Transitive tournament tilings in oriented graphs with large minimum total degree

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TRANSITIVE TOURNAMENT TILINGS IN ORIENTED GRAPHS WITH LARGE MINIMUM TOTAL DEGREE*

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Abstract. Let $\vec{T}_k$ be the transitive tournament on k vertices. We show that every oriented graph on $n = 4m$ vertices with minimum total degree $(11/12 + o(1))n$ can be partitioned into vertex disjoint $\vec{T}_4$'s, and this bound is asymptotically tight. We also improve the best known bound on the minimum total degree for partitioning oriented graphs into vertex disjoint $\vec{T}_k$'s.

Key words. tournaments, linear programming, oriented ramsey, absorbing, fractional matching, tiling ramsey

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1. Introduction.

For a pair of (di)graphs G and F, we call a collection of vertex disjoint copies of F in G an *F-tiling*. We say that an F-tiling is *perfect* if it consists of exactly $|V(G)|/|V(F)|$ copies of F. Perfect F-tilings are sometimes referred to as *perfect F-packings, perfect F-matchings*, or *F-factors*.

The classic Hajnal–Szemerédi theorem [8] states that if G is a graph on $n \in k\mathbb{N}$ vertices with minimum degree at least $(1-1/k)n$, then G contains a perfect K_k-tiling. Moreover, there are n-vertex graphs with minimum degree $(1-1/k)n-1$ that do not contain a perfect K_k-tiling.

Recall that *digraphs* are graphs such that every pair of vertices has at most two edges between them, one oriented in each direction; *oriented graphs* are orientations of simple graphs (so there is at most one directed edge between any pair of vertices). Note that oriented graphs are a subclass of digraphs.

Recently the study of tilings in digraphs has proven fruitful, and a number of papers have focused on developing analogues of the Hajnal–Szemerédi theorem. In this setting there is more than one natural notion of degree: The *minimum semidegree* $\delta^0(G)$ of a digraph G is the minimum of its minimum outdegree $\delta^+(G)$ and its minimum indegree $\delta^-(G)$. The *minimum total degree* $\delta(G)$ of G is the minimum number of edges incident to a vertex in G. Thus, for oriented graphs G, $0 \le 2\delta^0(G) \le \delta(G) \le n-1$. When there is no possibility of confusion, we often refer to the minimum total degree as the minimum degree.

Let $\vec{T}_k$ denote the transitive tournament on k vertices and C_3 denote the cyclic triangle. In [5] it was proven that every digraph on $n \in k\mathbb{N}$ vertices with minimum

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total degree at least $2(1-1/k)n-1$ contains a perfect $\vec{T}_k$-tiling. This degree condition is best possible, and the result implies the original Hajnal–Szemerédi theorem. A minimum semidegree version of the Hajnal–Szemerédi theorem was proven in [18] for large digraphs; this result considers perfect T-tilings for any fixed tournament T. Finally, Czygrinow et al. [6] gave a general result which, together with a result of Wang [20], determines the minimum total degree threshold for perfect T-tilings in a digraph for any tournament T.

For oriented graphs, the situation is much more difficult. First, notice that one can have arbitrarily large minimum total degree and still avoid even a single copy of an oriented graph. Indeed, a transitive tournament G on n vertices has $\delta(G) = n-1$ but contains no oriented graph with a directed cycle. Further, there are n-vertex tournaments (i.e., complete oriented graphs) with minimum semidegree at least $(n-4)/2$ (i.e., almost as large as possible) that do not contain a perfect C_3-tiling (see [9, 10]). Note though that Keevash and Sudakov [9] did prove that there exists a $c > 0$ so that every sufficiently large oriented graph with minimum semidegree at least $(1/2-c)n$ contains a C_3-tiling covering all but at most three vertices. Additionally, Li and Molla [10] recently proved that if n is a sufficiently large odd multiple of 3, every regular tournament on n vertices has a perfect C_3-tiling, thereby verifying a conjecture of Cuckler [4] and Yuster [22].

More is known for the perfect $\vec{T}_k$-tiling problem in oriented graphs, though understanding the general behavior of the minimum degree threshold remains a significant challenge. Yuster [21] observed that if G is an oriented graph on $n \in 3\mathbb{N}$ vertices with minimum total degree at least $5n/6$, then G has a perfect $\vec{T}_3$-tiling. Furthermore, this bound is the best possible. Balogh, Lo, and Molla [2] later proved an analogous result for the minimum semidegree threshold.

Yuster [21] gave a bound on the total degree threshold for *nearly perfect* tiling with $\vec{T}_k$. That is, if G is an oriented graph on n vertices with minimum total degree at least $\left(1-2^{-(k+\log k)}\right)n$, then G has vertex disjoint copies of $\vec{T}_k$ covering all but $o(n)$ vertices.[1] Yuster also showed that if G is an oriented graph on $n \in k\mathbb{N}$ vertices with minimum total degree at least $(1-4^{-k})n$, then G has a perfect $\vec{T}_k$-tiling.

Our main result is to asymptotically determine the minimum total degree threshold for a perfect $\vec{T}_4$-tiling.

THEOREM 1.1. *For all $\varepsilon > 0$, there exists n_0 such that if G is an oriented graph on $n \geq n_0$ vertices, n is divisible by 4, and $\delta(G) \geq \left(\frac{11}{12}+\varepsilon\right)n$, then G has a perfect $\vec{T}_4$-tiling. Furthermore, for every n divisible by 4, there exists an oriented graph G on n vertices with $\delta(G) = \lceil \frac{11n}{12} \rceil - 1$ such that G does not contain a perfect $\vec{T}_4$-tiling.*

Moreover, we improve the general bounds on the minimum total degree threshold for perfect $\vec{T}_k$-tiling, showing that a slight improvement on Yuster's above-mentioned bound for *nearly* perfect $\vec{T}_k$-tiling in fact ensures that G has a perfect $\vec{T}_k$-tiling. Let $\vec{r}(k)$ be the smallest integer n such that every tournament on n vertices contains a copy of $\vec{T}_k$.

THEOREM 1.2. *For every $k \geq 4$ and $\varepsilon > 0$, there exists n_0 such that when $n \geq n_0$ and n is divisible by k the following holds. If G is an oriented graph on n vertices and*

$$\delta(G) \geq \left(1 - \frac{1}{k(2\vec{r}(k-1)-k+1)} + \varepsilon\right)n,$$

[1] Here and elsewhere log has base 2.

then G contains a perfect $\vec{T}_k$-tiling. In particular, $\delta(G) \geq \left(1 - 2^{-(k+\log k)} + \varepsilon\right) n$ *suffices here.*

Roughly, we obtain both of our results by splitting the problem into two parts: determining the minimum degree threshold for "fractional $\vec{T}_k$-tiling" (which is related to "nearly perfect $\vec{T}_k$-tiling") and determining the minimum degree threshold for "$\vec{T}_k$-absorbing." When $k = 4$, we are able to determine these two thresholds exactly, which is why we obtain an asymptotically tight bound in that case.

As discussed in the following section, one can obtain a bound for the minimum degree threshold for perfect $\vec{T}_k$-tilings via an application of the Hajnal–Szemerédi theorem. Indeed, this is where Yuster's aforementioned bounds came from. However, the bound in Theorem 1.1 is lower than that obtained via the Hajnal–Szemerédi theorem, demonstrating that the problem in the oriented graph setting is genuinely different. In order to discuss more precisely where our bounds come from, we must first discuss their connection to some more parameters in the next two sections.

In section 3 we give a minimum degree condition that ensures an oriented graph has a perfect fractional $\vec{T}_k$-tiling (and thus a nearly perfect $\vec{T}_k$-tiling); see Theorem 3.2. This theorem will be applied in the proofs of both Theorems 1.1 and 1.2. In section 4 we introduce an absorbing result which, combined with our results from section 3, yields Theorem 1.2. Theorem 1.1 is then proved in section 5. We finish the paper with some concluding remarks and open questions.

2. Oriented Ramsey numbers and perfect tilings.

Recall $\vec{r}(k)$ is the smallest integer n such that every tournament on n vertices contains a copy of $\vec{T}_k$. Erdős and Moser [7] proved that $2^{(1/2+o(1))k} \leq \vec{r}(k) \leq 2^{k-1}$. The following result provides $\vec{r}(k)$ for small values of k.

THEOREM 2.1 (see [16]). *$\vec{r}(3) = 4$, $\vec{r}(4) = 8$, $\vec{r}(5) = 14$, and $\vec{r}(6) = 28$.*

One can consider Turán-type questions in oriented graphs. The following observation shows that the Turán number of $\vec{T}_k$ in an oriented graph is completely determined by $\vec{r}(k)$ and Turán's theorem. Here we let $t(n, r)$ be the number of edges in a Turán graph on n vertices with r parts, i.e., $t(n, r)$ is the number of edges in a complete r-partite graph on n vertices with parts of size either the ceiling or floor of n/r.

Observation 2.2. The maximum number of edges in an oriented graph on n vertices that does not contain a copy of $\vec{T}_k$ is $t(n, \vec{r}(k) - 1)$.

Proof. If G is an oriented graph on n vertices with more than $t(n, \vec{r}(k)-1)$ edges, then, by Turán's theorem, G must contain a tournament on $\vec{r}(k)$ vertices, which implies that G contains a copy of $\vec{T}_k$.

Let T be a tournament on $\vec{r}(k) - 1$ vertices that does not contain a $\vec{T}_k$. Blowing up each vertex of T equitably to form an oriented graph on n vertices produces a graph without a copy of $\vec{T}_k$ whose underlying simple graph is the Turán graph on n vertices with $\vec{r}(k) - 1$ parts. ∎

For every positive integer n, let $\mathcal{T}_n$ be the collection of tournaments with vertex set $[n]$. Let $\vec{tr}(k)$ be the smallest integer n such that every $T \in \mathcal{T}_n$ has a perfect $\vec{T}_k$-tiling. Note that, by induction, for $n > \vec{tr}(k)$ and divisible by k, every tournament $T \in \mathcal{T}_n$ has a perfect $\vec{T}_k$-tiling. A folklore result, which can be verified with a straightforward case analysis, is that $\vec{tr}(3) = 6$ (see [14]), and, with a computer search,[2] it has been shown that $\vec{tr}(4) = 16$. Caro [3] proved that

$$\vec{tr}(k) \leq \vec{r}(2k-1) + (2k-1)\vec{r}(k) < 4^k,$$

but the determination of $\vec{tr}(k)$ is open for every $k \geq 5$. (See [17, Proposition 10] for a concise proof of Caro's upper-bound.)

For $n \geq \vec{tr}(k)/k$, let $\vec{\delta}_n(k)$ be the minimum integer such that every oriented graph G on nk vertices with $\delta(G) \geq \vec{\delta}_n(k)$ has a perfect $\vec{T}_k$-tiling, and define $\vec{\delta}(k) := \limsup_n \frac{\vec{\delta}_n(k)}{nk}$. The following straightforward consequence of the Hajnal–Szemerédi theorem, together with any bounds on $\vec{tr}(k)$, gives a bound on $\vec{\delta}(k)$.

Observation 2.3 (Yuster [21], Treglown [17]). Given any $k, n \in \mathbb{N}$, $\vec{\delta}_n(k) \leq (1 - \frac{1}{\vec{tr}(k)})kn$ and so

$$\vec{\delta}(k) \leq 1 - \frac{1}{\vec{tr}(k)} < 1 - \frac{1}{4^k}.$$

Since $\vec{\delta}(3) = 5/6 = 1-1/6 = 1-1/\vec{tr}(3)$, it was conceivable that $\vec{\delta}(k) = 1-1/\vec{tr}(k)$ for all k. However, Theorem 1.1 shows that $\vec{\delta}(4) = 11/12$, whereas $\vec{tr}(4) = 16$, which means that Theorem 1.1 does not follow directly from the Hajnal–Szemerédi theorem.

3. Linear programming and fractional tilings.

3.1. Linear programming. Let H be a k-uniform hypergraph. A *matching* in H is a collection of vertex disjoint edges in H. A *fractional matching* in H is a function $w : E(H) \to [0,1]$ so that for each $v \in V(H)$, $\sum_{e \ni v} w(e) \leq 1$. The *size* of the fractional matching is $\sum_{e \in E(H)} w(e)$. By definition, the largest fractional matching in H has size at most $|H|/k$. (If it has size exactly $|H|/k$, we say it is *perfect*.) Define $\nu(H)$ and $\nu^*(H)$ to be the size of the largest matching and fractional matching in H, respectively.

A *vertex cover* for H is a set of vertices in H that together contain at least one vertex from each edge in H. A *fractional vertex cover* for H is a function $w : V(H) \to [0,1]$ so that for each $e \in E(H)$, $\sum_{v \in e} w(v) \geq 1$. The *size* of the fractional vertex cover is $\sum_{v \in V(H)} w(v)$. Let $\tau(H)$ and $\tau^*(H)$ be the size of the smallest vertex cover and fractional vertex cover of H, respectively. By the duality theorem of linear programming, we have

$$\nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H).$$

For a pair of graphs or directed graphs G and F, we let $H_F(G)$ be the $|V(F)|$-uniform hypergraph on the vertex set $V(G)$ in which $U \in \binom{V(G)}{|V(F)|}$ is an edge if and only if $G[U]$ contains a copy of F. If G is a graph, we define $H_k(G) := H_{K_k}(G)$, and if G is a directed graph, we set $H_k(G) := H_{\vec{T}_k}(G)$. We set $\nu_F(G) := \nu(H_F(G))$ and $\nu_k(G) := \nu(H_k(G))$. We define $\nu^*_F(G)$, $\tau^*_F(G)$, $\nu_F(G)$, $\nu^*_k(G)$, $\tau^*_k(G)$, and $\nu_k(G)$ analogously.

A *fractional F-tiling of G* is a weight function on the copies of F in G that corresponds to a fractional matching in $H_F(G)$, i.e., for every vertex $v \in V(G)$, the

[2]Using the `nauty` and `Traces` software package [13], we determined that there are 43 tournaments on 12 vertices which do not have a perfect $\vec{T}_4$-tiling. These tournaments are listed in Appendix 7. Later, Lidický [11] was able to use this list to determine that every tournament on 16 vertices has a perfect $\vec{T}_4$-tiling.

sum of the weights on the copies of F that contain v is at most one. It is a *perfect fractional F-tiling of* G if the sum of the weights is equal to $|V(G)|/|V(F)|$. We call a weight function on the vertices of G a *fractional F-cover* if the weight function is a vertex cover of $H_F(G)$, that is, if the sum of the weights on the vertices of every copy of F in G is at least one. For both a fractional F-tiling of G and a fractional F-cover of G, the *size* of the weight function is defined to be the sum of the weights (i.e., analogous to the notion of the size of a fractional matching and a fractional vertex cover).

Let $\vec{tr}^*(k)$ denote the smallest integer n such that for every $T \in \mathcal{T}_n$ we have $\nu_k^*(T) = n/k$. We clearly have that $\vec{tr}^*(k) \le \vec{tr}(k)$. Also, every tournament T on $n \ge \vec{tr}^*(k)$ vertices satisfies $\nu_k^*(T) = n/k$. Indeed, by induction on n, we may assume that $n > \vec{tr}^*(k)$ and, for each vertex $v \in V(T)$, there is a perfect fractional $\vec{T}_k$-tiling w_v in $T \setminus \{v\}$. Then $w := \frac{1}{n-1}\sum_{v\in V(T)} w_v$ is a perfect fractional $\vec{T}_k$-tiling in T.

3.2. Forcing fractional tilings and bounds on $\vec{tr}^*(k)$. For every $n \ge \vec{tr}^*(k)$, define $\vec{\delta}_n^*(k)$ to be the smallest integer such that every oriented graph on n vertices with $\delta(G) \ge \vec{\delta}_n^*(k)$ has a perfect fractional $\vec{T}_k$-tiling, and let $\vec{\delta}^*(k) := \limsup_n \vec{\delta}_n^*(k)/n$. Let $\vec{\delta}^0(k)$ be the infimum of the set of numbers $\delta \in [0,1]$ such that for every $\gamma > 0$ there exists n_0 such that every oriented graph G on $n \ge n_0$ vertices with $\delta(G) > \delta n$ has a $\vec{T}_k$-tiling of G missing at most γn vertices.

Using our notation, we now rewrite (a slightly weaker[3] version of) Yuster's result [21, Theorem 3.1].

THEOREM 3.1 (Yuster [21]). *For $k \ge 4$, $\vec{\delta}^0(k) \le 1 - \frac{1}{k(2\vec{r}(k-1)-2)+2} \le 1 - 2^{-(k+\log k)}$.*

Later in this section we prove the following bounds on $\vec{\delta}^0(k)$ and $\vec{\delta}^*(k)$ in terms of $\vec{tr}^*(k)$.

THEOREM 3.2. $1 - \frac{1}{\vec{tr}^*(k)-1} < \vec{\delta}^0(k) \le \vec{\delta}^*(k) \le 1 - \frac{1}{\vec{tr}^*(k)}$.

We also obtain the following bounds on $\vec{tr}^*(k)$.

THEOREM 3.3. *For all $k \ge 3$,*

$$\max\left\{2\vec{r}(k-1), \frac{k}{k-2}\left(\vec{r}(k)-2\right)\right\} \le \vec{tr}^*(k) \le k(2\vec{r}(k-1)-k+1).$$

Note that the upper bound in Theorem 3.3 together with Theorem 3.2 yields a slight strengthening of Theorem 3.1; they also can be combined with an absorbing result (Lemma 4.3) to give Theorem 1.2 (see section 4.2). Theorems 3.2 and 3.3 will also be applied in the proof of Theorem 1.1.

We now prove Theorem 3.2.

Proof of Theorem 3.2. Let G be an oriented graph on n vertices with $\delta(G) \ge (1 - \frac{1}{\vec{tr}^*(k)})n$. Blow up each vertex of G to a set of size $\vec{tr}^*(k)$ and call the resulting

[3]There are three differences to note. First, we ignore the case in which $k = 2$ and $k = 3$, which Yuster considers. Second, Yuster proves that one can almost tile an oriented graph that meets the minimum degree condition with the blow-up of $\vec{T}_k$, but with the regularity lemma, this version of the theorem implies the original version. Third, Yuster writes the minimum degree condition in terms of the function $f^*(k)$ which is defined to be the smallest integer m such that every tournament on at least m vertices has the property that every vertex is contained in a copy of $\vec{T}_k$, but it is not hard to see that $f^*(k) = 2\vec{r}(k-1)$ (see Example 3.5).

oriented graph G'. By the Hajnal–Szemerédi theorem, the simple graph underlying G' has a perfect $K_{\vec{tr}^*(k)}$-tiling. Note that each $K_{\vec{tr}^*(k)}$ has a perfect fractional $\vec{T}_k$-tiling in G'. Hence G' has a perfect fractional $\vec{T}_k$-tiling and so does G. So we have established that $\vec{\delta^*}(k) \le 1 - \frac{1}{\vec{tr}^*(k)}$.

Assume $\vec{\delta^0}(k) \le 1 - \frac{1}{\vec{tr}^*(k)-1}$. Let T be a tournament on $\vec{tr}^*(k) - 1$ vertices that does not have a perfect fractional $\vec{T}_k$-tiling, i.e., $\nu_k^*(T) < |T|/k$. Let $\gamma := \frac{|T|/k - \nu_k^*(T)}{|T|/k}$ and note that $\gamma > 0$. For s sufficiently large, blow up each of the vertices of T into a set of s vertices to form an oriented graph G on $n = s \cdot (\vec{tr}^*(k) - 1)$ vertices. Since $\delta(G)/n = 1 - s/n = 1 - 1/(\vec{tr}^*(k) - 1) \ge \vec{\delta^0}(k)$, and n is sufficiently large, we can assume that there exists a $\vec{T}_k$-tiling $\mathcal{T}$ of G that covers all but at most $0.9\gamma n$ vertices. Because every $\vec{T}_k$ in G corresponds to a $\vec{T}_k$ in T, we can create a fractional $\vec{T}_k$-tiling of T by giving each $\vec{T}_k$ in T weight equal to the number of times a $\vec{T}_k$ that corresponds to it appears in $\mathcal{T}$ divided by s. This fractional $\vec{T}_k$-tiling of T has size

$$\frac{|\mathcal{T}|}{s} \ge \frac{(1 - 0.9\gamma)\, n}{ks} = (1 - 0.9\gamma)\,\frac{|T|}{k} > (1 - \gamma)\,\frac{|T|}{k} = \nu_k^*(T),$$

a contradiction. So, we have established that $\vec{\delta^0}(k) > 1 - \frac{1}{\vec{tr}^*(k)-1}$.

To complete the proof, we need to show that $\vec{\delta^0}(k) \le \vec{\delta^*}(k)$. This can be shown by following a standard application of Szemerédi's regularity lemma.[4] Since the argument is standard we only sketch the proof. It suffices to show that given any $\delta > \vec{\delta^*}(k)$ and any $\gamma > 0$, there exists n_0 such that every oriented graph G on $n \ge n_0$ vertices with $\delta(G) > \delta n$ has a $\vec{T}_k$-tiling missing at most γn vertices.

Let G be such an oriented graph. Applying the regularity lemma one can obtain an oriented spanning subgraph R' of the so-called reduced digraph R of G where $\delta(R') > \vec{\delta^*}(k)|R'|$. Thus, (as R' is sufficiently large) R' contains a perfect fractional $\vec{T}_k$-tiling. Using this fractional tiling as a framework, the counting lemma associated with the regularity lemma now ensures that G contains a $\vec{T}_k$-tiling missing at most γn vertices. ∎

3.3. Proof of Theorem 3.3. The following example gives a lower bound on $\vec{\delta^0}(k)$, which together with Theorem 3.2 gives a lower bound on $\vec{tr}^*(k)$.

EXAMPLE 3.4. *Let $k \ge 3$. For every $n \ge \vec{r}(k)$ and $0 < \gamma < 1$, there exists an oriented graph G on n vertices with*

$$\delta(G) \ge \left\lfloor \left(1 - \frac{k-2}{k\,(\vec{r}(k) - 2)}\right) n - \frac{2\gamma n + k}{k(\vec{r}(k) - 2)} \right\rfloor,$$

such that no $\vec{T}_k$-tiling covers more than $(1-\gamma)n$ vertices of G. In particular, this implies that $\vec{\delta^0}(k) \ge 1 - \frac{k-2}{k(\vec{r}(k)-2)}$, which implies $\vec{tr}^*(k) \ge \frac{k}{k-2}\,(\vec{r}(k) - 2)$ by Theorem 3.2.

Proof. Take the largest tournament which does not contain $\vec{T}_k$; note that it has exactly $\vec{r}(k) - 1$ vertices. For $\gamma > 0$, blow up one of the vertices to a set X of size $\lfloor (1-\gamma)2n/k \rfloor$ and inside the set add all possible edges (oriented arbitrarily). Blow up the other $\vec{r}(k) - 2$ parts to independent sets of size either the floor or ceiling of

$$(n - |X|) \cdot \frac{1}{\vec{r}(k) - 2} \le \frac{(k-2)n}{k\,(\vec{r}(k) - 2)} + \frac{2\gamma n + k}{k(\vec{r}(k) - 2)},$$

[4]It is also possible to establish this fact without appealing to the regularity lemma, e.g., see [1].

while ensuring that the resulting oriented graph G has n vertices. Note that every $\vec{T}_k$ must use at least two vertices from X, so there is only space for at most $(1-\gamma)n/k$ vertex disjoint copies of $\vec{T}_k$ in G. $\square$

The next example gives a different lower bound on $\vec{tr}^*(k)$, which together with Example 3.4 implies the lower bound in Theorem 3.3.

EXAMPLE 3.5. *For every $k \geq 3$, $\vec{tr}^*(k) \geq 2\vec{r}(k-1)$.*

Proof. To see that $\vec{tr}^*(k) \geq 2\vec{r}(k-1)$, consider a tournament T on $n = 2\vec{r}(k-1)-1$ vertices in which there exists a vertex $u \in V(T)$ such that $|N^+(u)| = |N^-(u)| = (n-1)/2 = \vec{r}(k-1)-1$; both $N^+(u)$ and $N^-(u)$ induce a tournament on $\vec{r}(k-1)-1$ vertices that does not contain a $\vec{T}_{k-1}$; all of the edges between $N^+(u)$ and $N^-(u)$ are directed from $N^+(u)$ to $N^-(u)$. This ensures that T does not contain a transitive tournament that contains u and elements from both $N^+(u)$ and $N^-(u)$. Thus, u is not contained in a $\vec{T}_k$; this immediately implies T does not have a perfect fractional $\vec{T}_k$-tiling. $\square$

To prove the upper bound of Theorem 3.3, we first collect together some useful observations.

For a hypergraph H and for every $v \in V(H)$, we let $H(v)$ be the *link graph of* v, i.e., $H(v)$ is the hypergraph with vertex set $V(H)$ and edge set $\{e \setminus \{v\} : e \in E(H) \text{ and } v \in e\}$. The following lemma is well-known. We provide a proof for completeness.

LEMMA 3.6. *If H is a k-uniform hypergraph on n vertices and, for every $v \in V(G)$, $\nu^*(H(v)) \geq n/k$, then $\nu^*(H) = n/k$.*

Proof. Suppose that $\nu^*(H(v)) \geq n/k$ for every $v \in V(G)$, and $\nu^*(H) < n/k$. In a fractional matching of H of size $\nu^*(H)$, there must exist a vertex v in which the sum of the weights on the edges incident to v is strictly less than 1. By the complementary slackness theorem from linear programming, this implies that if w is a fractional vertex cover of H of size $\tau^*(H) = \nu^*(H)$, then $w(v) = 0$. This means that w is a fractional vertex cover of $H(v)$, so

$$\nu^*(H(v)) = \tau^*(H(v)) \leq \tau^*(H) = \nu^*(H) < n/k,$$

a contradiction. $\square$

Let G and F either be a pair of graphs or a pair of directed graphs such that $|G| = n$ and $|F| = k$ and let $H := H_F(G)$. For a vertex $v \in V(G)$, a weight function w on the $(k-1)$-subsets of $V(G)$ is a *v-extendable fractional F-tiling of size r* if it corresponds to a fractional matching of size r in the hypergraph $H(v)$. We have the following corollary to Lemma 3.6.

COROLLARY 3.7. *Let G and F either be a pair of graphs or a pair of directed graphs such that $|G| = n$ and $|F| = k$. If, for every $v \in V(G)$, there exists a v-extendable fractional F-tiling of size at least n/k, then there exists a perfect fractional F-tiling of G.*

Proof. This follows from Lemma 3.6 if we consider the hypergraph $H_F(G)$. $\square$

We now prove the upper bound in Theorem 3.3.

LEMMA 3.8. *For $k \geq 3$, $\vec{tr}^*(k) \leq k(2\vec{r}(k-1)-k+1)$.*

Proof. Let T be a tournament on $n := k(2\vec{r}(k-1) - k + 1)$ vertices. For an arbitrary $v \in V(T)$, we aim to prove that there exists a v-extendable fractional $\vec{T}_k$-tiling of size at least n/k. By Corollary 3.7, this will then prove the lemma. To do this, we first prove the following claim.

CLAIM 3.8.1. *If S is a tournament on $s \geq \vec{r}(k-1)$ vertices, then $\nu^*_{k-1}(S) \geq \frac{s-(\vec{r}(k-1)-k+1)}{k-1}$.*

Proof. Let w be a fractional $\vec{T}_{k-1}$-cover of S of size $\tau^*_{k-1}(S) = \nu^*_{k-1}(S)$ and let $v_1, \dots, v_s$ be an ordering of $V(S)$ such that $w(v_1) \leq w(v_2) \leq \cdots \leq w(v_s)$. Note that $S[\{v_1, \dots, v_{\vec{r}(k-1)}\}]$ contains at least one $\vec{T}_{k-1}$, so $\sum_{i=1}^{\vec{r}(k-1)} w(v_i) \geq 1$ and $w(v_{\vec{r}(k-1)}) \geq \frac{1}{k-1}$. Therefore,

$$\tau^*_{k-1}(S) = \sum_{i=1}^{\vec{r}(k-1)} w(v_i) + \sum_{i=\vec{r}(k-1)+1}^{s} w(v_i) \geq 1 + \frac{s - \vec{r}(k-1)}{k-1} = \frac{s - (\vec{r}(k-1) - k + 1)}{k-1}.$$

$\square$

Recall that

$$2\vec{r}(k-1) - k + 1 = \frac{n}{k}. \tag{1}$$

Let $\nu^{*+} := \nu^*_{k-1}(T[N^+(v)])$ and $\nu^{*-} := \nu^*_{k-1}(T[N^-(v)])$. Note that v forms a copy of $\vec{T}_k$ with any copy of $\vec{T}_{k-1}$ in $T[N^+(v)]$ or $T[N^-(v)]$. In particular, a lower bound on $\nu^{*+} + \nu^{*-}$ gives a lower bound on the size of the largest v-extendable fractional $\vec{T}_k$-tiling.

Suppose that $d^+_T(v) \geq d^-_T(v)$. If $d^-_T(v) \leq \vec{r}(k-1) - 1$, then $d^+_T(v) \geq n - \vec{r}(k-1)$. So, by (1) and the claim,

$$\nu^{*+} \geq \frac{d^+_T(v) - (\vec{r}(k-1) - k + 1)}{k-1} \geq \frac{n - (2\vec{r}(k-1) - k + 1)}{k-1} = \frac{n}{k}.$$

If $d^-_T(v) \geq \vec{r}(k-1)$, then by the claim, (1), the fact that $k \geq 3$, and the fact that $d^+_T(v) + d^-_T(v) = n - 1$ we have

$$\nu^{*+} + \nu^{*-} \geq \frac{d^+_T(v) + d^-_T(v) - 2(\vec{r}(k-1) - k + 1)}{k-1} \geq \frac{n - (2\vec{r}(k-1) - k + 1)}{k-1} = \frac{n}{k}.$$

An analogous argument applies if $d^-_T(v) \geq d^+_T(v)$. So there exists a v-extendable fractional $\vec{T}_k$-tiling of size at least n/k. $\square$

3.4. Remarks. Note that Example 3.5 and Theorem 3.2 together imply that $\vec{\delta^0}(k) \geq 1 - \frac{1}{2\vec{r}(k-1)-1}$. If it can be shown that the lower bound on $\vec{tr}^*(k)$ from Example 3.4 is also an upper bound, i.e., $\vec{tr}^*(k) = \frac{k}{k-2}(\vec{r}(k) - 2)$ (which is true for $k = 3$ and $k = 4$), then we have $2\vec{r}(k-1) \leq \frac{k}{k-2}(\vec{r}(k) - 2)$, or

$$\vec{r}(k) \geq \frac{2(k-2)}{k} \cdot \vec{r}(k-1) + 2,$$

which would imply that $\vec{r}(k) \geq (2 - o(1))^k$, which almost matches the Erdős–Moser bound of $\vec{r}(k) \leq 2^{k-1}$. In fact, even proving that $\vec{tr}^*(k) \leq (\sqrt{2} - c)\vec{r}(k)$ for some absolute constant $c > 0$ would improve the best known lower bound on $\vec{r}(k)$. It is also worthwhile to note that $\vec{r}(k)$ provides a lower bound on the classical Ramsey number $R(k,k)$. Indeed, let T be a tournament on $n := \vec{r}(k) - 1$ vertices which contains no copy of $\vec{T}_k$ and let $v_1, \dots, v_n$ be an ordering of $V(T)$. Let G be the graph on $V(T)$ where for all $1 \le i < j \le n$, $\{v_i, v_j\} \in E(G)$ if and only if $(v_i, v_j) \in E(T)$. Since T has no copy of $\vec{T}_k$, G has no clique or independent set of order k and thus $R(k,k) \ge \vec{r}(k) - 1$. Therefore, it is possible that a substantial improvement to the upper bound on $\vec{tr}^*(k)$ could give an improvement on the best known lower bound for the diagonal Ramsey numbers.

Note that when $n \ge \frac{k}{k-2}\left(\vec{r}(k) - 2\right)$,

$$\frac{n - (\vec{r}(k) - 2)}{2} \ge \frac{n - \frac{k-2}{k} \cdot n}{2} = \frac{n}{k}.$$

A way one might attempt to prove that $\vec{tr}^*(k) = \frac{k}{k-2}\left(\vec{r}(k) - 2\right)$ would be to first prove that equality holds in the following.

EXAMPLE 3.9. *For $k \ge 3$, if $\vec{r}(k) \le n \le \frac{k}{k-2}\left(\vec{r}(k) - 2\right)$, then*

$$\min_{T \in \mathcal{T}_n} \{\nu^*(T)\} \le \frac{n - (\vec{r}(k) - 2)}{2}.$$

Proof. Construct a tournament T on n vertices by starting with a tournament on $\vec{r}(k) - 1$ vertices that does not contain a $\vec{T}_k$ and then blow up one of the vertices to a set X of size $n - (\vec{r}(k) - 2)$. Then place edges between all vertices in X and orient them arbitrarily. Because every $\vec{T}_k$ has at least two vertices in X, we can cover all of the copies of $\vec{T}_k$ in T by assigning weight $1/2$ to the vertices in X and 0 to the vertices in $V(T) \setminus X$. Therefore,

$$\nu^*(T) = \tau^*(T) \le \frac{|X|}{2} = \frac{n - (\vec{r}(k) - 2)}{2}. \qquad \square$$

Example 3.9 is quite similar to Example 3.4. We have verified that, when $\vec{r}(k) \le n \le \frac{k}{k-2}\left(\vec{r}(k) - 2\right)$, equality holds in Example 3.9 when k is either 3 or 4. We have no evidence that equality holds when $k \ge 5$, and in light of the discussion above, it is, if true, likely extremely challenging to prove!

4. The absorbing method and the proof of Theorem 1.2.

4.1. Absorbing. We will apply the absorbing method of Rödl, Ruciński, and Szemerédi (see, e.g., [15]). The basic idea of the method is to prove that a randomly constructed small set can serve as an "absorber," i.e., we prove that there exists a small set that has the property that if, after removing this set from the graph, we can almost tile what is left of the oriented graph, then, using the absorbing set, we can extend this partial tiling into a perfect tiling over the entire original oriented graph.

To prove that our absorbing sets exist, we will use the following lemma, which follows immediately from a lemma of Lo and Markström [12, Lemma 1.1]. Here we write $0 < \alpha \ll \eta < 1$ to mean that α is chosen to be sufficiently small compared to η so that all constraints in the proof of the lemma hold.

LEMMA 4.1. *For every $k \ge 3$, $i \ge 1$, and $0 < \alpha \ll \eta < 1$, there exists n_0 such that for every directed graph G on $n \ge n_0$ vertices the following holds. If, for every $x, y \in V(G)$, there are at least ηn^{ik-1} sets $L \subseteq V(G)$ such that $|L| = ik - 1$ and both $G[L \cup \{x\}]$ and $G[L \cup \{y\}]$ contain perfect $\vec{T}_k$-tilings, then there exists $A \subseteq V(G)$ such that*

- *$|A| \leq \alpha n$; $|A|$ is divisible by k; and*
- *for every $W \subseteq V(G) \setminus A$, such that $|W| \leq \alpha^2 n$ and $|W|$ is divisible by k, we have that $G[A \cup W]$ has a perfect $\vec{T}_k$-tiling.*

Let $k \geq 3$ and $i \geq 1$. Define $\mathcal{A}(k,i)$ to be the set of all $\beta > 0$ with the following property: there exists $\eta > 0$ and $n_0 \in \mathbb{N}$ so that for each $n \geq n_0$, every n-vertex oriented graph G with $\delta(G) \geq \beta n$, and any pair $x, y \in V(G)$, there are at least ηn^{ik-1} sets $L \subseteq V(G)$ such that $|L| = ik-1$ and both $G[L \cup \{x\}]$ and $G[L \cup \{y\}]$ contain perfect $\vec{T}_k$-tilings. Let $A(k,i)$ be the infinimum of $\mathcal{A}(k,i)$. Write $A(k) := \inf_{i \geq 1} A(k,i)$. We call $A(k)$ the *absorbing threshold* for $\vec{T}_k$-tiling.

We will make use of the following simple fact.

FACT 4.2. *For every r, s, and c such that $1 \leq s \leq r$, and $|c| < 1/r$, the following holds. If G is a graph or oriented graph on n vertices and $\delta(G) \geq (\frac{r-1}{r} + c)n$, then for every $U \subseteq V(G)$ such that $|U| \geq \frac{s}{r} n$ we have $\delta(G[U]) \geq (\frac{s-1}{s} + c \cdot \frac{r}{s})|U|$.*

Proof. Because $n \leq \frac{r}{s}|U|$, we have that $\delta(G[U])$ is at least

$$|U| - (n - \delta(G)) \geq |U| - \left(\frac{1}{r} - c\right) n \geq |U| - \left(\frac{1}{r} - c\right) \frac{r}{s}|U| = \left(\frac{s-1}{s} + c \cdot \frac{r}{s}\right) |U|.$$

∎

LEMMA 4.3. *For all $k \geq 3$, $A(k,1) \leq 1 - \frac{1}{4\vec{r}(k-1)-2}$.*

Proof. Let $0 < \eta \ll \varepsilon \ll 1/k$, let n be sufficiently large, and let G be an oriented graph on n vertices with $\delta(G) \geq \left(1 - \frac{1}{4\vec{r}(k-1)-2} + \varepsilon\right) n$. Let $x, y \in V(G)$ and set $U := N(x) \cap N(y)$, $r := 4\vec{r}(k-1) - 2$, and $s := 4\vec{r}(k-1) - 4$. Since $|U| \geq sn/r$, Fact 4.2 (with $c = \varepsilon$) implies that

$$\delta(G[U]) \geq \left(\frac{s-1}{s} + \varepsilon \cdot \frac{r}{s}\right) |U|.$$

So by supersaturation[5] there exist at least ηn^{s+1} tournaments T on $(s+1)$ vertices in $G[U]$. Since $s + 1 = 4\vec{r}(k-1) - 3$, by the pigeonhole principle, for every such tournament T, there exists a subtournament of size at least $\vec{r}(k-1)$ in one of the four sets: $N^+(x) \cap N^+(y)$, $N^+(x) \cap N^-(y)$, $N^-(x) \cap N^+(y)$, and $N^-(x) \cap N^-(y)$, which partition U. This, in turn, implies that there exists $L \subseteq V(T)$ such that $G[L \cup \{x\}]$ and $G[L \cup \{y\}]$ are $\vec{T}_k$. Therefore, we have at least ηn^{k-1} of the desired sets.

The choice of $\varepsilon > 0$ can be made arbitrarily small, and thus we obtain that $A(k,1) \leq 1 - \frac{1}{4\vec{r}(k-1)-2}$. ∎

LEMMA 4.4. *For every $k \geq 3$, $i \geq 1$, and $\varepsilon > 0$, there exists n_0 such that for every $n \geq n_0$ that is divisible by k the following holds. If G is an oriented graph on n vertices and*

$$\delta(G) \geq \max\left\{\vec{\delta^0}(k) + \varepsilon, A(k,i) + \varepsilon\right\} n,$$

then G has a perfect $\vec{T}_k$-tiling.

Proof. Let $0 < \alpha \ll \eta \ll \varepsilon, 1/k, 1/i$. Let G be a sufficiently large oriented graph as in the statement of the lemma.

[5]That is, as G has minimum degree significantly above the threshold for containing a tournament on $s+1$ vertices.

By the degree condition we may apply Lemma 4.1 to get a set A such that $|A| \leq \alpha n$, $|A|$ is divisible by k, and, for every $W \subseteq V(G) \setminus A$ such that $|W| \leq \alpha^2 n$ and $|W|$ is divisible by k, the oriented graph $G[A \cup W]$ has a perfect $\vec{T}_k$-tiling. Since n is sufficiently large and $\delta(G - A) \geq (\vec{\delta^0}(k) + \varepsilon/2)|G - A|$, we can tile $G - A$ so that if W is the set of uncovered vertices, then $|W| \leq \alpha^2 n$. Since then $G[A \cup W]$ has a perfect $\vec{T}_k$-tiling, we obtain a perfect $\vec{T}_k$-tiling of G. ∎

4.2. Proof of Theorem 1.2. With the absorbing lemma at hand, it is now straightforward to deduce Theorem 1.2 from our previous results.

Proof of Theorem 1.2. From Theorem 3.2 and Lemma 3.8 we have that $\vec{\delta^0}(k) \leq 1 - \frac{1}{k(2\vec{r}(k-1)-k+1)}$. Since $k(2\vec{r}(k-1) - k + 1) \geq k \cdot \vec{r}(k-1) \geq 4\vec{r}(k-1) - 2$, the first part of Theorem 1.2 then follows from Lemmas 4.3 and 4.4.

The second part of the theorem follows by the inequality in the statement of Theorem 3.1. ∎

5. $\vec{T}_4$-tiling—Proof of Theorem 1.1. Note that $\vec{r}(4) = 8$. Example 3.4 with $(\gamma = 1/n)$ implies the second part of the theorem. For the first part of the theorem, we will show that $\vec{\delta^0}(4) \leq \frac{11}{12}$ (Proposition 5.1) and $A(4, 2) \leq \frac{11}{12}$ (Corollary 5.3), which together with Lemma 4.4 will complete the result.

Note that we sometimes call $\vec{T}_3$ the *transitive triangle.*

PROPOSITION 5.1. $\vec{tr}^*(4) = 12$ *and* $\vec{\delta^0}(4) = \frac{11}{12}$.

Proof. As $\vec{r}(4) = 8$, the lower bound in Theorem 3.3 gives $\vec{tr}^*(4) \geq 12$ and Example 3.4 gives $\vec{\delta^0}(4) \geq \frac{11}{12}$. Thus, it suffices to show that $\vec{tr}^*(4) \leq 12$ as together with Theorem 3.2 this implies $\vec{\delta^0}(4) \leq \frac{11}{12}$. Let T be a tournament on 12 vertices. It suffices, by Corollary 3.7, to show that for every $v \in V(T)$ there exists a v-extendable fractional $\vec{T}_4$-tiling of size at least 3. Recall that $\vec{tr}(3) = 6$, so every tournament on $3k \geq 6$ vertices has a perfect $\vec{T}_3$-tiling.

Let $v \in V(T)$ and suppose without loss of generality that $d^+(v) \geq d^-(v)$. If $d^+(v) \geq 9$, then we have three disjoint $\vec{T}_3$'s in $N^+(v)$ and we are done. If $d^-(v) \geq 4$, then since $d^+(v) \geq 6$, we have two disjoint $\vec{T}_3$'s in $N^+(v)$ and one $\vec{T}_3$ in $N^-(v)$. So the only case left to deal with is when $d^-(v) = 3$ and $d^+(v) = 8$. We would be done as before if there exists a $\vec{T}_4$ that contains v and has exactly one vertex in $N^-(v)$ and two vertices in $N^+(v)$, so assume such a $\vec{T}_4$ does not exist. This implies that

$$\text{every vertex in } N^-(v) \text{ has at most one out-neighbor in } N^+(v). \tag{2}$$

In this case we find a perfect $\vec{T}_4$-tiling of T directly. By (2), there exists a $\vec{T}_4$, say, F, such that F has two vertices in both $N^-(v)$ and $N^+(v)$. Let F_1 and F_2 be two disjoint transitive triangles contained in $N^+(v) \setminus V(F)$ and let u be the vertex in $N^-(v) \setminus V(F)$. By (2), for either F_1 or F_2, say, F_1, we have that $\{u\} \cup F_1$ induces a $\vec{T}_4$ (since u has only in-neighbors in F_1). Then, F, $T[\{u\} \cup F_1]$, and $T[\{v\} \cup F_2]$ form the desired perfect $\vec{T}_4$-tiling of the tournament T. ∎

To complete the proof of Theorem 1.1, we show that $A(4, 2) \leq \frac{11}{12}$. Let G be an oriented graph on n vertices and $\delta(G) \geq \left(\frac{11}{12} + \varepsilon\right) n$. It is sufficient to show that, for every pair of distinct vertices x and y in G, there are at least $\Omega(n^7)$ sets L, each of order 7, such that both $G[L \cup \{x\}]$ and $G[L \cup \{y\}]$ contain two disjoint copies of $\vec{T}_4$. At a high-level, we achieve this by noting that, by the minimum total degree condition, Fact 4.2, and supersaturation, there are $\Omega(n^{11})$ tournaments on 11 vertices in $G[N(x) \cap N(y)]$. The following lemma then implies that there are $\Omega(n^7)$ of the desired sets L. We provide the details of this argument in our proof of Corollary 5.3, which appears after the proof of Lemma 5.2.

LEMMA 5.2. *Let G be an oriented graph and let $x, y \in V(G)$ and $T \subseteq V(G) \setminus \{x, y\}$. If $\{x\} \cup T$ and $\{y\} \cup T$ each induce a tournament on 12 vertices, then there exists $Z \subseteq T$ such that $|Z| = 7$ and $G[\{x\} \cup Z]$ and $G[\{y\} \cup Z]$ both contain a perfect $\vec{T}_4$-tiling.*

Proof. For clarity, we will write $u \to v$ if the edge $uv \in E(G)$ is directed from u to v. Let K be the tournament induced in G by the vertex set T. Call $Z \subseteq T$ a *linking set* if $|Z| \in \{3, 7\}$ and $G[\{x\} \cup Z]$ and $G[\{y\} \cup Z]$ both have a perfect $\vec{T}_4$-tiling. Suppose Z is a linking set. If $|Z| = 7$, then we clearly satisfy the conclusion of the lemma. If $|Z| = 3$, then, because $|T \setminus Z| = 8 = \vec{r}(4)$, there exists $T' \subseteq T \setminus Z$ such that $G[T']$ is a $\vec{T}_4$, so with $Z \cup T'$ playing the role of Z we satisfy the conclusion of the lemma. Suppose, for a contradiction, that no linking set exists.

Let $N^{\sigma_x, \sigma_y} := N_T^{\sigma_x}(x) \cap N_T^{\sigma_y}(y)$ for $\sigma_x, \sigma_y \in \{+, -\}$, and let

$$\mathcal{P} := \{N^{+,+}, N^{+,-}, N^{-,+}, N^{-,-}\},$$

and note that $\mathcal{P}$ is a partition of $V(T)$.

Let $<$ be the partial order of $\mathcal{P}$ given by $N^{-,-} < N^{-,+} < N^{+,+}$ and $N^{-,-} < N^{+,-} < N^{+,+}$, and let $uw \in E(K)$. We say that *uw violates the partial order* if $u \in U$ and $w \in W$ for distinct sets $U, W \in \mathcal{P}$ and either

- U and W are incomparable or
- $U < W$ and $w \to u$.

Otherwise, we say that *uw satisfies the partial order*. Note that, for every edge $uw \in E(K)$,

(3) both xuw and yuw are transitive triangles $\iff$ uw satisfies the partial order.

CLAIM 5.2.1. *Every transitive triangle abc in K contains at least one edge that violates the partial order.*

Proof. Otherwise, by (3), both $\{a, b, c, x\}$ and $\{a, b, c, y\}$ induce copies of $\vec{T}_4$, so $\{a, b, c\}$ is a linking set. ∎

This immediately implies the following.

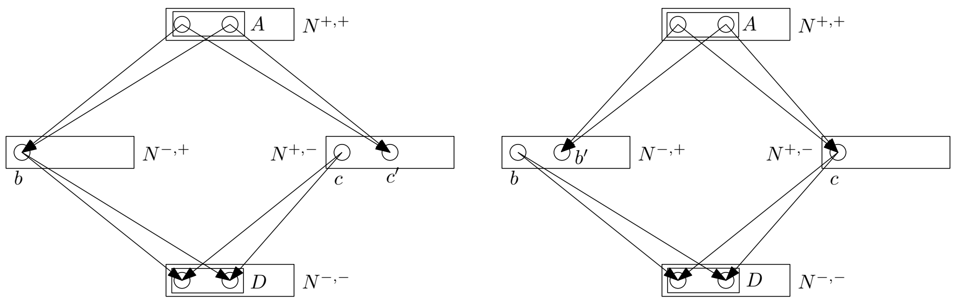
CLAIM 5.2.2. *For every pair of distinct sets $U, W \in \mathcal{P}$ that are comparable, the edges between U and W that satisfy the partial order form a matching.*

Because every tournament on four vertices contains a transitive triangle, and edges that violate the partial order must intersect two sets in $\mathcal{P}$, Claim 5.2.1 implies the following.

CLAIM 5.2.3. *For every $U \in \mathcal{P}$, we have $|U| \leq 3$. In particular, exactly one set in $\mathcal{P}$ has order 2 and the other sets in $\mathcal{P}$ each have order 3. Furthermore, if $U \in \mathcal{P}$ and $|U| = 3$, then U induces a cyclic triangle.*

Without loss of generality, suppose $|N^{-,-}| + |N^{-,+}| \leq |N^{+,+}| + |N^{+,-}|$, so either $N^{-,-}$ or $N^{-,+}$ is the set in $\mathcal{P}$ of order 2.

CLAIM 5.2.4. *There exists $b \in N^{-,+}$, $c \in N^{+,-}$, and $D \subseteq N^{-,-}$ such that $|D| = 2$, and, for every $d \in D$, we have $b \to d$ and $c \to d$, i.e., all of the edges between b and D and all of the edges between c and D violate the partial order. Moreover, $D \cup \{b, c\}$ induces a copy of $\vec{T}_4$ in T.*



(a) $c' \in N^{+,-} \setminus \{c\}$ (b) $b' \in N^{-,+} \setminus \{b\}$

FIG. 1. *Forbidden pairs from the proof of Claim* 5.2.5.

Proof. Recall that $|N^{-,-}| \in \{2, 3\}$. If $|N^{-,-}| = 2$, then let $D := N^{-,-}$. Since $|D| < |N^{-,+}| = |N^{+,-}| = 3$, Claim 5.2.2 implies that there exists $b \in N^{-,+}$ and $c \in N^{+,-}$ such that, for every $d \in D$, we have $b \to d$ and $c \to d$.

Suppose $|N^{-,-}| = 3$ and let $b \in N^{-,+}$. By Claim 5.2.2, there exists $D \subseteq N^{-,-}$ such that $|D| = 2$ and $b \to d$ for every $d \in D$. Since $|D| < |N^{+,-}|$, Claim 5.2.2 implies that there exists $c \in N^{+,-}$ such that $c \to d$ for every $d \in D$. ∎

CLAIM 5.2.5. *Suppose $b \in N^{-,+}$, $c \in N^{+,-}$, and $D \subseteq N^{-,-}$ such that $|D| = 2$, and, for every $d \in D$, we have $b \to d$ and $c \to d$.*

(C1) *There exists $a' \in N^{+,+}$ such that $b \to a'$ and $c \to a'$.*

(C2) *For every $v \in N^{-,+} \cup N^{+,-}$ such that $v \notin \{b, c\}$, there exists $a \in N^{+,+}$ such that $v \to a$ and there exists $d \in D$ such that $d \to v$.*

Proof. Call (A, c') a *forbidden pair* if A is a 2-subset of $N^{+,+}$, $c' \in N^{+,-} \setminus \{c\}$, and, for every $a \in A$, we have $a \to b$ and $a \to c'$. Note that no forbidden pairs can exist because if (A, c') is a forbidden pair, then $\{x, c'\} \cup A$, $\{b, c\} \cup D$, $\{y, c\} \cup D$ and $\{b, c'\} \cup A$ each induce a $\vec{T}_4$, so the set $\{b, c, c'\} \cup A \cup D$ is a linking set (see Figure 1(a)). By similar logic, a pair (A, b') where A is a 2-subset of $N^{+,+}$, $b' \in N^{-,+} \setminus \{b\}$ and $a \to b'$ and $a \to c$ for every $a \in A$ cannot exist. Therefore, we also call such a pair (A, b') a *forbidden pair* (see Figure 1(b)).

We will first show that

$$\text{for every } v \in N^{-,+} \cup N^{+,-} \text{ there exists } a \in N^{+,+} \text{ such that } v \to a. \tag{4}$$

Assume the contrary, so $N^-(v) \supseteq N^{+,+}$ for some $v \in N^{-,+} \cup N^{+,-}$. Suppose $v \in N^{-,+}$. If $v \neq b$, then by Claim 5.2.2, there exists $A \subseteq N^-(v) \cap N^-(c) \cap N^{+,+}$ such that $|A| = 2$, and (A, v) is a forbidden pair, a contradiction. If $v = b$, then, by Claim 5.2.2, for every $c' \in N^{+,-} \setminus \{c\}$, there exists $A \subseteq N^-(v) \cap N^-(c') \cap N^{+,+}$ such that $|A| = 2$, and (A, c') is a forbidden pair, a contradiction. Similar logic leads to a contradiction when $v \in N^{+,-}$, so (4) holds.

By (4), there exists $a', a'' \in N^{+,+}$ such that $b \to a'$ and $c \to a''$. To prove (C1), we need to show that $a' = a''$, so assume the contrary. Note that because $|N^{+,-}| = |N^{+,+}| = 3$, Claim 5.2.2 and (4) imply that the edges between $N^{+,-}$ and $N^{+,+}$ that satisfy the partial order form a matching of size 3. Therefore, because $c \to a''$ and $a'' \neq a'$, there exists $c' \in N^{+,-} \setminus \{c\}$ such that $c' \to a'$. Then $(N^{+,+} \setminus \{a'\}, c')$ is a forbidden pair, a contradiction.

Now assume that (C2) does not hold. With (4), this implies that there exists $v \in N^{-,+} \cup N^{+,-}$ such that $v \notin \{b, c\}$, and, for every $d \in D$, we have $v \to d$. If

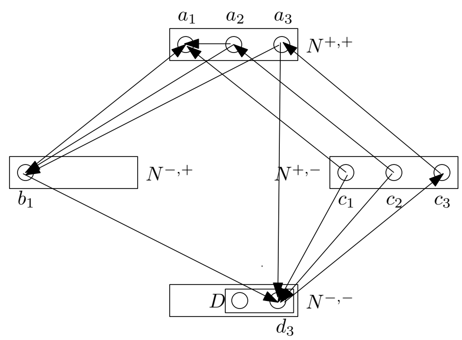


FIG. 2. *The selected vertices at the end of the proof of Lemma* 5.2. *Note that for* $i, j \in [3]$ *and* $i \neq j$ *we have* $a_i \to c_j$.

$v \in N^{-,+}$, then, since $b \to a'$, Claim 5.2.2 implies that $a' \to v$. This, with Claim 5.2.2, violates (C1) with v, c, and D playing the roles of b, c, and D, respectively. Similarly, if $v \in N^{+,-}$, we violate (C1) with b, v, and D playing the roles of b, c, and D, respectively. ∎

We now select vertices in the following order (see Figure 2):

- By Claim 5.2.4, we can select $b_1 \in N^{-,+}$, $c_1 \in N^{+,-}$, and $D \subseteq N^{-,-}$ so that $|D| = 2$ and, for every $d \in D$, we have $c_1 \to d$ and $b_1 \to d$.
- By Claim 5.2.5(C1) we can select $a_1 \in N^{+,+}$ so that $c_1 \to a_1$ and $b_1 \to a_1$.
- By Claim 5.2.3, we can label $\{a_2, a_3\} = N^{+,+} \setminus \{a_1\}$ so that $a_2 \to a_1$.
- By Claims 5.2.2 and 5.2.5(C2), we can label $\{c_2, c_3\} = N^{+,-} \setminus \{c_1\}$ so that $c_2 \to a_2$ and $c_3 \to a_3$.
- By Claim 5.2.5(C2), we can select $d_3 \in D$ such that $d_3 \to c_3$. By Claim 5.2.2, this implies that $c_2 \to d_3$. Furthermore, by Claim 5.2.1 applied to $G[\{a_3, c_3, d_3\}]$, we have $a_3 \to d_3$.

First note that $N^+(a_2) \supseteq \{a_1, b_1, c_1\}$, so both $\{a_1, a_2, b_1\}$ and $\{a_1, a_2, c_1\}$ induce transitive triangles. Since $N^+(y) \supseteq \{a_1, a_2, b_1\}$ and $N^+(x) \supseteq \{a_1, a_2, c_1\}$, both $\{y, a_1, a_2, b_1\}$ and $\{x, a_1, a_2, c_1\}$ induce copies of $\vec{T}_4$. Furthermore, $N^+(a_3) \supseteq \{b_1, c_1, c_2, d_3\}$, and $N^-(d_3) \supseteq \{a_3, b_1, c_1, c_2\}$, so both $\{a_3, c_1, c_2, d_3\}$ and $\{a_3, b_1, c_2, d_3\}$ induce copies of $\vec{T}_4$. Therefore, $\{a_1, a_2, a_3, b_1, c_1, c_2, d_3\}$ is a linking set. This contradiction completes the proof of the lemma. ∎

COROLLARY 5.3. $A(4, 2) \leq \frac{11}{12}$.

Proof. Let $0 < 1/n_0 \ll \eta \ll \varepsilon \ll 1$. Let G be an oriented graph on $n \geq n_0$ vertices with $\delta(G) \geq \left(\frac{11}{12} + \varepsilon\right) n$. Consider any distinct vertices x and y in G. Let $U := N(x) \cap N(y)$ and note that $|U| \geq 2\delta(G) - n \geq (10/12 + 2\varepsilon)n$. By Fact 4.2 (with $r = 12$, $s = 10$, and $c = \varepsilon$), we have that

$$\delta(G[U]) \geq \left(\frac{9}{10} + \frac{12}{10}\varepsilon\right) |U|,$$

so, by supersaturation, there exists at least ηn^{11} tournaments on 11 vertices in $G[U]$. By Lemma 5.2, in every such tournament, there exists a set Z on 7 vertices such that $G[\{x\} \cup Z]$ and $G[\{y\} \cup Z]$ both contains a perfect $\vec{T}_4$-tiling. Since each such set Z is contained in at most n^4 tournaments on 11 vertices in G, there are at least ηn^7 such sets Z. As $\varepsilon > 0$ can be chosen arbitrarily small, $A(4, 2) \leq \frac{11}{12}$, as required. ∎

6. Concluding remarks and open questions. In this paper we have asymptotically determined the minimum degree required to force a perfect $\vec{T}_4$-tiling in an oriented graph (Theorem 1.1). We also obtained bounds for the general perfect $\vec{T}_k$-tiling problem (Theorem 1.2) and the perfect fractional $\vec{T}_k$-tiling problem (Theorem 3.2). In light of Theorem 3.2 it would be interesting to determine whether one can ensure a perfect $\vec{T}_k$-tiling in an oriented graph G of minimum degree $(1-1/\vec{tr}^*(k)+o(1))|G|$.

QUESTION 6.1. *Let $n, k \in \mathbb{N}$, where k divides n and $k \geq 4$. Does every n-vertex graph with*

$$\delta(G) > (1-1/\vec{tr}^*(k)+o(1))n$$

contain a perfect $\vec{T}_k$-tiling?

Note that the $k=4$ case of Question 6.1 is answered in the affirmative by Theorem 1.1. If one can show that, for all $k \geq 5$,

$$A(k) \leq 1-1/\vec{tr}^*(k),$$

then together with Theorem 3.2 and Lemma 4.4 this would positively answer Question 6.1.

For large k, Theorem 3.2 gives rather close upper and lower bounds on the threshold for perfect fractional $\vec{T}_k$-tiling in oriented graphs (recall that $\vec{tr}^*(k)$ grows exponentially with k). We suspect that it is possible one can improve on the lower bound in Theorem 3.2 (perhaps the upper bound is in fact tight).

It would also be interesting to close the bounds on $\vec{tr}^*(k)$ in Theorem 3.3; indeed as discussed in section 3.4 this could even lead to improvements on the lower bounds on $\vec{r}(k)$ and the classical Ramsey numbers $R(k,k)$. It is also natural to seek structural information on $\vec{T}_k$-free tournaments on $\vec{r}(k)-1$ vertices. When $k=3,4,5,6$, the unique $\vec{T}_k$-free tournament on $\vec{r}(k)-1$ vertices is regular (see [16]). This leads to the following question.

QUESTION 6.2. *Let $k \geq 3$. Is every $\vec{T}_k$-free tournament on $\vec{r}(k)-1$ vertices a regular tournament?*

As noted by a referee, it is not even clear that $\vec{r}(k)$ is even for all $k \geq 3$ (a necessary condition for Question 6.2 to have an affirmative answer). So this in itself is an interesting question.

Answering Question 6.2 may also provide insight on the problem (raised in [17]) of determining the minimum *semidegree* that forces an oriented graph to contain a perfect $\vec{T}_k$-tiling. Indeed, given a fixed $k \geq 3$, let $\mathrm{reg}(k)$ denote the size of the largest $\vec{T}_k$-free regular tournament. Construct an oriented graph $G_{n,k}$ as follows. The vertex set of $G_{n,k}$ consists of a set A of $n/k-1$ vertices and a set B of $(1-1/k)n+1$ vertices; $G_{n,k}[A]$ induces a tournament so that for every vertex in this tournament, its in- and outdegrees differ by at most one. Further $G_{n,k}[B]$ is a blow-up of a $\vec{T}_k$-free regular tournament T on $\mathrm{reg}(k)$ vertices where the independent sets in B corresponding to vertices in T are as equally sized as possible. (More generally, we could let $G_{n,k}[B]$ be a $\vec{T}_k$-free oriented graph on $|B|$ vertices having the largest possible minimum semidegree; however, we suspect that such an oriented graph will come from the blow-up of a $\vec{T}_k$-free regular tournament T on $\mathrm{reg}(k)$ vertices.) Finally, add all possible edges between A and B in $G_{n,k}$, oriented to ensure that for every vertex v in $G_{n,k}$, $d^+_{G_{n,k}}(v)$ and $d^-_{G_{n,k}}(v)$ are as close as possible. Notice that every copy of $\vec{T}_k$ in $G_{n,k}$ must use at least one vertex from A; thus as $|A| = n/k-1$, $G_{n,k}$ does not contains a perfect $\vec{T}_k$-tiling. Further, certainly $\delta^0(G_{n,k}) \geq (\frac{1}{2} - \frac{(k-1)}{2k\cdot\mathrm{reg}(k)} - o(1))n$.

Note that $G_{n,k}$ is a generalization of the example given in [17, Proposition 6] (which deals with the case when $k = 3$). Further, in [2] it was proven that $G_{n,3}$ is an extremal example for the minimum semidgree problem for perfect $\vec{T}_3$-tilings. That is, all sufficiently large oriented graphs on n vertices whose minimum semidegree is above that of $G_{n,k}$ contain a perfect $\vec{T}_3$-tiling. Thus, it is natural to ask the following question.

QUESTION 6.3. *Let $k, n \geq 3$ so that k divides n. Does every oriented graph G on n vertices with*

$$\delta^0(G) > \left(\frac{1}{2} - \frac{(k-1)}{2k \cdot \mathrm{reg}(k)} + o(1)\right) n$$

contain a perfect $\vec{T}_k$-tiling?

7. Appendix: Tournaments on 12 vertices that do not have a perfect $\vec{T}_4$-tiling. In Figure 3, we list 43 tournaments on 12 vertices that do not have a perfect $\vec{T}_4$-tiling. This is an exhaustive list (up to isomorphism) of such tournaments. The $66 = \binom{12}{2}$ numbers in each line represent the entries in the upper triangle of the $n \times n$ matrix $(a_{i,j})$, where $a_{i,j} = 1$ if the edge incident to v_i and v_j is directed from v_i to v_j and $a_{i,j} = 0$ otherwise. These entries are listed in the following order:

$$a_{1,2}a_{1,3}\cdots a_{1,12}a_{2,3}a_{2,4}\cdots a_{2,12}\ldots a_{10,11}a_{10,12}a_{11,12}.$$

(This is the default output format for the program `gentourng`, which is a program

```
110011001001111001001111010110101101111101010111101110101100111111
111011111111010101100111010100110010101111001110110110101101111111
111011000001010110110111011010110101011101100111011111011111110111
110011010001110100100111011001101110101110101110011111111111110101
111011100001010110110111001010110101011101100111011111011111110111
110011101111110010100111001010101110011110110101100110101101111111
111100010001101010100110110011111011011111010111001101111110111101
101011000101111000100110101000110110111110101111110111011111110111
110011101001110110010111001100101101011101010111001111111111110111
101011001001111001001110111111110101101101010111101110101100111111
111100101001101011001110101111111100101101100110101110101111110111
101010011001111010000110100101111010111110110111101111111010110111
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110100110001101101001111100101100110011101011110111011111110111101
110101110001101010010111100100110110011101011110110111011111111101
110010101001110011001111110010101101011101111101100110101111110111
101010110011110101111110010100111100101101100110101110101111110111
101010111111110101000110010101111100101101101110100110111110110111
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101010110011110101000110010100111100101101101110100110101111111111
101010110011110101000110010101111100101101100110100110101111111111
101010111001011110001001101010001101101111101011111101110111111110111
101010110011110101000111010100111100111101101110111110101110111101
```

FIG. 3. 43 *tournaments on* 12 *vertices that do not have a perfect* $\vec{T}_4$*-tiling.*

that is distributed with `nauty` and `Traces` [13] that can be used to generate all small tournaments.)

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