SL}_3(3)$, $|O_3(P)| = 3^6$ and $Z(O_3(P)) = \Phi(O_3(P))$ has order 3^3.*

PROOF. Let P be the stabiliser of an isotropic 3-space in the natural module for $\Omega_7(3)$. Then $P/O_3(P) \cong \mathrm{SL}_3(3)$. Now $|O_3(P)| = 3^6$. By [**27**, Table 3.3.1] the 3-rank of X is 5 hence $O_3(P)$ cannot be elementary abelian. □

LEMMA D.28. *Let $X \cong \mathrm{PSL}_4(3)$ or $\mathrm{PSU}_4(3)$ and S be a Sylow 3-subgroup of X. Then*

(i) *$J(S)$ is elementary abelian of order 3^4 and*
  (ia) *$N_X(J(S))/J(S) \cong (\mathrm{SL}_2(3) \circ \mathrm{SL}_2(3)){:}2$ if $X \cong \mathrm{PSL}_4(3)$ and*
  (ib) *$N_X(J(S))/J(S) \cong \mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$ if $X \cong \mathrm{PSU}_4(3)$.*

(ii) *if $X \cong \mathrm{PSL}_4(3)$, then there is an involution $i \in X$ with $E(C_X(i)) \cong \mathrm{PSL}_2(9) \cong \mathrm{PSU}_2(9)$.*

PROOF. By [**14**, Table 8.8] in the linear case and [**14**, Table 8.9] in the unitary case, there is an elementary abelian subgroup E of S of order 3^4, such that $N_X(E)/E$ has the structure given in (i). To prove (i), it remains to show that $E = J(S)$. By Lemma D.1 we have that $|Z(S)| = 3$ and $Q = O_3(C_X(Z(S)))$ is extraspecial of order 3^5. This shows that there are no elementary abelian subgroups of S of order 3^5. Let F be an elementary abelian subgroup of S of order 3^4. Then from the action of $\mathrm{SL}_2(3)$ on $Q/Z(Q)$ given in Lemma D.1 we have that $|C_{Q/Z(Q)}(F)| = 9$. Hence $C_Q(F)/Z(Q) = C_{Q/Z(Q)}(F)$ and so F is uniquely determined, in particular $J(S) = E$.

(ii) follows from [**27**, Table 4.5.1, page 172]. □

E. Miscellanea

This final appendix contains a collection of unrelated results about simple groups which do not belong in any of the other appendices.

LEMMA E.1. *Suppose that $X \cong \mathrm{PSU}_5(2)$. Then $\mathrm{Aut}(X)$ has three conjugacy classes of involutions. If $i \in \mathrm{Aut}(X)$ is an involution which* *induces an outer automorphism on X, then 5 divides $|C_X(i)|$ while $C_X(z)$ is a $5'$-group for all involutions z in X.*

Proof. Application of [**5**, (6.1),(6.2)] yields $\mathrm{PSU}_5(2)$ has exactly two classes of involutions and the centralizer in X of any involution has a $5'$ order. The assertion about the outer involutions follows from Lemma A.16. $\square$

Lemma E.2. *Let $X \cong \mathrm{PSp}_4(5)$, $i \in X$ be an involution and $w \in X$ have order 5. Then the following hold*

(i) *$|C_X(i)|$ is divisible by 5;*
(ii) *if $|C_X(i)|$ is divisible by 25, then i is 2-central;*
(iii) *$|C_X(w)|$ is even if and only if 5^3 divides $|C_X(w)|$; and*
(iv) *there are three X-conjugacy classes of subgroups of X of order 5 which are centralized by an involution. Furthermore, if H_1, H_2, H_3 are representatives of these conjugacy classes and $|N_X(H_i)| = |N_X(H_j)|$, then $i = j$.*

Proof. This is taken from the table in [**71**, page 489-491]. $\square$

Lemma E.3. *Let $X \cong \mathrm{PSL}_3(3)$. Then the maximal subgroups of X whose order are divisible by 6 but not by 9 are isomorphic to $\mathrm{Sym}(4)$.*

Proof. This can be found in [**14**, Tables 8.3 and 8.4]. $\square$

Lemma E.4. *Suppose that $X \cong \mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(3)$ or ${}^2\mathrm{B}_2(8)$ and H be a proper subgroup of X. Then H is soluble.*

Proof. This is well-known. $\square$

Lemma E.5. *Suppose $X \cong \mathrm{Sp}_4(3)$ and $\overline{X} = X/Z(X)$. Then the following statements hold.*

(i) *There is a maximal subgroup of $\overline{X}$ which is isomorphic to $\mathrm{GU}_3(2) \sim 3_+^{1+2}{:}\mathrm{SL}_2(3)$.*
(ii) *Let E be an elementary abelian subgroup of order 27 in $\overline{X}$ and U be a subgroup of $\overline{X}$ with $E \le U$. If E is not normal in U, then $U \cong \mathrm{GU}_3(2) \sim 3_+^{1+2}{:}\mathrm{SL}_2(3)$ or $U = \overline{X}$.*
(iii) *Suppose that $U \le X$ and $|U| = 2^a \cdot 3$ for some $a \ge 4$. If U has a proper subgroup Y with $Y \cong \mathrm{SL}_2(3)$ and U has no normal subgroup isomorphic to Y, then U is contained in a group of shape $(\mathrm{Q}_8 \times \mathrm{Q}_8).\mathrm{Sym}(3)$.*

Proof. (i) can be seen in [**14**, Table 8.12].

Let E be an elementary abelian subgroup of order 27 in a Sylow 3-subgroup of X. As $|Z(S)| = 3$ by Lemma A.3 and $|S : E| = 3$, we see that $E = J(S)$ is uniquely determined. The result now follows from [**14**, Table 8.12].

Recall that $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ by [**37**, Proposition 2.9.1]. As U is a $\{2,3\}$-group and $O_3(U) = 1$, $\overline{U}$ is a 2-local subgroup of $\overline{X}$ and the Borel–Tits Theorem [**27**, Theorem 3.1.3] yields $\overline{U}$ is contained in one of the two maximal parabolic subgroups $\overline{P_1}$ and $\overline{P_2}$ of $\overline{X}$ containing a given Sylow 2-subgroup of $\overline{X}$. Choose notation so that $\overline{P}_1 \sim 2^4.\mathrm{Alt}(5)$. Then $P_1 \sim (\mathrm{Q}_8 \circ \mathrm{Dih}(8)).\mathrm{Alt}(5)$. Then $UO_2(P_1)/O_2(P_1)$ is either contained in a subgroup isomorphic to $\mathrm{Sym}(3)$ or U is contained in the normalizer of a Sylow 2-subgroup of X. In the first case we find $U \leq W \sim (\mathrm{Q}_8 \circ \mathrm{Dih}(8)).\mathrm{Sym}(3)$ and so Y is normal in U, a contradiction. In the second case we may suppose that $U \leq P_2$. Thus in any case $U \leq P_2 \sim (\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)).2$ and, as 9 does not divide $|U|$, we have that U is as claimed. □

Lemma E.6. *Let $X \cong \mathrm{Sp}_6(3)$ and V be the natural module. Suppose that $W \cong (\mathrm{SL}_2(3) \times \Omega_3(3)).2$ is a subgroup of X which acts irreducibly on V. If U is an over-group of W different from X, then U is isomorphic to a subgroup of $(\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)) : 2$. Furthermore, $\Omega_1(W)$ is normal in U.*

Proof. We consider the maximal subgroups of $\mathrm{Sp}_6(3)$ given in [**14**, Table 8.28 and Table 8.29] testing which ones could contain W. Let M be a maximal subgroup of X containing W. Then by comparing $|W|$ and $|M|$, we see that the only possibilities for M are $\mathrm{GU}_3(3){:}2$, $\mathrm{GL}_3(3){:}2$, $\mathrm{Sp}_2(3^3){:}3$ and $(\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)){:}2$. The Sylow 2-subgroups of $\mathrm{Sp}_2(3^3)$ are quaternion, so W cannot be contained in such a group. The group $\mathrm{GL}_3(3)$ fixes an isotropic 3-space of V, but W' acts irreducibly, a contradiction. Hence we are left with $M = \mathrm{GU}_3(3){:}2$ or the target example. As $\mathrm{GU}_3(3)$ is a rank one group, by the Borel–Tits Theorem [**27**, Theorem 3.1.3], the centralizer of any element of order three in a given Sylow 3-subgroup of M is contained in the Borel subgroup, hence W' is contained in a Borel subgroup of M, which is absurd. This completes the proof. □

Lemma E.7. *Suppose that $X \cong \mathrm{SL}_2(p^e)$ and V is the natural X-module. Let $T \in \mathrm{Syl}_p(X)$ and $V_1 = C_V(T)$. If $v \in V \setminus C_V(T)$, then $\langle T, C_X(v)\rangle = X$.*

Proof. We have that $C_V(T)$ is a 1-dimensional subspace of V regarded as a $\mathrm{GF}(p^e)$-space. As X acts transitively on these subspaces we have that v is centralized by T^g, for some $g \in X$, where $T \neq T^g$. Now application of [**33**, Satz 8.27] yields $X = \langle T, T^g\rangle$. □

LEMMA E.8. *Suppose that $H \leq X$ with $X \cong \mathrm{Alt}(7)$ and $H \cong \mathrm{Sym}(5)$. Assume that V is an irreducible 4-dimensional $\mathrm{GF}(2)X$-module. Then the elements of order three in H act fixed-point-freely on V. In particular H, does not induce the orthogonal $\mathrm{O}_4^-(2)$-module on V.*

PROOF. A Sylow 3-subgroup of $\mathrm{GL}_4(2)$ is elementary abelian of order 9 and so by coprime action just one class of elements of order three act fixed-point-freely. Let $\nu \in X$ be of order 7, then $\dim C_V(\nu) = 1$. Hence an element of order three, which normalizes $\langle \nu \rangle$ has a fixed point. These elements are products of two 3-cycles. Thus the 3-cycles in X operate fixed point freely on V. As $H \cong \mathrm{Sym}(5)$ contains an element ρ of order three, which is centralized by an involution and involutions are products of two transpositions, the element ρ is a 3-cycle and therefore it acts fixed point freely on V, the assertion. Since elements of order 3 in H have fixed points on the orthogonal module for H, we have V restricted to H is not the orthogonal module. $\square$

LEMMA E.9. *Let $H_1 \cong \mathrm{O}_6^+(2)$ and $H_2 \cong \mathrm{O}_6^-(2)$. Then H_1 and H_2 have isomorphic Sylow 2-subgroups. Furthermore, $F^*(H_1)$ and $F^*(H_2)$ have isomorphic Sylow 2-subgroups.*

PROOF. To prove the main claim, we show that both groups have a Sylow 2-subgroup isomorphic to a Sylow 2-subgroup of $H_3 \cong \mathrm{Alt}(10)$. This is plain to see for $\mathrm{O}_6^+(2) \cong \mathrm{Sym}(8)$, so consider H_2. Consider the subgroup $J = 2 \wr \mathrm{Sym}(5) \leq \mathrm{Sym}(10)$. Then J has shape $2 \times 2^4{:}\mathrm{Sym}(5)$ and any two subgroups of J of shape $2^4{:}\mathrm{Sym}(5)$ are isomorphic. Note that $J \cap H_3$ has shape $2^4{:}\mathrm{Sym}(5)$ and contains a Sylow 2-subgroup of H_3. Then the stabiliser L of an isotropic point in the natural representation of H_2 has shape $2^4{:}\mathrm{Sym}(5)$ and it has a subgroup of index 10 of shape $2^3{:}\mathrm{Sym}(4)$ which is contained in a maximal subgroup $2^4{:}\mathrm{Sym}(4)$ of index 5. It follows that L is isomorphic to a subgroup of $2 \wr \mathrm{Sym}(5)$ and so L and $\mathrm{Alt}(10)$ have isomorphic Sylow 2-subgroups. This proves the first claim.

Next consider H_2'. This group contains a subgroup of shape $J = 2^4{:}\mathrm{Alt}(5)$ which is a subgroup of $2 \wr \mathrm{Sym}(5)$. Now consider the subgroup J_1 of J of shape $2^4{:}\mathrm{Alt}(4)$ (which may be considered as a subgroup of $2 \wr \mathrm{Sym}(4)$ for ease of calculation). Suppose that $O_2(J) = \langle e_1, e_2, e_3, e_4 \rangle$ is the base group of J_1. Then $S = \langle e_1, e_2, e_3, e_4, (1,2)(3,4), (1,3)(2,4) \rangle$ is a Sylow 2-subgroup of J_1. The subgroup $\langle e_1, e_2, (1,2)(3,4) \rangle$ has index 8 in S and the representations on the cosets of this subgroup embed S into $\mathrm{Alt}(8)$. $\square$

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