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Limit theory for isolated and extreme points in hyperbolic random geometric graphs

Nikolaos Fountoulakis[*][†] Joseph Yukich[‡][§]

Abstract

Given $\alpha \in (0,\infty)$ and $r \in (0,\infty)$, let $\mathcal{D}_{r,\alpha}$ be the disc of radius r in the hyperbolic plane having curvature $-\alpha^2$. Consider the Poisson point process having uniform intensity density on $\mathcal{D}_{R,\alpha}$, with $R = 2\log(n/\nu)$, $n \in \mathbb{N}$, and $\nu < n$ a fixed constant. The points are projected onto $\mathcal{D}_{R,1}$, preserving polar coordinates, yielding a Poisson point process $\mathcal{P}_{\alpha,n}$ on $\mathcal{D}_{R,1}$. The hyperbolic geometric graph $\mathcal{G}_{\alpha,n}$ on $\mathcal{P}_{\alpha,n}$ puts an edge between pairs of points of $\mathcal{P}_{\alpha,n}$ which are distant at most R. This model has been used to express fundamental features of complex networks in terms of an underlying hyperbolic geometry.

For $\alpha \in (1/2,\infty)$ we establish expectation and variance asymptotics as well as asymptotic normality for the number of isolated and extreme points in $\mathcal{G}_{\alpha,n}$ as $n \to \infty$. The limit theory and renormalization for the number of isolated points are highly sensitive on the curvature parameter. In particular, for $\alpha \in (1/2,1)$, the variance is super-linear, for $\alpha = 1$ the variance is linear with a logarithmic correction, whereas for $\alpha \in (1,\infty)$ the variance is linear. The central limit theorem fails for $\alpha \in (1/2,1)$ but it holds for $\alpha \in (1,\infty)$.

Keywords: random geometric graphs; hyperbolic plane; complex networks; central limit theorem.
MSC2020 subject classifications: Primary 05C80, Secondary 05C12; 05C82.
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1 Introduction and main results

1.1 Hyperbolic random geometric graphs

We study in this paper the random geometric graph on the hyperbolic plane H^2_{-1}, as introduced by Krioukov et al. [16]. The standard Poincaré disk representation of H^2_{-1} is the open unit disk $\mathcal{D} := \{(u,v) \in \mathbb{R}^2 \ : \ u^2+v^2 < 1\}$ equipped with the hyperbolic (Riemannian) metric d_H given by $ds^2 = 4\,\frac{du^2+dv^2}{(1-u^2-v^2)^2}$. Recall that the arclength of the

*School of Mathematics, University of Birmingham, United Kingdom. E-mail: n.fountoulakis@bham.ac.uk
†Research partially supported by the Alan Turing Institute, grant EP/N510129/1.
‡Department of Mathematics, Lehigh University, United States of America. E-mail: jey0@lehigh.edu
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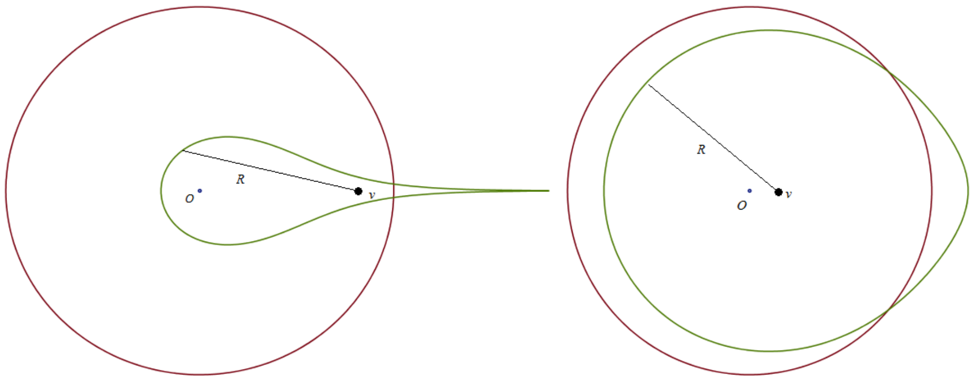

Figure 1: The disc $\mathcal{B}_R(v)$ around the point $v \in \mathcal{D}_R$.

boundary of a disk $\mathcal{D}_r \subset \mathcal{D}$ of radius r and centered at the origin is $2\pi \sinh(r)$, whereas the area of $\mathcal{D}_r$ is $2\pi(\cosh(r) - 1)$.

Given $\nu \in (0, \infty)$ a fixed constant and a natural number $n > \nu$, we let

$$R := 2 \log(n/\nu),$$

i.e., $n = \nu \exp(R/2)$. For every $\alpha \in (0, \infty)$, consider the probability density function

$$\rho_{\alpha,n}(r) := \begin{cases} \alpha \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1} & 0 \le r \le R \\ 0 & \text{otherwise} \end{cases}. \tag{1.1}$$

Let θ be uniformly distributed on $(-\pi, \pi]$. When $\alpha = 1$ the distribution of (r, θ) given by (1.1) is the uniform distribution on $\mathcal{D}_R$ under the metric d_H. For general $\alpha \in (0, \infty)$ Krioukov et al. [16] call this the *quasi-uniform* distribution on $\mathcal{D}_R$, since it arises as the projection of the uniform distribution on a disc of hyperbolic radius R in $H^2_{-\alpha^2}$, the hyperbolic plane having curvature $-\alpha^2$ and equipped with the metric $\frac{4}{\alpha^2} \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2}$.

Denote by $\kappa_{\alpha,n}$ the Borel measure on $\mathcal{D}_R$ given by

$$\kappa_{\alpha,n}(A) := \frac{1}{2\pi} \int_A \rho_{\alpha,n}(r) dr d\theta, \tag{1.2}$$

where A is a Borel subset of $\mathcal{D}_R$. We let $\mathcal{P}_{\alpha,n}$ denote the Poisson point process on $\mathcal{D}_R$ with intensity measure $n\kappa_{\alpha,n}$. Denote by $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ the probability space on which the point process $\mathcal{P}_{\alpha,n}$ is realised. Let $\mathbb{E} := \mathbb{E}_n$ denote expectation with respect to $\mathbb{P} := \mathbb{P}_n$.

We join two points in $\mathcal{P}_{\alpha,n}$ with an edge if and only if they are within hyperbolic distance R of each other. The resulting *hyperbolic random geometric graph on* $\mathcal{D}_R$ is denoted by $\mathcal{G}_{\alpha,n} := \mathcal{G}_{\alpha,n,\nu}$. Figure 1 illustrates the disc $\mathcal{B}_R(v)$ of radius R centered at $v \in \mathcal{D}_R$. An equivalent construction of $\mathcal{G}_{\alpha,n}$ goes as follows. Given $\alpha \in (0, \infty)$ and $r \in (0, \infty)$, let $\mathcal{D}_{r,\alpha}$ be a disc of radius r in $H^2_{-\alpha^2}$. Consider the Poisson point process having uniform intensity density on $\mathcal{D}_{R,\alpha}$. The points are projected onto $\mathcal{D}_{R,1}$, preserving polar coordinates, and the hyperbolic geometric graph on $\mathcal{D}_{R,\alpha}$ is created by putting an edge between the points of the Poisson point process whose projections are distant at most R. The projection of this graph onto $\mathcal{D}_{R,1}$ is $\mathcal{G}_{\alpha,n,\nu}$.

When $\mathcal{P}_{\alpha,n}$ is replaced by n i.i.d. random variables having density $\rho_{\alpha,n}(r)/2\pi$, we obtain the model of Krioukov et al. [16]. The underlying hyperbolic geometry gives rise to a power-law degree distribution tuned by the parameter α, whereas the parameter

ν determines the average degree of $\mathcal{G}_{\alpha,n}$ [16]. The model realises the assumption that there are intrinsic hierarchies in a complex network that induce a tree-like structure. This set-up provides a geometric framework describing the inherent inhomogeneity of complex networks and suggests that the geometry of complex networks is hyperbolic.

The graph $\mathcal{G}_{\alpha,n}$ also arises as a cosmological model. As noted in [15], the higher-dimensional analogue of $\mathcal{G}_{\alpha,n}$ asymptotically coincides as $n \to \infty$ with the graph encoding the large scale causal structure of the de Sitter spacetime representation of the universe. Roughly speaking, the latter graph is obtained by sprinkling Poisson points in de Sitter spacetime (the hyperboloid) and then joining two points if they lie within each other's light cones. The light cones are then mapped to the hyperbolic plane, where they are approximated (for large times) by the hyperbolic balls of a certain radius (see Fig. 2 in [15]). Graph properties of $\mathcal{G}_{\alpha,n}$ thus yield information about the causal structure of de Sitter spacetime.

1.2 Main results

For any $p \in \mathcal{D}_R$ we let $r(p)$ denote its radius (hyperbolic distance to the origin) and

$$y(p) := R - r(p)$$

its defect radius. Given a point process $\mathcal{P}$ on $\mathcal{D}_R$ and $p \in \mathcal{P} \cap \mathcal{D}_R$, we say that p is *isolated* with respect to $\mathcal{P}$ if and only if there is no $p' \in \mathcal{P}$, $p' \neq p$, such that $d_H(p, p') \leq R$. We say that p is *extreme* with respect to $\mathcal{P}$ if and only if there is no $p' \in \mathcal{P}$, $p' \neq p$, such that $d_H(p, p') \leq R$ and $y(p') \in [0, y(p))$.

Given $p \in \mathcal{P}$, define the score $\xi^{iso}(p, \mathcal{P})$ to be 1 if p is isolated with respect to $\mathcal{P}$ and zero otherwise. Likewise, define $\xi^{ext}(p, \mathcal{P})$ to be 1 if p is extreme with respect to $\mathcal{P}$ and zero otherwise. Our main goal is to establish the limit theory for the number of isolated and extreme points in $\mathcal{G}_{\alpha,n}$, given respectively by

$$S^{iso}(\mathcal{P}_{\alpha,n}) := \sum_{p \in \mathcal{P}_{\alpha,n}} \xi^{iso}(p, \mathcal{P}_{\alpha,n})$$

and

$$S^{ext}(\mathcal{P}_{\alpha,n}) := \sum_{p \in \mathcal{P}_{\alpha,n}} \xi^{ext}(p, \mathcal{P}_{\alpha,n}).$$

Isolated vertices are well-studied in the setting of Euclidean graphs, where they feature in the connectivity properties of certain random graph models. The paper [22] elaborates on this when the graph in question is either the geometric graph on i.i.d. points in $[0,1]^d$ or even a soft version of this graph. In the cosmological set-up [15], in the large time limit, isolated points are precisely those whose past and future light cones are empty, i.e., the set of points neither accessible by the past nor having access to the future. Extreme points are those whose future light cones are empty, i.e., points which do not causally influence other points.

Extreme points are the analog of maximal points of a sample, of broad interest in computational geometry, networks, and analysis of linear programming. Recall that if $K \subset \mathbb{R}^d$ is a cone with apex at the origin of $\mathbb{R}^d$, then given $\mathcal{X} \subset \mathbb{R}^d$ locally finite, $x \in \mathcal{X}$ is K-maximal if $(K \oplus x) \cap \mathcal{X} = x$. If $\mathcal{X}$ is an i.i.d. sample uniformly distributed on a smooth convex body B in $\mathbb{R}^d$ of volume 1, $n := \text{card}(\mathcal{X})$, then both the expectation and variance of the number of maximal points in $\mathcal{X}$ are asymptotically $\Theta(n^{(d-1)/d})$, the order of the expected number of points close to the boundary of B [25]. In the present paper, the expectation and variance of the number of extreme points are shown to grow linearly with $n := \text{card}(\mathcal{X})$, which likewise is of the order of the expected number of points close to the boundary of $\mathcal{D}_R$.

The second order limit theory for the number of isolated points is altogether different. Our first main result shows that the growth rates of the variance of $S^{iso}(\mathcal{P}_{\alpha,n})$ decrease with increasing $\alpha \in (1/2, \infty)$ and undergo a double jump when α crosses 1. Moreover, there is a logarithmic correction at $\alpha = 1$. The variance grows faster than the expectation for $\alpha \in (1/2, 1]$, but it is always sub-quadratic with respect to input size. The asymptotics for the range $\alpha \in (1/2, 1]$ contrast markedly with the second order limit behavior of isolated points in the random geometric graph in the Euclidean plane [21], where asymptotics grow linearly with input. This phenomenon, which gives rise to non-standard renormalization growth rates, appears to be linked to the high connectivity properties of $\mathcal{G}_{\alpha,n}$ for small α, as described in Section 1.3.

The limit constants appearing in our first and second order results (1.3), (1.5), and (1.6) are given in terms of expectations and covariances of scores involving isolated and extreme points of a Poisson point process on the upper half-plane, which appears to be a natural setting for studying such problems. Put $\gamma := 8\nu\alpha/\pi(2\alpha - 1)$.

Theorem 1.1. *We have for all* $\alpha \in (1/2, \infty)$

$$\lim_{n\to\infty} \frac{\mathbb{E}[S^{iso}(\mathcal{P}_{\alpha,n})]}{n} = 2\alpha \int_0^\infty \exp\left(-\gamma e^{y/2}\right) \exp(-\alpha y) dy, \tag{1.3}$$

and

$$\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})] = \begin{cases} \Theta(n^{3-2\alpha}) & \alpha \in (\frac{1}{2}, 1) \\ \Theta(nR) = \Theta(n \log n) & \alpha = 1 \\ \Theta(n) & \alpha \in (1, \infty) \end{cases}. \tag{1.4}$$

On the other hand, for all $\alpha \in (1/2, \infty)$, the expectation and variance asymptotics for the number of extreme points exhibit linear scaling in n, that is to say the renormalization is the standard one in stochastic geometric models.

Theorem 1.2. *We have for all* $\alpha \in (1/2, \infty)$

$$\lim_{n\to\infty} \frac{\mathbb{E}[S^{ext}(\mathcal{P}_{\alpha,n})]}{n} = \mu, \tag{1.5}$$

and

$$\lim_{n\to\infty} \frac{\mathrm{Var}[S^{ext}(\mathcal{P}_{\alpha,n})]}{n} = \sigma^2, \tag{1.6}$$

where $\mu, \sigma^2 \in (0, \infty)$ *are given by* (5.2) *and* (5.10)*, respectively, below.*

Denote by N the standard normal random variable with mean zero and variance one. One might expect that $S^{iso}(\mathcal{P}_{\alpha,n})$, after centering and renormalizing, converges in distribution to N for *all* $\alpha \in (1/2, \infty)$. The next result shows that this is false.

Theorem 1.3. *As* $n \to \infty$*, for any* $\alpha \in (1, \infty)$ *we have*

$$\frac{S^{iso}(\mathcal{P}_{\alpha,n}) - \mathbb{E}[S^{iso}(\mathcal{P}_{\alpha,n})]}{\sqrt{\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})]}} \xrightarrow{\mathcal{D}} N. \tag{1.7}$$

The limit (1.7) *fails for* $\alpha \in (1/2, 1)$. *As* $n \to \infty$*, for any* $\alpha \in (1/2, \infty)$ *we have*

$$\frac{S^{ext}(\mathcal{P}_{\alpha,n}) - \mathbb{E}[S^{ext}(\mathcal{P}_{\alpha,n})]}{\sqrt{\mathrm{Var}[S^{ext}(\mathcal{P}_{\alpha,n})]}} \xrightarrow{\mathcal{D}} N. \tag{1.8}$$

The proofs of these results depend on mapping the point process $\mathcal{P}_{\alpha,n}$ in the disc $\mathcal{D}_R$ to a Poisson point process hosted by a rectangle in the upper half-plane. This mapping, introduced in [9], transforms the graph $\mathcal{G}_{\alpha,n}$ into an analytically more tractable graph, as seen in Section 2. In particular, it facilitates the evaluation of the probability content of the intersection of two radius R balls, which is essential to evaluating the covariance of scores at distinct points. Variance calculations are based on the covariance formula for two points. In the case of isolated points, the upper bound is based on the Poincaré inequality. The lower bound, which turns out to be tight, is based on a careful analysis of the intersection of the balls of radius R around two typical points. We refer the reader to Sections 3 and 4 for all details.

The determination of variance asymptotics for $S^{ext}(\mathcal{P}_{\alpha,n})$ is handled by extending stabilization methods. That is, for a given point p, we define a radius of stabilization $R^{\xi} := R^{\xi^{ext}}$ for ξ^{ext}, in the sense that points at distance farther than R^{ξ} from p do not affect the property of p being extreme. We show that the covariance of two points (depending on their interpoint distance and heights) is rather small. By stabilization, the covariance converges to the covariance of two points in the infinite hyperbolic plane. We show in Section 5 that though the constants describing the tail behavior of R^{ξ} grow exponentially fast with the height of p, it is still possible to extract an explicit integral formula for the scaled variance.

To prove the asymptotic normality (1.7) we use the Poincaré inequality for Poisson functionals [17], which bounds the Wasserstein distance in terms of first- and second-order difference operators. When $\alpha \in (1,\infty)$ there is a high probability event on which these difference operators may be controlled, as vertices of high degree are fewer in this regime. For $\alpha \in (1/2, 1)$, conditional under the likely event of having no vertex sufficiently close to the origin (equivalently having no vertex of significantly high degree), the variance is much smaller than the unconditional variance, and the convergence to the standard normal fails in this regime. Intuitively, vertices close to the origin generate radius R balls which cover a relatively large part of $\mathcal{D}_R$ and any vertex lying therein will not be isolated. Surprisingly, when $\alpha \in (1,\infty)$, this phenomenon stops having a significant effect and, therefore one can deduce the asymptotic normality of $S^{iso}(\mathcal{P}_{\alpha,n})$.

The case of extreme points is different, since the extremality status of a point is influenced only by points of larger radius lying in the ball of radius R around it. This region is typically quite small and makes the corresponding score functions almost independent. To prove the central limit theorem (1.8), we cut the plane into rectangles and define a dependency graph on the vertex set of such rectangles, so that no points in non-adjacent vertices in this dependency graph can be connected, and we use the central limit theorem of Baldi and Rinott [3].

Remarks. (i) We are unaware of results treating the limit theory for statistics of $\mathcal{G}_{\alpha,n}$ in the regime $\alpha \in (1/2, 1)$. The paper [24] establishes variance asymptotics and asymptotic normality for the number of copies of trees in $\mathcal{G}_{\alpha,n}$ with at least two vertices, but the authors require $\alpha \in (1,\infty)$, save for when counting trees close to the boundary of $\mathcal{D}_R$. The methods of [24] do not appear to treat the limit theory of $S^{iso}(\mathcal{P}_{\alpha,n})$ and $S^{ext}(\mathcal{P}_{\alpha,n})$, as $n \to \infty$.

(ii) It is an interesting problem whether the number of isolated points asymptotically follows a normal law when $\alpha = 1$. The methods in this paper do not apply, as they give estimates that are useless. To deal with this case, one likely needs a more detailed treatment of the variance of $S^{iso}(\mathcal{P}_{\alpha,n})$, giving not only the order of magnitude but the multiplicative constant.

(iii) As seen in [5, 10], the expected number of cliques of order $k \geq 2$ is $\Theta(n)$ if $\alpha \in (1-1/k, \infty)$, whereas the expected number of cliques of order $k \geq 3$ is $\Theta(n^{(1-\alpha)k})$ if $\alpha \in (1/2, 1-1/k)$.

(iv) In dimension $d \geq 3$ we expect that the central limit theorem (1.8) holds for all $\alpha \in (1/2, \infty)$, where R is suitably defined so that the average degree of the random graph is $\Theta(1)$. It is unclear for which α the central limit (1.7) holds in dimension $d \geq 3$.

(v) It is unclear whether there exists a limiting distribution for $S^{iso}(\mathcal{P}_{\alpha,n})$ when $\alpha \in (1/2, 1)$. As we are going to see in Section 6, the variance of $S^{iso}(\mathcal{P}_{\alpha,n})$ is highly sensitive when conditioning on the high probability event of having no points within a certain radius in $\mathcal{D}_R$. It is plausible that a central limit theorem holds in such a conditional space.

(vi) As elaborated upon in the next subsection, the degree distribution in $\mathcal{G}_{\alpha,n}$ follows a power-law with exponent $2\alpha+1$ when $\alpha \in (1/2, \infty)$. In particular, the degree distribution belongs to L^2 when $\alpha \in (1, \infty)$. It would be worthwhile to better understand the connection between asymptotic normality of the number of isolated vertices and moments of the degree distribution.

Notation and terminology We say that a sequence of events $E_n \in \mathcal{F}_n$ occur *asymptotically almost surely (a.a.s.)* if $\lim_{n\to\infty} \mathbb{P}(E_n) = 1$. Given a_n and b_n two sequences of positive real numbers, we write $a_n \sim b_n$ to denote that $a_n/b_n \to 1$, as $n \to \infty$.

1.3 Degree and connectivity properties of the graph $\mathcal{G}_{\alpha,n}$

For $\alpha \in (1/2, \infty)$, the tails of the distribution of the degrees in $\mathcal{G}_{\alpha,n}$ follow a power law with exponent $2\alpha + 1$; see Krioukov et al. [16]. This was verified rigorously in [12]. For $\alpha \in (1/2, 1)$, the exponent is between 2 and 3, as is the case in a number of networks arising in applications (see for example [2] for a list of experimental observations). The paper [16] observes that the average degree of $\mathcal{G}_{\alpha,n}$ is determined through the parameter ν for $\alpha \in (1/2, \infty)$. This was rigorously shown in [12]. In particular, they show that the average degree tends to $8\alpha^2\nu/\pi(2\alpha-1)^2$ in probability. However, when $\alpha \in (0, 1/2]$, the average degree tends to infinity as $n \to \infty$. Thus, in this sense, the regime $\alpha \in (1/2, \infty)$ corresponds to the thermodynamic regime in the context of random geometric graphs on the Euclidean plane [21]. In [8] the degree distribution of a *soft* version of this model is determined. Here, pairs of points that are distant at most R are joined with some probability that is not identically equal to 1.

When α is small, one expects more points of $\mathcal{P}_{\alpha,n}$ to be near the origin and one may expect increased graph connectivity. The paper [6] establishes that $\alpha = 1$ is the critical point for the emergence of a giant component in $\mathcal{G}_{\alpha,n}$. In particular, when $\alpha \in (0, 1)$, the fraction of the vertices contained in the largest component is bounded away from 0 a.a.s. [6], whereas if $\alpha \in (1, \infty)$, the largest component is sublinear in n a.a.s. For $\alpha = 1$, the component structure depends on ν. If ν is large enough, then a giant component exists a.a.s., but if ν is small enough, then a.a.s. all components have sublinear size [6]. Figures 2 and 3 illustrate these transitions.

The paper [9] strengthens these results and shows the fraction of vertices belonging to the largest component converges in probability to a constant which depends on α and ν. Furthermore, for $\alpha = 1$, there exists a critical value $\nu_0 \in (0, \infty)$ such that when ν crosses ν_0 a giant component emerges a.a.s. [9]. For $\alpha \in (0, 1)$ the second largest component has polylogarithmic order a.a.s.; see [13] and [14]. For $\alpha \in (0, 1/2)$ we have that $\mathcal{G}_{\alpha,n}$ is a.a.s. connected, whereas $\mathcal{G}_{\alpha,n}$ is disconnected for $\alpha \in (1/2, \infty)$ [7]. For $\alpha = 1/2$, the probability of connectivity tends to a certain constant given explicitly in [7].

Apart from the component structure, the geometry of this model has been also considered. In [13] and [11] polylogarithmic upper bounds on the diameter are shown. These were improved shortly afterwards in [19] where a logarithmic upper bound on

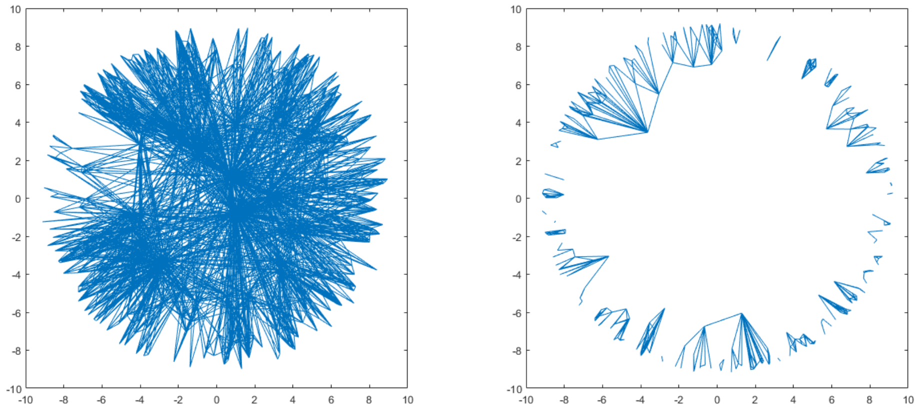

Figure 2: Samples of $\mathcal{G}_{\alpha,n}$ for $n = 300$, $\nu = 3$ and $\alpha = 0.7$ and 2, respectively, from left to right. Average degree decreases as α increases.

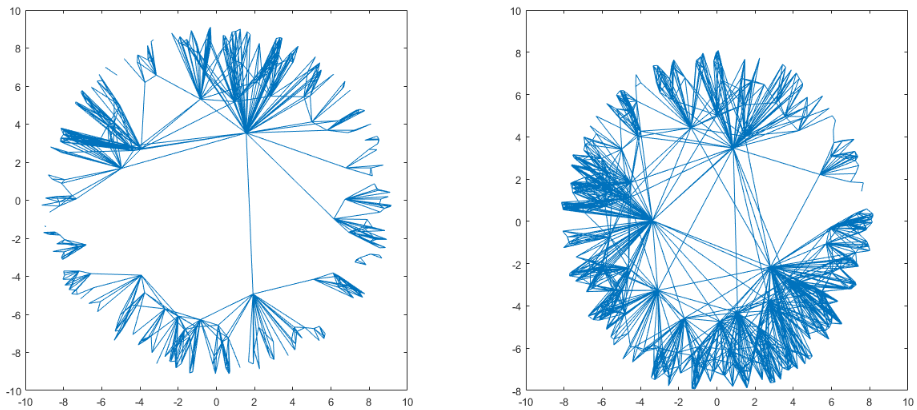

Figure 3: Samples of $\mathcal{G}_{\alpha,n}$ for $n = 300$, $\alpha = 1$ and $\nu = 3$ and 5, respectively, from left to right. Average degree increases as ν increases.

the diameter is established. Furthermore, in [1] it is shown that for $\alpha \in (1/2, 1)$ the largest component has doubly logarithmic typical distances and it forms what is called an ultra-small world.

2 Auxiliary results

2.1 Approximating a hyperbolic ball

We characterize when two points in $\mathcal{D}_R$ are within hyperbolic distance R. In particular the next lemma approximates hyperbolic balls by analytically more tractable sets, reducing a statement about hyperbolic distances between two points to a statement about their relative angle. For a point $p \in \mathcal{D}_R$, we let $\theta(p) \in (-\pi, \pi]$ be the angle $\widehat{pOq}$ between p and a (fixed) reference point $q \in \mathcal{D}_R$ (where positive angle is determined by moving from q to p in the anti-clockwise direction). For points $p, p' \in \mathcal{D}_R$ we denote by $\theta(p, p')$ their relative angle:

$$\theta(p, p') := \min\{|\theta(p) - \theta(p')|,\ 2\pi - |\theta(p) - \theta(p')|\}.$$

For any $p \in \mathcal{D}_R$ recall that $r(p)$ denotes its radius (hyperbolic distance to the origin) whereas $y(p) := R - r(p)$, or more succinctly $y := R - r$. Thus for $p \in \mathcal{D}_R$, we shall write $p := (\theta(p), y(p))$. The hyperbolic law of cosines relates the relative angle $\theta(p, p')$ between two points with their hyperbolic distance:

$$\cosh(d_H(p, p')) = \cosh(r(p)) \cosh(r(p')) - \sinh(r(p)) \sinh(r(p')) \cos(\theta(p, p')). \tag{2.1}$$

For $r, r' \in [0, R]$, we let $\theta_R(r, r')$ be the value of $\theta(p, p') \in [0, \pi]$ satisfying (2.1), having set $d_H(p, p') = R$, for two points $p, p' \in \mathcal{D}_R$ with $r(p) = r$ and $r(p') = r'$. As $\cos(\cdot)$ is decreasing in $[0, \pi]$, it follows that $d_H(p, p') \in [0, R]$ if and only if $\theta(p, p') \leq \theta_R(r(p), r(p'))$.

When $y(p)$ and $y(p')$ are not too large, our next result estimates $\theta_R(r(p), r(p'))$ as a function of $y(p)$ and $y(p')$. To prepare for mapping $\mathcal{D}_R$ to a rectangle in $\mathbb{R} \times \mathbb{R}^+$ having length proportional to $\frac{1}{2}e^{R/2}$, we re-scale $\theta_R(r(p), r(p'))$ by a factor of $\frac{1}{2}e^{R/2}$. The following lemma appears in a stronger form in [9]. Here and elsewhere we put

$$H := 4 \log R. \tag{2.2}$$

The proof of the next lemma is in Section A.

Lemma 2.1. *Given p and p' in $\mathcal{D}_R$, $y := y(p) := R - r$, and $y' := y(p') := R - r'$, with $r, r' \in [0, R]$ we set*

$$\Delta(r, r') := \frac{1}{2} e^{R/2} \theta_R(r, r') = \frac{1}{2} e^{R/2} \arccos\left(\frac{\cosh r \cosh r' - \cosh R}{\sinh r \sinh r'} \right).$$

For every $\varepsilon \in (0, 1/3)$ there exists a $C := C(\varepsilon) \in (0, R)$ such that the following holds.

(i) If $r + r' \in (R + C, 2R]$, i.e., if $y(p) + y(p') \in [0, R - C)$, then

$$(1 - \varepsilon) e^{\frac{1}{2}(y + y')} \leq \Delta(r, r') \leq (1 + \varepsilon) e^{\frac{1}{2}(y + y')}. \tag{2.3}$$

(ii) If $r, r' \in [R - H, R]$, i.e., if $y(p), y(p') \in [0, H]$, then

$$\Delta(r, r') = (1 + \lambda_n(r, r')) e^{\frac{1}{2}(y + y')} \tag{2.4}$$

where $\lambda_n(r, r') = o(1)$ as $n \to \infty$, uniformly over all $r, r' \in [R - H, R]$.

(iii) In part (i) above, one can take $\varepsilon := \varepsilon(n) \to 0$ and $C := C(n) \to \infty$ as $n \to \infty$. In particular we may relate ε and C by $\varepsilon = \Theta(e^{-C})$.

Recall that $\mathcal{D}_r$ denotes the disc of (hyperbolic) radius r centered at the origin O. For any $h \in [0, R)$ we let $\mathcal{A}_h$ denote the annulus $\mathcal{D}_R \setminus \mathcal{D}_{R-h}$. Throughout we shall use caligraphic letters to denote subsets of $\mathcal{D}_R$. For $p \in \mathcal{D}_R$ we let $\mathcal{B}(p) := \mathcal{B}_R(p) \cap \mathcal{D}_R$. We now approximate $\mathcal{B}(p)$ whenever $r(p) \in (C, R]$, with C as in Lemma 2.1. This goes as follows.

By the triangle inequality, given $p \in \mathcal{D}_R$, any point with defect radius at most $y(p) := R - r(p)$ is also within distance R from p. To approximate $\mathcal{B}(p)$ from above, we will take a superset of this set, namely the set of points of radius at most $y(p) - C$, with $C := C(\varepsilon)$ as in Lemma 2.1. We set $1_{+\varepsilon} := 1 + \varepsilon$ and $1_{-\varepsilon} := 1 - \varepsilon$ and put

$$\begin{aligned} \mathcal{B}^+(p) &:= \mathcal{B}^+(p, \varepsilon) \\ &:= \{p' \in \mathcal{D}_R : y(p') + y(p) \in [0, R - C),\ \theta(p, p') \leq 1_{+\varepsilon} \cdot 2e^{\frac{1}{2}(y(p) + y(p') - R)}\} \\ &\qquad \cup \{p' \in \mathcal{D}_R : y(p') + y(p) \in [R - C, 2R]\} \end{aligned}$$

and

$$\mathcal{B}^-(p) := \mathcal{B}^-(p,\varepsilon) := \{p' \in \mathcal{D}_R \ : y(p') + y(p) \in [0, R-C),\ \theta(p,p') \le 1_{-\varepsilon} \cdot 2e^{\frac{1}{2}(y(p)+y(p')-R)}\}.$$

For $\varepsilon \in (0,1/3)$, $C := C(\varepsilon) > 0$ as in Lemma 2.1(i), and $p \in \mathcal{D}_R$ with $r(p) \in (C,R]$, the inequality (2.3) yields the following inclusions:

$$\mathcal{B}^-(p) \subset \mathcal{B}(p) \subset \mathcal{B}^+(p). \tag{2.5}$$

In our calculations for $\mathbb{E}[\xi^{ext}(p,\mathcal{P}_{\alpha,n})]$ we will need the truncated subset of $\mathcal{B}(p)$ consisting of points with height coordinates at most $y(p)$, namely

$$\mathcal{D}(p) := \{p' \in \mathcal{B}(p) \ : \ y(p') \le y(p)\}. \tag{2.6}$$

A point $p \in \mathcal{D}_R \cap \mathcal{P}_{\alpha,n}$ is extreme with respect to $\mathcal{P}_{\alpha,n}$ if and only if $\mathcal{D}(p) \cap \mathcal{P}_{\alpha,n} = \{p\}$. Lemma 2.1(ii) implies that if $y(p) \in [0,H]$, then

$$\mathcal{D}(p) := \{p' : \ y(p') \le y(p),\ \theta(p,p') \le (1+\lambda_n(p,p'))e^{\frac{1}{2}(y(p)+y(p')-R)}\}. \tag{2.7}$$

2.2 Properties of $\mathcal{G}_{\alpha,n}$

The density of the defect radius is close to the exponential density with parameter α. The proof of this fact is based on elementary algebraic manipulations and appears in Section A.

Lemma 2.2. *Let $\bar\rho_{\alpha,n}(y) := \rho_{\alpha,n}(R-y),\ y \in [0,R)$, be the probability density of the defect radii. For all $\alpha \in (1/2,\infty)$ we have*

$$|\bar\rho_{\alpha,n}(y) - \alpha e^{-\alpha y}| < \frac{2\alpha}{e^{\alpha R}-2} = O(n^{-2\alpha}),\ y \in [0,R]. \tag{2.8}$$

One does not expect to observe isolated and extreme points close to the origin. The following lemma makes this precise and shows that the isolated and extreme points a.a.s. have defect radii less than $H := 4\log R$.

Lemma 2.3. *Let $p \in \mathcal{D}_R$. If $y(p) \in (H,R]$ then for all $\alpha \in (1/2,\infty)$ we have*

$$\mathbb{E}[\xi^{iso}(p,\mathcal{P}_{\alpha,n}\cup\{p\})] = n^{-\Omega(\log n)},\ \ \mathbb{E}[\xi^{ext}(p,\mathcal{P}_{\alpha,n}\cup\{p\})] = n^{-\Omega(\log n)}.$$

Proof. Let $\varepsilon \in (0,1/3)$ and $C := C(\varepsilon)$ be as in Lemma 2.1. First assume that $y(p) \in [0,R-C)$. We may bound $\mathbb{E}[\xi^{iso}(p,\mathcal{P}_{\alpha,n}\cup\{p\})]$ by the probability that $\mathcal{B}(p)\cap\mathcal{A}_H$ is empty. Using the first inclusion in (2.5) and recalling the definition of $\kappa_{\alpha,n}$ at (1.2), we have (for n sufficiently large):

$$\begin{aligned}\mathbb{E}[\xi^{iso}(p,\mathcal{P}_{\alpha,n}\cup\{p\})] &\le \exp\left(-n\kappa_{\alpha,n}(\mathcal{B}(p)\cap\mathcal{A}_H)\right)\\ &= \exp\left(-n\frac{2(1-\varepsilon)}{2\pi}e^{-R/2}e^{y(p)/2}\int_0^{4\log R} e^{y/2}\bar\rho_{\alpha,n}(y)dy\right)\\ &< \exp\left(-\frac{\nu(1-2\varepsilon)}{\pi}e^{y(p)/2}\int_0^{4\log R} e^{(1/2-\alpha)y}dy\right)\\ &= \exp\left(-\Omega(e^{2\log R})\right) = \exp\left(-\Omega(R^2)\right) = n^{-\Omega(\log n)},\end{aligned}$$

where the inequality follows by Lemma 2.2.

Suppose now that $y(p) \in [R-C, R]$, i.e, $r(p) \in [0, C]$. By the triangle inequality any point of $\mathcal{D}_R$ of radius less than $R-C$ is within hyperbolic distance R from p. This implies that

$$\mathbb{E}[\xi^{iso}(p, \mathcal{P}_{\alpha,n} \cup \{p\})] \le \exp\left(-n\kappa_{\alpha,n}(\mathcal{D}_{R-C})\right).$$

Recalling (1.2) we obtain

$$\kappa_{\alpha,n}(\mathcal{D}_{R-C}) = \int_0^{R-C} \alpha \frac{\sinh \alpha r}{\cosh(\alpha R) - 1} dr = \frac{\cosh(\alpha(R-C) - 1}{\cosh(\alpha R) - 1} = \Theta(1),$$

whereby $\mathbb{E}[\xi^{iso}(p, \mathcal{P}_{\alpha,n} \cup \{p\})] = \exp(-\Omega(n))$.

These upper bounds are also valid for $\mathbb{E}[\xi^{ext}(p, \mathcal{P}_{\alpha,n} \cup \{p\})]$, with the exception of the last integral which would start from $r(p) \in (0, R]$ instead of from $r(p) = 0$. However, the asymptotic growth of this integral is still $\Theta(1)$. □

2.3 Mapping $\mathcal{D}_R$ to $\mathbb{R}^2$

To further simplify our calculations, we will transfer our analysis from $\mathcal{D}_R$ to $\mathbb{R}^2$, making use of a mapping introduced in [9]. We set

$$I_n := \frac{\pi}{2} e^{R/2} = \frac{\pi}{2\nu} \cdot n.$$

For any subset $E \subseteq [0, R]$, define the rectangular domain

$$D(E) := (-I_n, I_n] \times E$$

and we put

$$D := D([0, R]) := (-I_n, I_n] \times [0, R]. \tag{2.9}$$

For $p \in \mathcal{D}_R$, recall that we write $p := (y(p), \theta(p))$, with $y(p)$ the defect radius and $\theta(p)$ the angle with respect to a reference point. We re-scale the angle $\theta(p)$ by $2e^{-R/2}$, setting $x(p) := \frac{1}{2}\theta(p)e^{R/2}$. This defines the map $\Phi : \mathcal{D}_R \to D$, mapping $(\theta(p), y(p)) \mapsto (x(p), y(p))$.

Put $\beta := 2\nu\alpha/\pi$. The map Φ sends $\mathcal{P}_{\alpha,n}$ to the Poisson point process $\tilde{\mathcal{P}}_{\alpha,n}$ on D with intensity density

$$d\tilde{\mathcal{P}}_{\alpha,n}(x, y) = (\beta e^{-\alpha y} + \epsilon_n) dy dx, \ (x, y) \in D, \tag{2.10}$$

where, recalling Lemma 2.2, we have $\epsilon_n = O(n^{-2\alpha}) = o(n^{-1})$, since $\alpha \in (1/2, \infty)$.

The analogue of the relative angle is defined as follows. For $x, x' \in (-I_n, I_n]$, we let

$$|x - x'|_\Phi := \min\{|x - x'|, \ 2I_n - |x - x'|\}.$$

When considering the geometry of hyperbolic balls inside D, it will be convenient to use arithmetic on the x-axis modulo $2I_n$. In particular, for $x_1, x_2 \in (-I_n, I_n]$, we write $x_1 <_\Phi x_2$, if $x_1 < x_2$ and $|x_1 - x_2|_\Phi = |x_1 - x_2|$ or $x_1 > x_2$ and $|x_1 - x_2|_\Phi = 2I_n - |x_1 - x_2|$. This definition naturally extends to all other types of inequalities. Also, for any $x_1, x_2 \in \mathbb{R}$, we write $x_1 =_\Phi x_2$, if $x_1 = x_2 \mod 2I_n$.

Mapping balls in $\mathcal{D}$ to balls in D We set

$$B(p) := \Phi(\mathcal{B}(p)), \ B^-(p) := \Phi(\mathcal{B}^-(p)), \text{ and } B^+(p) := \Phi(\mathcal{B}^+(p)).$$

Thus, for $p \in D$ with $y(p) \in [0, R-C)$ and $\varepsilon > 0$, we have

$$B^-(p) := B^-(p, \varepsilon) := \{(x, y) : \ y + y(p) \in [0, R-C), \ |x - x(p)|_\Phi < 1_{-\varepsilon} \cdot e^{\frac{1}{2}(y + y(p))}\}$$

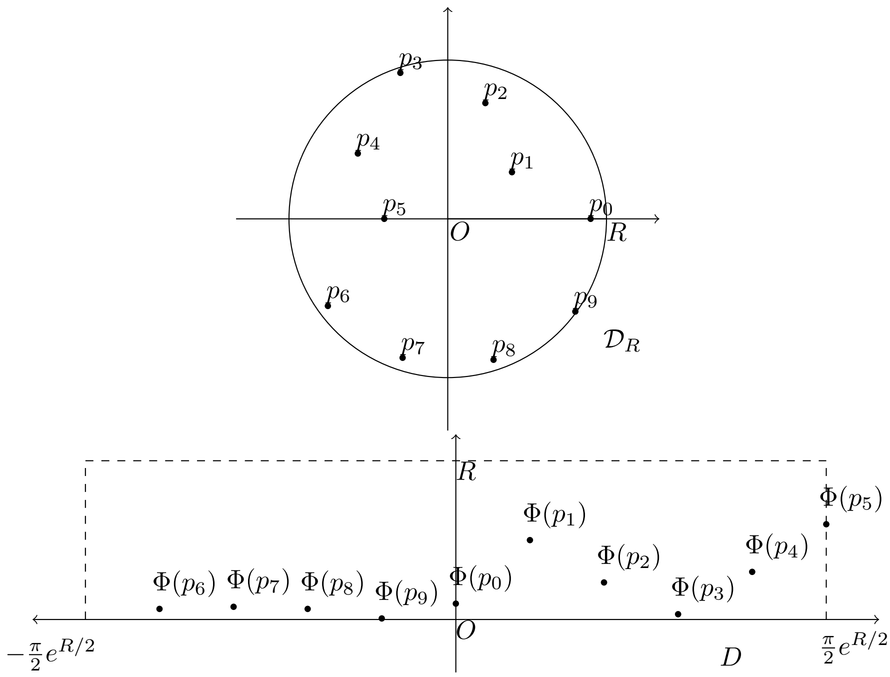

Figure 4: The mapping $\Phi : \mathcal{D}_R \to D$. (Angles in $\mathcal{D}_R$ are taken with respect to a reference point located on the positive x-axis.)

and

$$B^+(p) := B^+(p,\varepsilon) := \{(x,y) :\ |x - x(p)|_\Phi < 1_{+\varepsilon} \cdot e^{\frac{1}{2}(y+y(p))}\} \cup \{(x,y) :\ y + y(p) \in (R-C, 2R]\}.$$

For $p \in \mathcal{D}_R$ with $y(p) \in [0, R-C)$ note that Φ transforms the set inclusion (2.5) into

$$B^-(p) \subset B(p) \subset B^+(p). \tag{2.11}$$

Approximating $S^{iso}(\mathcal{P}_{\alpha,n})$ and $S^{ext}(\mathcal{P}_{\alpha,n})$ on D Let $\tilde{p} := (x(\tilde{p}), y(\tilde{p}))$ be the image of p by Φ. For $\tilde{p} \in \tilde{\mathcal{P}}_{\alpha,n}$ we define $\tilde{\xi}^{iso}(\tilde{p}, \tilde{\mathcal{P}}_{\alpha,n}) = \xi^{iso}(p, \mathcal{P}_{\alpha,n})$. In other words,

$$\tilde{S}^{iso}(\tilde{\mathcal{P}}_{\alpha,n}) := \sum_{\tilde{p} \in \tilde{\mathcal{P}}_{\alpha,n}} \tilde{\xi}^{iso}(\tilde{p}, \tilde{\mathcal{P}}_{\alpha,n}) = S^{iso}(\mathcal{P}_{\alpha,n}).$$

Regarding $S^{ext}(\mathcal{P}_{\alpha,n})$, recall from (2.7) that $p \in \mathcal{P}_{\alpha,n}$ is extreme if and only if $\mathcal{D}(p) \cap \mathcal{P}_{\alpha,n} = \{p\}$. The image under Φ of the truncated ball $\mathcal{D}(p)$ at (2.6) is

$$D(y(\tilde{p})) := \{(x,y) \in B((x(\tilde{p}), y(\tilde{p}))) :\ y \in [0, y(\tilde{p})]\}.$$

Note that $\xi^{ext}(p, \mathcal{P}_{\alpha,n}) = 1$ if and only if $D(y(\tilde{p})) \cap \tilde{\mathcal{P}}_{\alpha,n} = \{\tilde{p}\}$. For $\tilde{p} \in \tilde{\mathcal{P}}_{\alpha,n}$ define

$$\tilde{\xi}^{ext}(\tilde{p}, \tilde{\mathcal{P}}_{\alpha,n}) := \begin{cases} 1 & \tilde{\mathcal{P}}_{\alpha,n} \cap D(y(\tilde{p})) = \{\tilde{p}\} \\ 0 & \tilde{\mathcal{P}}_{\alpha,n} \cap D(y(\tilde{p})) \neq \{\tilde{p}\} \end{cases}.$$

By definition, we thus have $\xi^{ext}(p, \mathcal{P}_{\alpha,n}) = \tilde{\xi}^{ext}(\tilde{p}, \tilde{\mathcal{P}}_{\alpha,n})$ and

$$\tilde{S}^{ext}(\tilde{\mathcal{P}}_{\alpha,n}) = \sum_{\tilde{p} \in \tilde{\mathcal{P}}_{\alpha,n}} \tilde{\xi}^{ext}(\tilde{p}, \tilde{\mathcal{P}}_{\alpha,n}) =: S^{ext}(\mathcal{P}_{\alpha,n}).$$

From now on, when the context is clear, we write p instead of $\tilde{p}$ for a generic point in $\tilde{\mathcal{P}}_{\alpha,n}$.

Lemma 2.4. *We have for all* $\alpha \in (1/2, \infty)$

$$\mathbb{E}[\tilde{\xi}^{iso}(p, \tilde{\mathcal{P}}_{\alpha,n} \cup \{p\})\mathbf{1}(p \in D((H,R)))] = \exp(-\Omega(R^2)) = n^{-\Omega(\log n)},$$

and similarly when $\tilde{\xi}^{iso}$ *is replaced by* $\tilde{\xi}^{ext}$.

Proof. This follows from Lemma 2.3. □

We put

$$\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n}) := \sum_{p \in \tilde{\mathcal{P}}_{\alpha,n} \cap D([0,H])} \tilde{\xi}^{iso}(p, \tilde{\mathcal{P}}_{\alpha,n}) \tag{2.12}$$

and

$$\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_{\alpha,n}) := \sum_{p \in \tilde{\mathcal{P}}_{\alpha,n} \cap D([0,H])} \tilde{\xi}^{ext}(p, \tilde{\mathcal{P}}_{\alpha,n}). \tag{2.13}$$

Lemma 2.5. *We have for all* $\alpha \in (1/2, \infty)$

$$\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})] = \mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n})] + o(1), \;\; \mathrm{Var}[S^{ext}(\mathcal{P}_{\alpha,n})] = \mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_{\alpha,n})] + o(1),$$

as well as

$$\mathbb{E}[S^{iso}(\mathcal{P}_{\alpha,n})] = \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n})] + o(1), \;\; \mathbb{E}[S^{ext}(\mathcal{P}_{\alpha,n})] = \mathbb{E}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_{\alpha,n})] + o(1).$$

Proof. For brevity we write S_n for $S^{iso}(\mathcal{P}_{\alpha,n})$ and $\tilde{S}_n$ for $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n})$. We first assert that

$$\mathbb{P}(S_n \neq \tilde{S}_n) = O(n^{-15}). \tag{2.14}$$

To see this, we condition on the event that $|\tilde{\mathcal{P}}_{\alpha,n}| \leq 2n$ (note that the complementary event has probability which is generously bounded by $O(n^{-16})$) and then use Boole's inequality together with Lemma 2.4.

Now write

$$\mathrm{Var} S_n = \mathrm{Var}[S_n \mathbf{1}(S_n = \tilde{S}_n) + S_n \mathbf{1}(S_n \neq \tilde{S}_n)].$$

By Hölder's inequality and (2.14), we have

$$\begin{aligned}\mathrm{Var}[S_n \mathbf{1}(S_n \neq \tilde{S}_n)] &\leq \mathbb{E}[S_n^2 \mathbf{1}(S_n \neq \tilde{S}_n)] \\ &\leq (\mathbb{E}[|S_n|^3])^{2/3} \mathbb{P}(S_n \neq \tilde{S}_n)^{1/3} = O(n^{-3}).\end{aligned}$$

Using the inequality $\mathrm{Var}[X+Y] \leq \mathrm{Var}X + \mathrm{Var}Y + 2\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}$, together with $\mathrm{Var}[S_n \mathbf{1}(S_n = \tilde{S}_n)] = O(n^2)$, we see that $\mathrm{Var} S_n = \mathrm{Var} \tilde{S}_n + o(1)$, which proves the first assertion in Lemma 2.5. The remaining assertions are proved similarly. □

Define the Poisson point process $\tilde{\mathcal{P}}_\alpha$ on $\mathbb{R} \times \mathbb{R}^+$ with intensity measure μ_α given by

$$\mu_\alpha(S) := \beta \int_S e^{-\alpha y} dx dy, \tag{2.15}$$

where $S \subseteq \mathbb{R} \times \mathbb{R}^+$ is measurable. Recall from (2.9) that $D = (-I_n, I_n] \times [0, R]$. Put

$$\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) := \sum_{p \in \tilde{\mathcal{P}}_\alpha \cap D([0,H])} \tilde{\xi}^{iso}(p, \tilde{\mathcal{P}}_\alpha \cap D) \tag{2.16}$$

and

$$\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) := \sum_{p \in \tilde{\mathcal{P}}_\alpha \cap D([0,H])} \tilde{\xi}^{ext}(p, \tilde{\mathcal{P}}_\alpha \cap D). \tag{2.17}$$

The following lemma, together with Lemma 2.5, shows that to prove Theorems 1.1 and 1.2, it is enough to establish expectation and variance asymptotics for $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ and $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$.

Lemma 2.6. *We have for all* $\alpha \in (1/2, \infty)$

$$\left|\mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n}) - \tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]\right| = o(1), \quad \left|\mathbb{E}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_{\alpha,n}) - \tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]\right| = o(1)$$

and

$$|\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n})] - \mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]| = o(n), \quad |\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_{\alpha,n})] - \mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]| = o(n).$$

We will show that $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]$ and $\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]$ are both $\Omega(n)$. This, together with the following corollary of Lemmas 2.5 and 2.6, implies that the leading terms of $\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})]$ and $\mathrm{Var}[S^{ext}(\mathcal{P}_{\alpha,n})]$ are given by $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]$ and $\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]$, respectively.

Corollary 2.7. *We have for all* $\alpha \in (1/2, \infty)$

$$\left|\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})] - \mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]\right| = o(n) \tag{2.18}$$

and

$$\left|\mathrm{Var}[S^{ext}(\mathcal{P}_{\alpha,n})] - \mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]\right| = o(n). \tag{2.19}$$

Proof of Lemma 2.6. Let $\check{X}_n$ be $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n})$ and let $\widehat{X}_n$ be $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$. We denote by F_n the event that $\tilde{\mathcal{P}}_{\alpha,n} \neq \tilde{\mathcal{P}}_\alpha \cap D$. By Lemma 2.2, there is a coupling of the point processes $\tilde{\mathcal{P}}_{\alpha,n}$ and $\tilde{\mathcal{P}}_\alpha \cap D$ such that $\mathbb{P}(F_n) = O(n^{-2\alpha}) = o(n^{-1})$, since $\alpha \in (1/2, \infty)$. We let $A_n := \{|\tilde{\mathcal{P}}_{\alpha,n}| > 2n\}$ and $B_n := \{|\tilde{\mathcal{P}}_\alpha \cap D| > 2n\}$. Then $\mathbb{P}(A_n \cup B_n) = o(n^{-1})$. Setting $\check{Y}_n := \check{X}_n - \mathbb{E}\check{X}_n$ and $\widehat{Y}_n := \widehat{X}_n - \mathbb{E}\widehat{X}_n$ gives

$$\begin{aligned}\mathrm{Var}\check{X}_n &= \mathbb{E}[\check{Y}_n^2 \mathbf{1}(F_n^c)] + \mathbb{E}[\check{Y}_n^2 \mathbf{1}(F_n)] \\ &= \mathbb{E}[\widehat{Y}_n^2 \mathbf{1}(F_n^c)] + \mathbb{E}[\check{Y}_n^2 \mathbf{1}(F_n)] \\ &\le \mathbb{E}\widehat{Y}_n^2 + \mathbb{E}[\check{Y}_n^2 \mathbf{1}(F_n)].\end{aligned}$$

Now,

$$\begin{aligned}\mathbb{E}[\check{Y}_n^2 \mathbf{1}(F_n)] &\le \mathbb{E}[|\tilde{\mathcal{P}}_{\alpha,n}|^2 \mathbf{1}(F_n)] \\ &= \mathbb{E}[|\tilde{\mathcal{P}}_{\alpha,n}|^2 \mathbf{1}(F_n)\mathbf{1}(A_n^c)] + \mathbb{E}[|\tilde{\mathcal{P}}_{\alpha,n}|^2 \mathbf{1}(F_n)\mathbf{1}(A_n)] \\ &\le 4n^2 \mathbb{P}(F_n) + \mathbb{E}[|\tilde{\mathcal{P}}_{\alpha,n}|^2 \mathbf{1}(A_n)] \\ &= o(n) + \mathbb{E}[|\tilde{\mathcal{P}}_{\alpha,n}|^2 \mathbf{1}(A_n)] = o(n),\end{aligned}$$

since $|\tilde{\mathcal{P}}_{\alpha,n}|$ is Poisson-distributed with parameter equal to n. Thus $\mathrm{Var}\check{X}_n \le \mathrm{Var}\widehat{X}_n + o(n)$. The bound remains valid if we interchange $\check{X}_n$ with $\widehat{X}_n$, $\tilde{\mathcal{P}}_{\alpha,n}$ with $\tilde{\mathcal{P}}_\alpha \cap D$, and A_n with B_n. We thus obtain $|\mathrm{Var}\check{X}_n - \mathrm{Var}\widehat{X}_n| = o(n)$. The proof of the bound for $|\mathbb{E}[\check{X}_n - \widehat{X}_n]|$ is identical, except that second moments are replaced by first moments and this yields $|\mathbb{E}[\check{X}_n - \widehat{X}_n]| = o(1)$. This completes the proof of the estimates involving $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_{\alpha,n})$ and $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$. The proofs of the assertions involving $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_{\alpha,n})$ and $\tilde{S}^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$ are identical. □

3 Preparing for the proof of Theorem 1.1

We provide several lemmas needed to estimate $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]$.

3.1 A covariance formula for $\tilde{\xi}^{iso}$

We establish a basic covariance formula needed for the calculation of $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]$. If $\xi(p, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p\})$ is a score function, we define the *covariance of* ξ *at the points* p_1, p_2 as

$$\begin{aligned} c^\xi(p_1,p_2) &:= c^\xi(p_1,p_2;\tilde{\mathcal{P}}_\alpha \cap D) \\ &:= \mathbb{E}\left[\xi(p_1,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1,p_2\})\cdot \xi(p_2,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1,p_2\})\right] \\ &\quad - \mathbb{E}\left[\xi(p_1,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1\})\right]\ \mathbb{E}\left[\xi(p_2,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_2\})\right]. \end{aligned} \tag{3.1}$$

We will give an expression for $c^{\tilde{\xi}^{iso}}$. The number of points of $(\tilde{\mathcal{P}}_\alpha \cap D)$ inside $B(p)$ is Poisson-distributed with parameter $\mu_\alpha(B(p))$, implying that $\mathbb{E}[\tilde{\xi}^{iso}(p, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p\})] = \exp(-\mu_\alpha(B(p)))$. Thus

$$\mathbb{E}[\tilde{\xi}^{iso}(p_1,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1\})]\mathbb{E}[\tilde{\xi}^{iso}(p_2,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_2\})] = \exp\left(-\left(\mu_\alpha(B(p_1)) + \mu_\alpha(B(p_2))\right)\right).$$

Put

$$\mathrm{S}_{p_1p_2} := B(p_1) \cap B(p_2). \tag{3.2}$$

If $p_1 \notin B(p_2)$ and $p_2 \notin B(p_1)$, then we have

$$\begin{aligned} &\mathbb{E}[\tilde{\xi}^{iso}(p_1,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1,p_2\})\tilde{\xi}^{iso}(p_2,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1,p_2\})] \\ &= \exp\left(-\mu_\alpha(B(p_1)\cup B(p_2))\right) \\ &= \exp(-\left(\mu_\alpha(B(p_1)) + \mu_\alpha(B(p_2))\right) + \mu_\alpha(\mathrm{S}_{p_1p_2})) \\ &= \mathbb{E}[\tilde{\xi}^{iso}(p_1,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1\})]\mathbb{E}[\tilde{\xi}^{iso}(p_2,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_2\})]\cdot\exp\left(\mu_\alpha(\mathrm{S}_{p_1p_2})\right). \end{aligned}$$

Therefore, given $p_1, p_2 \in D([0,H])$, we have the *following basic covariance formula*:

$$c^{\tilde{\xi}^{iso}}(p_1,p_2) = \begin{cases} \mathbb{E}[\tilde{\xi}^{iso}(p_1,\tilde{\mathcal{P}}_\alpha \cap D)]\mathbb{E}[\tilde{\xi}^{iso}(p_2,\tilde{\mathcal{P}}_\alpha \cap D)]\left(\exp\left(\mu_\alpha(\mathrm{S}_{p_1p_2})\right)-1\right) & p_1 \notin B(p_2) \\ -\mathbb{E}[\tilde{\xi}^{iso}(p_1,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_1\})]\mathbb{E}[\tilde{\xi}^{iso}(p_2,(\tilde{\mathcal{P}}_\alpha \cap D)\cup\{p_2\})] & p_1 \in B(p_2) \end{cases}. \tag{3.3}$$

Consider the second case in (3.3), where the covariance is negative. By Lemma 2.1(ii), given points p_1 and p_2 with $y(p_1), y(p_2) \in [0,H]$, we have

$$p_1 \in B(p_2) \text{ if and only if } |x(p_2)-x(p_1)|_\Phi < (1+\lambda_n(p_1,p_2))e^{\frac{1}{2}(y(p_1)+y(p_2))},$$

where $\lambda_n(p_1,p_2) = o(1)$ uniformly for all $p_1, p_2 \in D([0,H])$. Setting

$$Y_{12} := (1+\lambda_n(p_1,p_2))e^{\frac{1}{2}(y(p_1)+y(p_2))} \tag{3.4}$$

we may re-state the above as

$$p_1 \in B(p_2) \text{ if and only if } |x(p_2)-x(p_1)|_\Phi < Y_{12}. \tag{3.5}$$

Before focussing on the first case in (3.3) we need some geometric preliminaries.

3.2 The geometry of balls with height coordinate at most H

Our aim now is to estimate $\mu_\alpha(\mathrm{S}_{p_1p_2})$. The set inclusion $B^-(p) \subseteq B(p) \subseteq B^+(p)$ at (2.11) implies

$$\mu_\alpha(B^-(p_1) \cap B^-(p_2)) \le \mu_\alpha(\mathrm{S}_{p_1p_2}) \le \mu_\alpha(B^+(p_1) \cap B^+(p_2)).$$

Given $l \in [0, R]$, $\varepsilon > 0$ and $p \in D([0, H))$ we set

$$B_l^-(p) := B_l^-(p, \varepsilon) := \{(x, y) \ : \ y \in [0, l), \ |x - x(p)|_\Phi < 1_{-\varepsilon} \cdot e^{\frac{1}{2}(y+y(p))}\} \tag{3.6}$$

$$B_l^+(p) := B_l^+(p, \varepsilon) := \{(x, y) \ : \ y \in [0, l), \ |x - x(p)|_\Phi < 1_{+\varepsilon} \cdot e^{\frac{1}{2}(y+y(p))}\} \tag{3.7}$$

and

$$Z_l(p) := \{(x, y) \in D \ : \ y \ge l\}. \tag{3.8}$$

We continue to assume that p_1 and p_2 belong to $D([0, H])$. We assume without loss of generality that $x(p_1) <_\Phi x(p_2)$ and $y(p_1) \in (y(p_2), H]$. Henceforth, for $C := C(\varepsilon)$ as in Lemma 2.1, we put

$$h := h(p_1) := R - y(p_1) - C. \tag{3.9}$$

Notice that $B^-(p_1) = B_h^-(p_1)$ and $B_h^-(p_2) \subseteq B^-(p_2)$. See Figure 5.

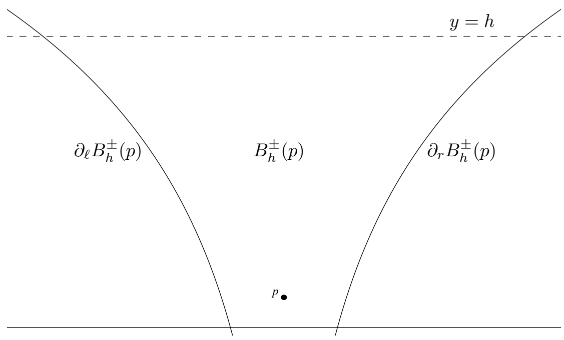

Figure 5: The ball $B_h^\pm(p)$ around a point p.

Furthermore, the definitions of $B^+(p)$ and $B^-(p)$ and the assumption $y(p_1) > y(p_2)$ imply

$$B_h^-(p_1) \cap B_h^-(p_2) \subseteq B^-(p_1) \cap B^-(p_2)$$

and

$$B^+(p_1) \cap B^+(p_2) \subseteq \big(B_h^+(p_1) \cap B_h^+(p_2)\big) \cup Z_h(p_1).$$

These inclusions yield

$$\mu_\alpha(\mathrm{S}_{p_1p_2}) \le \mu_\alpha(B_h^+(p_1) \cap B_h^+(p_2)) + \mu_\alpha(Z_h(p_1)) \tag{3.10}$$

and

$$\mu_\alpha(B_h^-(p_1) \cap B_h^-(p_2)) \le \mu_\alpha(B^-(p_1) \cap B^-(p_2)) \le \mu_\alpha(\mathrm{S}_{p_1p_2}), \tag{3.11}$$

whence

$$\mu_\alpha(B_h^-(p_1) \cap B_h^-(p_2)) \le \mu_\alpha(\mathrm{S}_{p_1p_2}) \le \mu_\alpha(B_h^+(p_1) \cap B_h^+(p_2)) + \mu_\alpha(Z_h(p_1)). \tag{3.12}$$

First, we notice that the definitions of h, β, and I_n give

$$\begin{aligned}\mu_\alpha(Z_h(p_1)) &= \beta \cdot 2I_n \int_{R-y(p_1)-C}^{R} e^{-\alpha y} dy \\ &= 2\nu e^{\frac{R}{2}} \left(e^{-\alpha(R-y(p_1)-C)} - e^{-\alpha R}\right) \\ &= 2\nu e^{(\frac{1}{2}-\alpha)R} \left(e^{\alpha y(p_1)+\alpha C} - 1\right) \\ &= \Theta(1) \cdot e^{(\frac{1}{2}-\alpha)R+\alpha y(p_1)}.\end{aligned} \tag{3.13}$$

Denote by $B_h^{\pm}(p)$ either of the balls $B_h^{+}(p)$ or $B_h^{-}(p)$ and denote by $1_{\pm\varepsilon}$ either $1_{+\varepsilon}$ or $1_{-\varepsilon}$, depending on which of the two cases we are considering. The following lemma characterises when two balls are disjoint.

Lemma 3.1. *Fix $\varepsilon \in (0,1)$ and assume $x(p_1) <_\Phi x(p_2)$. With h at (3.9) we have $B_h^{\pm}(p_1) \cap B_h^{\pm}(p_2) = \emptyset$ if and only if*

$$|x(p_2) - x(p_1)|_\Phi > 1_{\pm\varepsilon} \cdot e^{\frac{h}{2}}(e^{\frac{y(p_1)}{2}} + e^{\frac{y(p_2)}{2}}).$$

Proof. By the definition of $B_h^{\pm}(p_1)$, the right-most point of $B_h^{\pm}(p_1)$, denoted by $p' := (x(p'), y(p'))$ satisfies $x(p') =_\Phi x(p_1) + 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_1)+h)}$. Similarly, the left-most point of $B_h^{\pm}(p_2)$ (denoted by p'') satisfies $x(p'') =_\Phi x(p_2) - 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_2)+h)}$. Note that $y(p') = y(p'')$. Then $x(p') = x(p'')$ if and only if $x(p_2) - x(p_1) =_\Phi 1_{\pm\varepsilon} \cdot e^{h/2}(e^{y(p_1)/2} + e^{y(p_2)/2})$. If $|x(p_2) - x(p_1)|_\Phi > 1_{\pm\varepsilon} \cdot e^{h/2}(e^{y(p_1)/2} + e^{y(p_2)/2})$, then $B_h^{\pm}(p_1) \cap B_h^{\pm}(p_2) = \emptyset$. Likewise, if $|x(p_2) - x(p_1)|_\Phi \le 1_{\pm\varepsilon} \cdot e^{h/2}(e^{y(p_1)/2} + e^{y(p_2)/2})$, then $B_h^{\pm}(p_1) \cap B_h^{\pm}(p_2) \neq \emptyset$. □

We still assume $x(p_1) <_\Phi x(p_2)$ and we set $t :=_\Phi x(p_2) - x(p_1)$. With h at (3.9) we consider in the remainder of this sub-section the case $0 <_\Phi t <_\Phi 1_{\pm\varepsilon} \cdot e^{h/2}(e^{y(p_1)/2} + e^{y(p_2)/2})$. For t in this domain, Lemma 3.1 implies that $B_h^{\pm}(p_1) \cap B_h^{\pm}(p_2) \neq \emptyset$. Given $p \in D((0,H))$, denote the left and right boundaries of $B_h^{\pm}(p)$ by

$$\partial_\ell B_h^{\pm}(p) := \{p' \in D((0,h)) :\ x(p') =_\Phi x(p) - 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p)+y(p'))}\},$$

and

$$\partial_r B_h^{\pm}(p) := \{p' \in D((0,h)) :\ x(p') =_\Phi x(p) + 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p)+y(p'))}\};$$

cf. Figure 5. The first part of the next lemma shows that $\partial_r B_h^{\pm}(p_1)$ and $\partial_\ell B_h^{\pm}(p_2)$ intersect whenever the x-coordinates of p_1 and p_2 are far enough apart with respect to the exponentiated height coordinates.

Lemma 3.2. *(i) If $\partial_r B_h^{\pm}(p_1) \cap \partial_\ell B_h^{\pm}(p_2) \neq \emptyset$ then*

$$x(p_2) - x(p_1) >_\Phi 1_{\pm\varepsilon} \cdot \left(e^{\frac{y(p_1)}{2}} + e^{\frac{y(p_2)}{2}}\right). \tag{3.14}$$

(ii) $\partial_\ell B_h^{\pm}(p_1) \cap \partial_\ell B_h^{\pm}(p_2) = \emptyset$. (iii) If $p'_{12} \in \partial_r B_h^{\pm}(p_1) \cap \partial_r B_h^{\pm}(p_2) \neq \emptyset$ then

$$e^{\frac{y(p'_{12})}{2}} = \frac{1}{1_{\pm\varepsilon}} \cdot \frac{|x(p_2) - x(p_1)|_\Phi}{e^{y(p_1)/2} - e^{y(p_2)/2}}. \tag{3.15}$$

Proof. (i) If $p_{12} := \partial_r B_h^{\pm}(p_1) \cap \partial_\ell B_h^{\pm}(p_2) \neq \emptyset$ (cf. Figure 6), then $y(p_{12})$ satisfies the following equations:

$$x(p_{12}) - x(p_1) =_\Phi 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_1)+y(p_{12}))}$$

and

$$x(p_2) - x(p_{12}) =_\Phi 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_2)+y(p_{12}))}.$$

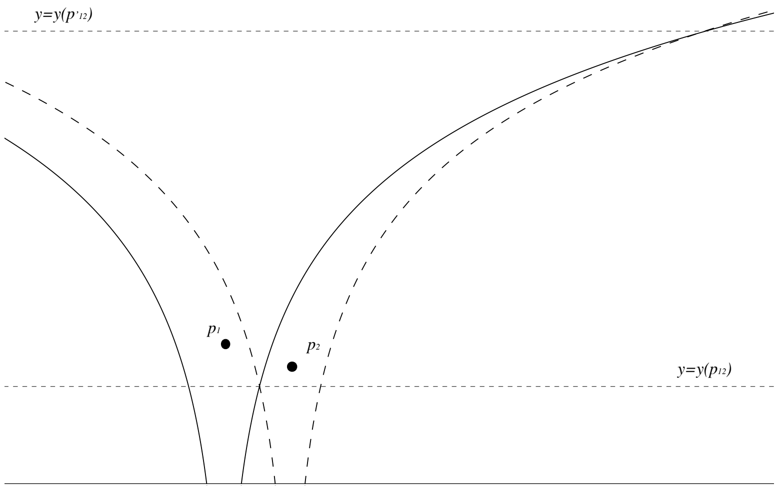

Figure 6: The intersection of two balls

Therefore

$$e^{\frac{y(p_{12})}{2}} = \frac{1}{1_{\pm\varepsilon}} \cdot \frac{|x(p_2) - x(p_1)|_\Phi}{e^{y(p_1)/2} + e^{y(p_2)/2}}. \tag{3.16}$$

Hence, p_{12} exists provided that $y(p_{12}) > 0$, which implies (3.14), as desired.

(ii) On the contrary, assume that there exists $p \in \partial_\ell B_h^\pm(p_1) \cap \partial_\ell B_h^\pm(p_2)$. Using the definition of the left boundary we have

$$x(p_1) - 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_1)+y(p))} =_\Phi x(p_2) - 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_2)+y(p))}.$$

We deduce that

$$e^{y(p)/2} = \frac{1}{1_{\pm\varepsilon}} \cdot \frac{|x(p_2) - x(p_1)|_\Phi}{e^{y(p_2)/2} - e^{y(p_1)/2}} < 0,$$

since $y(p_2) < y(p_1)$, which is impossible. Thus, such p cannot exist and $\partial_\ell B_h^\pm(p_1) \cap \partial_\ell B_h^\pm(p_2) = \emptyset$, proving (ii) as desired.

(iii) Assume $p'_{12} = \partial_r B_h^\pm(p_1) \cap \partial_r B_h^\pm(p_2) \neq \emptyset$ (cf. Figure 6). Then p'_{12} satisfies

$$x(p_1) + 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_1)+y(p'_{12}))} =_\Phi x(p_2) + 1_{\pm\varepsilon} \cdot e^{\frac{1}{2}(y(p_2)+y(p'_{12}))},$$

which yields (3.15) and completes the proof of Lemma 3.2. □

Note that (3.15) and (3.16) imply that $y(p_{12}) < y(p'_{12})$. For convenience, we will set

$$y(p_{12}) := y_L \quad \text{and} \quad y(p'_{12}) := y_U.$$

Consider now the union of the two balls $B_h^\pm(p_1) \cup B_h^\pm(p_2)$. For any $x >_\Phi x(p'_{12})$ let $p \in \partial_r B_h^\pm(p_1)$ and $p' \in \partial_r B_h^\pm(p_2)$ be such that $x(p) = x(p') = x$. Then $y(p') > y(p)$. Now for $x <_\Phi x(p_1) - 1_{\pm\varepsilon} \cdot e^{y(p_1)/2}$, consider two points $p \in \partial_\ell B_h^\pm(p_1)$ and $p' \in \partial_\ell B_h^\pm(p_2)$ such that $x(p) = x(p') = x$. Since $\partial_\ell B_h^\pm(p_1) \cap \partial_\ell B_h^\pm(p_2) = \emptyset$, it follows that $y(p) < y(p')$. In other words, the curves $\partial_\ell B_h^\pm(p_1)$ and $\partial_\ell B_h^\pm(p_2)$ do not intersect and $\partial_\ell B_h^\pm(p_2)$ stays "above" $\partial_\ell B_h^\pm(p_1)$.

3.3 The μ_α-content of $B_h^\pm(p_1)\cap B_h^\pm(p_2)$

We now focus on $\mu_\alpha(B_h^+(p_1)\cap B_h^+(p_2))$ and $\mu_\alpha(B_h^-(p_1)\cap B_h^-(p_2))$. The calculations of the μ_α-measure of these two intersections are similar, as the considered sets differ only by constant factors $1_{+\varepsilon}$ and $1_{-\varepsilon}$. We provide a generic calculation covering both cases. The inequality (3.12) shows that the μ_α-content of

$$\mathrm{S}^\pm_{p_1p_2} := B_h^\pm(p_1)\cap B_h^\pm(p_2)$$

controls the growth of $\mathrm{Cov}(p_1,p_2)$. The following lemma gives quantitative bounds on $\mu_\alpha(\mathrm{S}^\pm_{p_1p_2})$. We will use the first part of the lemma to lower bound $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha\cap D)]$. It turns out that this gives the main contribution to the variance bound of Theorem 1.1. We will give a matching upper bound on the variance through the Poincaré inequality. The second part of the lemma gives an upper bound on the intensity measure of $\mathrm{S}^\pm_{p_1p_2}$, which will be used in the proof of the central limit theorem for $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha\cap D), \alpha\in(1,\infty)$. Recall from (3.9) that we have set $h := R-y(p_1)-C$. Also, recall that we set $\gamma := 4\beta/(2\alpha-1)$.

Lemma 3.3. *Let $p_i := (x(p_i), y(p_i))$, $i=1,2$ as above. For $i=1,2$ we put $Y_i := e^{y(p_i)/2}$, we set $\eta(Y_2) := \gamma Y_2 e^{(1/2-\alpha)h}$, and we suppose that $t := x(p_2)-x(p_1) >_\Phi 0$.*

(i) If $\max\{1_{\pm\varepsilon}\cdot(Y_1+Y_2), Y_{12}\} < t \le 1_{\pm\varepsilon}\cdot e^{\frac{h}{2}}(Y_1-Y_2)$, then

$$\mu_\alpha(\mathrm{S}^\pm_{p_1p_2}) = \kappa t^{1-2\alpha} - 1_{\pm\varepsilon}\eta(Y_2), \tag{3.17}$$

where

$$\kappa := 1_{\pm\varepsilon}\cdot\frac{\gamma}{4\alpha}\left((Y_1+Y_2)^{2\alpha}-(Y_1-Y_2)^{2\alpha}\right). \tag{3.18}$$

(ii) If $e^{R/4}(Y_1+Y_2) <_\Phi t <_\Phi I_n$, then

$$\mu_\alpha(\mathrm{S}^\pm_{p_1p_2}) = O(1)\cdot\left(t^{1-2\alpha}\cdot(Y_1+Y_2)^{2\alpha} + n^{1-2\alpha}\cdot Y_1^{2\alpha}\right). \tag{3.19}$$

Proof. Part (i). We express $\mathrm{S}^\pm_{p_1p_2}$ as the disjoint union of the sets $D((y_U,R))\cap\mathrm{S}^\pm_{p_1p_2}$ and $D([y_L,y_U])\cap\mathrm{S}^\pm_{p_1p_2}$. The above analysis implies that $D((y_U,R))\cap\mathrm{S}^\pm_{p_1p_2} = D((y_U,R])\cap B_h^\pm(p_2)$. Let us consider the region $D([y_L,y_U])$. Let $y\in[y_L,y_U]$. Then any point p with $y(p)=y$ belongs to $\mathrm{S}^\pm_{p_1p_2}$ if and only if $x(p_2)-1_{\pm\varepsilon}\cdot e^{\frac12(y(p_2)+y)} \le_\Phi x(p) \le_\Phi x(p_1)+1_{\pm\varepsilon}\cdot e^{\frac12(y(p_1)+y)}$. Note that $D((0,y_L))\cap\mathrm{S}^\pm_{p_1p_2}=\emptyset$.

Recalling $\gamma = 4\beta/(2\alpha-1)$, these observations imply

$$\begin{aligned}
&\mu_\alpha(D((y_U,R))\cap\mathrm{S}^\pm_{p_1p_2})\\
&= 1_{\pm\varepsilon}\cdot 2\beta e^{\frac{y(p_2)}{2}}\int_{y_U}^h e^{(\frac12-\alpha)y}dy\\
&= 1_{\pm\varepsilon}\cdot\gamma\cdot\left(e^{\frac{y(p_2)}{2}+(\frac12-\alpha)y_U} - e^{\frac{y(p_2)}{2}}e^{(\frac12-\alpha)h}\right)\\
&= 1_{\pm\varepsilon}\cdot\gamma e^{\frac{y(p_2)}{2}+(\frac12-\alpha)y_U} - 1_{\pm\varepsilon}\cdot\gamma e^{\frac{y(p_2)}{2}}e^{(\frac12-\alpha)h}.
\end{aligned}\tag{3.20}$$

We also have

$$\begin{aligned}
&\mu_\alpha(D([y_L,y_U])\cap\mathrm{S}^\pm_{p_1p_2})\\
&\qquad= \beta\int_{y_L}^{y_U}\left((x(p_1)-x(p_2)) + 1_{\pm\varepsilon}\cdot e^{\frac{y}{2}}\left(e^{\frac{y(p_1)}{2}}+e^{\frac{y(p_2)}{2}}\right)\right)e^{-\alpha y}dy\\
&\qquad= \frac{\beta}{\alpha}(x(p_1)-x(p_2))\left(e^{-\alpha y_L}-e^{-\alpha y_U}\right)\\
&\qquad\quad + 1_{\pm\varepsilon}\cdot\frac{\gamma}{2}\left(e^{\frac{y(p_1)}{2}}+e^{\frac{y(p_2)}{2}}\right)\left(e^{(\frac12-\alpha)y_L}-e^{(\frac12-\alpha)y_U}\right).
\end{aligned}\tag{3.21}$$

Hence, by (3.20) and (3.21) we have

$$\begin{aligned}\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2}) &= \mu_\alpha(D((y_U,R])\cap \mathrm{S}^{\pm}_{p_1p_2}) + \mu_\alpha(D([y_L,y_U])\cap \mathrm{S}^{\pm}_{p_1p_2})\\ &= \frac{\beta}{\alpha}(x(p_1)-x(p_2))\left(e^{-\alpha y_L}-e^{-\alpha y_U}\right)\\ &\quad + 1_{\pm\varepsilon}\cdot\frac{\gamma}{2}e^{\frac{y(p_1)}{2}}\left(e^{(\frac12-\alpha)y_L}-e^{(\frac12-\alpha)y_U}\right)\\ &\quad + 1_{\pm\varepsilon}\cdot\frac{\gamma}{2}e^{\frac{y(p_2)}{2}}\left(e^{(\frac12-\alpha)y_L}+e^{(\frac12-\alpha)y_U}\right) - 1_{\pm\varepsilon}\cdot\gamma e^{\frac{y(p_2)}{2}}e^{(\frac12-\alpha)h}.\end{aligned} \tag{3.22}$$

By (3.15) and (3.16) we have

$$e^{\frac{y_L}{2}} = \frac{1}{1_{\pm\varepsilon}}\cdot\frac{t}{Y_1+Y_2} \quad\text{and}\quad e^{\frac{y_U}{2}} = \frac{1}{1_{\pm\varepsilon}}\cdot\frac{t}{Y_1-Y_2}. \tag{3.23}$$

Therefore,

$$e^{-\alpha y_L} = (1_{\pm\varepsilon})^{2\alpha}\cdot\left(\frac{t}{Y_1+Y_2}\right)^{-2\alpha} \text{ and } e^{-\alpha y_U} = (1_{\pm\varepsilon})^{2\alpha}\cdot\left(\frac{t}{Y_1-Y_2}\right)^{-2\alpha}. \tag{3.24}$$

Combining (3.23) and (3.24) yields

$$e^{(\frac12-\alpha)y_L} = (1_{\pm\varepsilon})^{2\alpha-1}\cdot\left(\frac{t}{Y_1+Y_2}\right)^{1-2\alpha} \text{ and } e^{(\frac12-\alpha)y_U} = (1_{\pm\varepsilon})^{2\alpha-1}\cdot\left(\frac{t}{Y_1-Y_2}\right)^{1-2\alpha}. \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.22) we have

$$\begin{aligned}\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2}) &= -(1_{\pm\varepsilon})^{2\alpha}\cdot\frac{\beta}{\alpha}t\left(\left(\frac{t}{Y_1+Y_2}\right)^{-2\alpha}-\left(\frac{t}{Y_1-Y_2}\right)^{-2\alpha}\right)\\ &\quad + (1_{\pm\varepsilon})^{2\alpha}\cdot\frac{\gamma}{2}\left(Y_1\left(\left(\frac{t}{Y_1+Y_2}\right)^{1-2\alpha}-\left(\frac{t}{Y_1-Y_2}\right)^{1-2\alpha}\right)\right.\\ &\quad \left.+Y_2\left(\left(\frac{t}{Y_1+Y_2}\right)^{1-2\alpha}+\left(\frac{t}{Y_1-Y_2}\right)^{1-2\alpha}\right)\right) - 1_{\pm\varepsilon}\cdot\gamma Y_2 e^{(\frac12-\alpha)h}\\ &= -(1_{\pm\varepsilon})^{2\alpha}\cdot\frac{\beta}{\alpha}t^{1-2\alpha}\left((Y_1+Y_2)^{2\alpha}-(Y_1-Y_2)^{2\alpha}\right)\\ &\quad + (1_{\pm\varepsilon})^{2\alpha}\cdot\frac{\gamma}{2}t^{1-2\alpha}\left[Y_1\left((Y_1+Y_2)^{2\alpha-1}-(Y_1-Y_2)^{2\alpha-1}\right)\right.\\ &\quad \left.+Y_2\left((Y_1+Y_2)^{2\alpha-1}+(Y_1-Y_2)^{2\alpha-1}\right)\right] - 1_{\pm\varepsilon}\cdot\gamma Y_2 e^{(\frac12-\alpha)h}.\end{aligned} \tag{3.26}$$

Notice that

$$\begin{aligned}&Y_1\left((Y_1+Y_2)^{2\alpha-1}-(Y_1-Y_2)^{2\alpha-1}\right)\\ &\qquad + Y_2\left((Y_1+Y_2)^{2\alpha-1}-(Y_1-Y_2)^{2\alpha-1}\right)\\ &= (Y_1+Y_2)(Y_1+Y_2)^{2\alpha-1}-(Y_1-Y_2)(Y_1-Y_2)^{2\alpha-1}\\ &= (Y_1+Y_2)^{2\alpha}-(Y_1-Y_2)^{2\alpha}.\end{aligned}$$

Substituting this into (3.26) yields

$$\begin{aligned}\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2}) &= -(1_{\pm\varepsilon})^{2\alpha}\cdot\frac{\beta}{\alpha}t^{1-2\alpha}\left((Y_1+Y_2)^{2\alpha}-(Y_1-Y_2)^{2\alpha}\right)\\ &\quad + (1_{\pm\varepsilon})^{2\alpha}\cdot\frac{\gamma}{2}t^{1-2\alpha}\left((Y_1+Y_2)^{2\alpha}-(Y_1-Y_2)^{2\alpha}\right) - 1_{\pm\varepsilon}\cdot\gamma Y_2 e^{(\frac12-\alpha)h}\\ &= (1_{\pm\varepsilon})^{2\alpha}\cdot\frac{\gamma}{4\alpha}t^{1-2\alpha}\left((Y_1+Y_2)^{2\alpha}-(Y_1-Y_2)^{2\alpha}\right) - 1_{\pm\varepsilon}\cdot\gamma Y_2 e^{(\frac12-\alpha)h}.\end{aligned}$$

Hence, the proof of part (i) is complete.

Part (ii). We will consider three different subsets of the interval $(e^{R/4}\cdot(Y_1+Y_2), I_n)$. For the case where $e^{R/4}\cdot(Y_1+Y_2) <_\Phi t \le_\Phi 1_{\pm\varepsilon}\cdot e^{h/2}(Y_1-Y_2)$ we will use part (i). (Note that $\max\{1_{\pm\varepsilon}\cdot(Y_1+Y_2), Y_{12}\} < e^{R/2}\cdot(Y_1+Y_2)$, since $Y_{12} < 2Y_1Y_2 < 2Y_1^2 \le 2R^4$.) Indeed, the expression for $\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2})$ immediately implies that for any such t we have

$$\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2}) = O(1)\cdot t^{1-2\alpha}(Y_1+Y_2)^{2\alpha}.$$

Now, assume that $1_{\pm\varepsilon}\cdot e^{h/2}(Y_1-Y_2) <_\Phi t \le_\Phi 1_{\pm\varepsilon}\cdot e^{h/2}(Y_1+Y_2)$. In this case, we have $y_U\in(h,R]$. Thus, any point p with $y(p)=y$ and $y\in[y_L,h]$ belongs to $\mathrm{S}^{\pm}_{p_1p_2}$ if and only if $x(p_2)-1_{\pm\varepsilon}\cdot e^{\frac{1}{2}(y(p_2)+y)} \le_\Phi x(p) \le_\Phi x(p_1)+1_{\pm\varepsilon}\cdot e^{\frac{1}{2}(y(p_1)+y)}$. Hence, we will use a modified version of (3.21):

$$\begin{aligned}\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2}) &= \mu_\alpha(D([y_L,R])\cap \mathrm{S}^{\pm}_{p_1p_2})\\ &= \beta\int_{y_L}^{h}\left((x(p_1)-x(p_2)) + 1_{\pm\varepsilon}\cdot e^{\frac{y}{2}}\left(e^{\frac{y(p_1)}{2}}+e^{\frac{y(p_2)}{2}}\right)\right)e^{-\alpha y}dy\\ &= \frac{\beta}{\alpha}(x(p_1)-x(p_2))\left(e^{-\alpha y_L}-e^{-\alpha h}\right)\\ &\quad + 1_{\pm\varepsilon}\cdot\frac{2\beta}{2\alpha-1}\left(e^{\frac{y(p_1)}{2}}+e^{\frac{y(p_2)}{2}}\right)\left(e^{(\frac{1}{2}-\alpha)y_L}-e^{(\frac{1}{2}-\alpha)h}\right).\end{aligned}\tag{3.27}$$

Using (3.24) and (3.25), the above becomes:

$$\begin{aligned}\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2}) &= -\frac{\beta}{\alpha}t\left((1_{\pm\varepsilon})^{2\alpha}\cdot\left(\frac{t}{Y_1+Y_2}\right)^{-2\alpha}-e^{-\alpha h}\right)\\ &\quad + 1_{\pm\varepsilon}\cdot\frac{\gamma}{2}(Y_1+Y_2)\left((1_{\pm\varepsilon})^{2\alpha-1}\cdot\left(\frac{t}{Y_1+Y_2}\right)^{1-2\alpha}-e^{(1/2-\alpha)h}\right)\\ &= (1_{\pm\varepsilon})^{2\alpha}\cdot t^{1-2\alpha}(Y_1+Y_2)^{2\alpha}\beta\left(-\frac{1}{\alpha}+\frac{2}{2\alpha-1}\right)+\frac{\beta}{\alpha}te^{-\alpha h}\\ &\quad - 1_{\pm\varepsilon}\cdot\frac{\gamma}{2}(Y_1+Y_2)\,e^{(\frac{1}{2}-\alpha)h}.\end{aligned}$$

(Note that when $t=(Y_1+Y_2)e^{h/2}$, the above expression is equal to 0.) Now, since $-\frac{1}{\alpha}+\frac{2}{2\alpha-1}=\frac{1}{\alpha(2\alpha-1)}>0$ we obtain

$$\mu_\alpha(\mathrm{S}^{\pm}_{p_1p_2}) = O(1)\cdot\left(t^{1-2\alpha}(Y_1+Y_2)^{2\alpha}+te^{-\alpha h}\right).$$

Recalling that $h=R-y(p_1)-C$ we deduce that $e^{-\alpha h}=O(1)\cdot e^{-\alpha R}Y_1^{2\alpha}=O(1)\cdot n^{-2\alpha}\cdot Y_1^{2\alpha}$. So

$$te^{-\alpha h}=O(1)\cdot t\cdot n^{-2\alpha}\cdot Y_1^{2\alpha}\overset{t<I_n=O(n)}{=}O(1)\cdot t^{1-2\alpha}(Y_1+Y_2)^{2\alpha},$$

which yields (3.19) when t satisfies $1_{\pm\varepsilon}\cdot e^{h/2}(Y_1-Y_2) <_\Phi t \le_\Phi 1_{\pm\varepsilon}\cdot e^{h/2}(Y_1+Y_2)$.

Finally, assume that $1_{\pm\varepsilon}\cdot e^{h/2}(Y_1+Y_2) <_\Phi t <_\Phi I_n$. By Lemma 3.1 we have that $\mathrm{S}^{\pm}_{p_1p_2}=\emptyset$ and therefore

$$\mu_\alpha(\mathrm{S}_{p_1p_2}) \le \mu_\alpha(Z_h(p_1)) = \Theta(1)\cdot e^{(\frac{1}{2}-\alpha)R+\alpha y(p_1)}.\tag{3.28}$$

Since $n=\nu e^{R/2}$, the above expression is $O(1)\cdot n^{1-2\alpha}\cdot Y_1^{2\alpha}$, which also yields (3.19). Combining the three cases together we deduce part (ii). □

4 Proof of Theorem 1.1

A central tool in the proof of our main results is the Palm theory for Poisson processes (see [21, 23, 18]). Let $\mathbf{S}$ be a measurable space and $\mathcal{N}(\mathbf{S})$ the set of all locally finite point configurations on $\mathbf{S}$. For a Poisson point process $\mathcal{P}$ on $\mathbf{S}$ with intensity ρ and a measurable non-negative function $h : \mathbf{S}^r \times \mathcal{N}(\mathbf{S}) \to [0,\infty)$ the Campbell-Mecke formula (cf. Theorems 4.1, 4.4 of [18]) gives

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x_1,\dots,x_r\in\mathcal{P}}^{\neq} h(x_1,\dots,x_r,\mathcal{P})\right] \\
&= \int_{\mathbf{S}}\cdots\int_{\mathbf{S}} \mathbb{E}\left[\, h(x_1,\dots,x_r,\mathcal{P}\cup\{x_1,\dots,x_r\})\,\right]\rho(x_1)\cdots\rho(x_r)dx_1\cdots dx_r,
\end{aligned} \tag{4.1}$$

where the sum ranges over all pairwise distinct r-tuples of points of $\mathcal{P}$.

Equation (4.1) can be used to calculate $\mathrm{Var}[X]$ where $X = \sum_{p\in\mathcal{P}} \xi(p,\mathcal{P})$ for some score function $\xi(p,\mathcal{P}\cup\{p\})$ on $\mathbf{S}$ taking values in $\{0,1\}$. With $c^{\xi}(x_1,x_2) = c^{\xi}(x_1,x_2;\mathcal{P})$ (cf. (3.1)), the definition of the variance together with (4.1) yield:

$$\begin{aligned}
\mathrm{Var}[X] &= \int_{\mathbf{S}}\int_{\mathbf{S}} c^{\xi}(x_1,x_2)\rho(x_1)\rho(x_2)dx_1dx_2 + \int_{\mathbf{S}} \mathbb{E}\left[\,\xi(x,\mathcal{P}\cup\{x\})^2\,\right]\rho(x)dx \\
&= \int_{\mathbf{S}}\int_{\mathbf{S}} c^{\xi}(x_1,x_2)\rho(x_1)\rho(x_2)dx_1dx_2 + \mathbb{E}[X],
\end{aligned} \tag{4.2}$$

where the last equality holds since $\xi^2 = \xi$ (in fact the first equality does not require that the score function is an indicator random variable, but this is the case throughout our paper).

4.1 Proof of expectation asymptotics (1.3)

Lemma 4.1. *Uniformly for $p \in D([0,H])$ we have*

$$\mu_\alpha(B(p)) = \gamma e^{y(p)/2} + o(1). \tag{4.3}$$

Proof. We use the inclusions in (2.11). By Lemma 2.1(iii) we may put $C := C(n) := 5\log R$ and $\varepsilon = O(e^{-5\log R})$. Now (2.11) yields:

$$\begin{aligned}
&\mu_\alpha(B(p)) \le \mu_\alpha(B^+(p)) \\
&= 2\cdot 1_{+\varepsilon}\cdot\beta\int_0^{R-y(p)-C} e^{\frac12(y(p)+y)}e^{-\alpha y}dy + \beta\cdot 2I_n\int_{R-y(p)-C}^{R} e^{-\alpha y}dy \\
&= 2\cdot 1_{+\varepsilon}\cdot\beta e^{\frac{y(p)}{2}}\int_0^{R-y(p)-C} e^{(\frac12-\alpha)y}dy + \frac{\beta\pi}{\alpha}e^{\frac{R}{2}}\left(e^{-\alpha(R-y(p)-C)} - e^{-\alpha R}\right) \\
&= 1_{+\varepsilon}\cdot\gamma e^{\frac{y(p)}{2}}\left(1 - e^{(\frac12-\alpha)(R-y(p)-C)}\right) + \frac{\beta\pi}{\alpha}e^{\frac{R}{2}}\left(e^{-\alpha(R-y(p)-C)} - e^{-\alpha R}\right).
\end{aligned} \tag{4.4}$$

where we recall that $2I_n = \pi e^{R/2}$ and where γ and β are related via $\gamma := 4\beta/(2\alpha-1)$. Recalling $\alpha \in (1/2,\infty)$, $y(p) \in (0,H)$, $H = o(R)$, and $C = o(R)$, it follows that uniformly over all such p

$$e^{\frac{y(p)}{2}+(\frac12-\alpha)(R-y(p)-C)} = o(1), \quad e^{\frac{R}{2}-\alpha(R-y(p)-C)} = o(1), \text{ and } e^{\frac{R}{2}-\alpha R} = o(1).$$

We conclude that $\mu_\alpha(B(p)) \le 1_{+\varepsilon}\cdot\gamma e^{\frac{y(p)}{2}} + o(1)$. (Recall that $1_{+\varepsilon} := 1+\varepsilon$ and $\varepsilon = O(e^{-5\log R})$.) Notice that $y(p)/2 \in [0, 2\log R)$ since $y(p) \in [0,H)$. Thus $\varepsilon\cdot e^{y(p)/2} =$

$o(e^{-\frac{1}{2}\log R})$, uniformly over all such p, whereby

$$\mu_\alpha(B(p)) \le \gamma e^{\frac{y(p)}{2}} + o(1). \tag{4.5}$$

To obtain a lower bound, we use the first inclusion in (2.11):

$$\begin{aligned}\mu_\alpha(B(p)) &\ge \mu_\alpha(B^-(p))\\ &= 2\cdot 1_{-\varepsilon}\cdot\beta\int_0^{R-y(p)-C} e^{\frac{1}{2}(y(p)+y)}e^{-\alpha y}dy\\ &= 2\cdot 1_{-\varepsilon}\cdot\beta e^{\frac{y(p)}{2}}\int_0^{R-y(p)-C} e^{(\frac{1}{2}-\alpha)y}dy\\ &= 1_{-\varepsilon}\cdot\gamma e^{\frac{y(p)}{2}}\left(1-e^{(\frac{1}{2}-\alpha)(R-y(p)-C)}\right).\end{aligned}$$

Using again that $\varepsilon = O(e^{-5\log R})$ and $y(p)/2 \in [0, 2\log R)$ we deduce a matching lower bound:

$$\mu_\alpha(B(p)) \ge \gamma e^{\frac{y(p)}{2}} + o(1). \tag{4.6}$$

Combining (4.5) with (4.6) shows (4.3), as desired. □

We now establish expectation asymptotics (1.3). Since

$$\mathbb{E}[\tilde{\xi}^{iso}(p, (\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p\})] = \exp(-\mu_\alpha(B(p)))$$

it follows from (4.3) that

$$\mathbb{E}[\tilde{\xi}^{iso}(p, (\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p\})] \sim \exp\left(-\gamma e^{\frac{y(p)}{2}}\right) \tag{4.7}$$

uniformly over all $p \in D([0, H])$. The Campbell-Mecke formula (4.1) and (4.7) yield

$$\begin{aligned}\mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha\cap D)] &= \beta\int_{-I_n}^{I_n}\int_0^H \mathbb{E}[\tilde{\xi}^{iso}((x,y), (\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x,y)\})]e^{-\alpha y}dxdy\\ &\sim \beta\cdot 2I_n\int_0^H e^{-\gamma e^{y/2}}e^{-\alpha y}dy \sim \beta\cdot\pi e^{R/2}\int_0^\infty e^{-\gamma e^{y/2}}e^{-\alpha y}dy\\ &= 2\alpha n\int_0^\infty e^{-\gamma e^{y/2}}e^{-\alpha y}dy,\end{aligned}$$

since $\beta = 2\nu\alpha/\pi$. By Lemmas 2.5 and 2.6, we deduce (1.3) as desired. □

4.2 Upperbounding $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha\cap D)]$

We derive the asymptotics for $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha\cap D)]$ in two steps. First, in this subsection we provide an upper bound via the Poincaré inequality. It turns out that this is tight up to multiplicative constants. The next subsection provides a matching lower bound for $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha\cap D)]$ using the geometry of the intersection of hyperbolic balls obtained in Section 3.

Let F be a functional on a space $\mathbf{S}$ hosting a Poisson process $\mathcal{P}$ of intensity measure λ. For a point $p \in \mathbf{S}$ we define the *first order linear operator* $\nabla_p F := F(\mathcal{P}\cup\{p\}) - F(\mathcal{P})$. Then the Poincaré inequality (inequality (1.1) in [17]) states that

$$\mathrm{Var}F \le \mathbb{E}\left[\int_{\mathbf{S}}(\nabla_p F(\mathcal{P}))^2\lambda(dp)\right]. \tag{4.8}$$

We now put

$$F := \tilde{S}_H^{iso}(\tilde{P}_\alpha \cap D),\ \mathbf{S} := D,\ \mathcal{P} := \tilde{\mathcal{P}}_\alpha \cap D \text{ and } \lambda := \mu_\alpha.$$

Note that $|\nabla_p F|$ is stochastically dominated from above by the number of points of $(\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p\}$ in $B(p)$. By Lemma 4.1 we deduce that $|B(p) \cap (\tilde{\mathcal{P}}_\alpha \cap D)|$ is Poisson-distributed with parameter equal to

$$\mu_\alpha(B(p)) \leq \gamma' e^{\frac{y(p)}{2}},$$

uniformly over all $p \in D$, for some constant $\gamma' > 0$. Thus,

$$\mathbb{E}\left[\,(\nabla_p F(\mathcal{P}))^2\,\right] = O(1) \cdot e^{y(p)}$$

which implies that

$$\mathbb{E}\left[\int_{\mathbf{S}} (\nabla_p F(\mathcal{P}))^2 \lambda(dp)\right] \leq \int_{\mathbf{S}} \mathbb{E}\left[\,(\nabla_p F(\mathcal{P}))^2\,\right] \lambda(dp) = O(1) \cdot n \int_0^R e^{(1-\alpha)y} dy.$$

In other words, evaluating the integral and using $e^{(1-\alpha)R} = n^{2(1-\alpha)}$ in the range $\alpha \in (1/2, 1)$, we get

$$\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{P}_\alpha \cap D)] = \begin{cases} O(n^{3-2\alpha}) & \alpha \in (\frac{1}{2}, 1) \\ O(nR) & \alpha = 1 \\ O(n) & \alpha \in (1, \infty) \end{cases}. \tag{4.9}$$

4.3 Lowerbounding $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{P}_\alpha \cap D)]$

Recall the definition of $c^{\tilde{\xi}^{iso}}(p_1, p_2)$ at (3.1). By (4.2), we have $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] = V_1 + V_2$ with

$$V_1 := \beta^2 \int_{-I_n}^{I_n} \int_0^H \int_{-I_n}^{I_n} \int_0^H c^{\tilde{\xi}^{iso}}((x_1, y_1), (x_2, y_2)) e^{-\alpha y_1} e^{-\alpha y_2} dy_2 dx_2 dy_1 dx_1$$

and

$$V_2 := \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] = \mathbb{E}\left[\sum_{p \in \tilde{\mathcal{P}}_\alpha \cap D([0,H])} \tilde{\xi}^{iso}(p, \tilde{\mathcal{P}}_\alpha \cap D)\right].$$

Since $V_2 \geq 0$ it suffices to provide a lower bound on V_1 matching that at (4.9). Put

$$\begin{aligned} &\mathrm{Cov}^-(x_1, y_1, y_2, t) \\ &:= \mathbb{E}[\tilde{\xi}^{iso}(p_1, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p_1\})]\mathbb{E}[\tilde{\xi}^{iso}(p_2, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p_2\})] \left(\exp\left(\mu_\alpha(\mathrm{S}^-_{p_1 p_2})\right) - 1\right), \end{aligned}$$

where $x(p_1) = x_1$, $y(p_1) = y_1$, $y(p_2) = y_2$ and $|x(p_2) - x(p_1)|_\Phi = t$. By symmetry, it suffices to consider the case where $y_2 \leq y_1$ and $x(p_2) >_\Phi x(p_1)$. Indeed, this is one of four possible cases regarding the relative positions of $p_1, p_2 \in D([0, H])$ and it accounts for the pre-factor 4 appearing in front of our upcoming lower bounds.

Note that if $x(p_1) = 0$ and $x(p_2) - x(p_1) >_\Phi 0$, then in fact $|x(p_2) - x(p_1)|_\Phi = x(p_2) - x(p_1)$. Considering points p_1 and p_2 such that $x(p_1) = 0$, $y(p_1) = y_1$, $y(p_2) = y_2$ and $x(p_2) - x(p_1) = t \in (0, I_n)$ we have by (3.3) the bound $c^{\tilde{\xi}^{iso}}(p_1, p_2) \geq \mathrm{Cov}^-(x_1, y_1, y_2, t)$.

Therefore,

$$V_1 \geq V_1^- := 4\beta^2 \int_{-I_n}^{I_n} \int_0^H \int_0^{y_1} \int_0^{I_n} \mathrm{Cov}^-(x_1, y_1, y_2, t) e^{-\alpha y_2} e^{-\alpha y_1} dt dy_2 dy_1 dx_1.$$

We will drop the $-$ sign and write $\mathrm{Cov}(x_1, y_1, y_2, t) := \mathrm{Cov}^-(x_1, y_1, y_2, t)$. Note that $\mathrm{Cov}(x_1, y_1, y_2, t)$ does not depend on x_1 as the Poisson process $\tilde{\mathcal{P}}_\alpha \cap D$ is stationary with respect to the spatial x-coordinate. Therefore, we can write

$$V_1^- = 8\beta^2 I_n \int_0^H \int_0^{y_1} \int_0^{I_n} \mathrm{Cov}(0, y_1, y_2, t) e^{-\alpha y_2} e^{-\alpha y_1} dt dy_2 dy_1.$$

We change variables and, as before, put $Y_i = e^{y_i/2}, i = 1, 2$. Hence, $2dY_i = e^{y_i/2} dy_i$ and $dy_i = 2Y_i^{-1} dY_i$. Also, $e^{-\alpha y_i} = Y_i^{-2\alpha}$. Moreover, as y_1 ranges from 0 to H, the variable Y_1 ranges from 1 to $e^{H/2} = e^{2\log R} = R^2$. Thus,

$$V_1^- = 32\beta^2 I_n \int_1^{R^2} \int_1^{Y_1} \int_0^{I_n} \mathrm{Cov}(0, Y_1, Y_2, t) \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dt dY_2 dY_1. \tag{4.10}$$

To simplify notation we shall write

$$e^{h/2} := 1_{-\varepsilon} \cdot e^{h/2}.$$

This amounts to transferring the term $1_{-\varepsilon}$ inside h changing the constant C to $C - 2\ln 1_{-\varepsilon}$. It will make no difference.

Let us observe that

$$\int_0^{I_n} \mathrm{Cov}(0, Y_1, Y_2, t) dt \geq J_1 + J_2, \tag{4.11}$$

where, recalling Y_{12} defined at (3.4), we have

$$J_1 := \int_0^{Y_{12}} \mathrm{Cov}(0, Y_1, Y_2, t) dt, \text{ and } J_2 := \int_{Y_{12} \vee 1_{+\varepsilon} \cdot (Y_1 + Y_2)}^{e^{h/2}(Y_1 - Y_2)} \mathrm{Cov}(0, Y_1, Y_2, t) dt.$$

By (3.5) the covariance is negative only when t belongs to the range covered by the J_1 integral. For $t \in (Y_{12}, I_n]$, the covariance is positive. Thus, for the range $(Y_{12}, I_n]$ it suffices to use the subset given by the smaller range of J_2 which in turn is covered by Lemma 3.3(i).

For $k = 1, 2$ we set

$$L_k := \int_1^{R^2} \int_1^{Y_1} J_k \alpha^2 Y_1^{-2\alpha-1} Y_2^{-2\alpha-1} dY_2 dY_1. \tag{4.12}$$

We now show that $|L_1| = O(1)$ for all $\alpha \in (1/2, \infty)$, whereas we derive lower bounds on L_2 which match the upper bounds in (4.9).

4.3.1 Calculating integral L_1

Formula (4.7) and the second part of the covariance formula (3.3) give for all $t \in [0, Y_{12}]$

$$\mathrm{Cov}(0, Y_1, Y_2, t) \sim -\exp(-\gamma e^{y_1/2}) \exp(-\gamma e^{y_2/2}) = -\exp\left(-\gamma\left(Y_1 + Y_2\right)\right), \tag{4.13}$$

uniformly for all $Y_2 \leq Y_1 \leq R^2$, whereby

$$J_1 = \int_0^{Y_{12}} \mathrm{Cov}(0, Y_1, Y_2, t) dt \sim -Y_{12} \exp\left(-\gamma\left(Y_1 + Y_2\right)\right).$$

Therefore, since $Y_{12} = Y_1 Y_2(1 + o(1))$ by (3.4), we eventually obtain:

$$L_1 = \int_1^{R^2} \int_1^{Y_1} J_1 \frac{2}{Y_1^{2\alpha+1}} \frac{2}{Y_2^{2\alpha+1}} dY_2 dY_1 \sim -4 \int_1^{R^2} \int_1^{Y_1} \frac{e^{-\gamma Y_1}}{Y_1^{2\alpha}} \frac{e^{-\gamma Y_2}}{Y_2^{2\alpha}} dY_2 dY_1.$$

Hence, we deduce for all $\alpha \in (1/2, \infty)$ that $|L_1| = O(1)$.

4.3.2 The lower bound on integral L_2

Let $s := \max\{5\alpha/(2\alpha - 1), 5\}$ and $r := R^s e^{-h/2}$. Given the domain $W := \{(Y_1, Y_2) \;:\; 1 \leq Y_2 \leq Y_1 \leq R^2\}$ consider the sub-domain

$$W' := \{(Y_1, Y_2) \in W \;:\; Y_1 - Y_2 \geq r\}.$$

It suffices to consider the contribution to L_2 that comes from the domain W'. That is, we will bound from below the integral

$$L_2' := \int_{W'} J_2 \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1.$$

Combining (3.3), Lemma 3.3, (4.7) and recalling $\eta(Y_2) := \gamma Y_2 e^{(1/2-\alpha)h}$ we obtain

$$\begin{aligned}
&\mathrm{Cov}(0, Y_1, Y_2, t) \\
&= \mathbb{E}[\tilde{\xi}^{iso}(p_1, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p_1\})]\mathbb{E}[\tilde{\xi}^{iso}(p_2, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p_2\})] \left(\exp\left(\kappa t^{1-2\alpha} - \eta(Y_2)\right) - 1\right) \\
&\sim \exp(-\gamma(Y_1 + Y_2)) \left(\exp\left(\kappa t^{1-2\alpha} - \eta(Y_2)\right) - 1\right).
\end{aligned}$$

For simplicity, we set $t_1 := Y_{12} \vee 1_{+\epsilon}(Y_1 + Y_2)$ and $t_2 := e^{h/2}(Y_1 - Y_2)$, whereby

$$\begin{aligned}
J_2 &= \int_{t_1}^{t_2} \mathrm{Cov}(0, Y_1, Y_2, t) dt \\
&\sim \exp(-\gamma(Y_1 + Y_2)) \int_{t_1}^{t_2} \left(\exp\left(\kappa t^{1-2\alpha} - \eta(Y_2)\right) - 1\right) dt.
\end{aligned} \tag{4.14}$$

Consider the integral in (4.14) when $(Y_1, Y_2) \in W'$. The following lemma shows that its value changes radically as α crosses 1. The regimes for this lemma induce three regimes for L_2'.

Lemma 4.2. *There is a $\delta > 0$ such that for any n sufficiently large, any $(Y_1, Y_2) \in W'$ and $\alpha \in (1/2, \infty)$, we have*

$$\int_{t_1}^{t_2} t^{1-2\alpha} dt \geq \begin{cases} 2\alpha(1+\delta) t_2^{2(1-\alpha)} & \alpha \in (\frac{1}{2}, 1) \\ \frac{1}{5} \ln t_2 & \alpha = 1 \\ \frac{1}{4(\alpha-1)} t_1^{2(1-\alpha)} & \alpha \in (1, \infty) \end{cases}.$$

Proof. Elementary integration gives the three different cases:

$$\int_{t_1}^{t_2} t^{1-2\alpha} dt = \begin{cases} \frac{1}{2(1-\alpha)} \left(t_2^{2(1-\alpha)} - t_1^{2(1-\alpha)}\right) & \alpha \in (\frac{1}{2}, 1) \\ \ln \frac{t_2}{t_1} & \alpha = 1 \\ \frac{1}{2(\alpha-1)} \left(t_1^{2(1-\alpha)} - t_2^{2(1-\alpha)}\right) & \alpha \in (1, \infty) \end{cases}.$$

By definition, for any $(Y_1, Y_2) \in W'$, we have that $t_2 > t_1 + R^5/2$, whereas $t_1 = Y_1 Y_2 \vee (Y_1 + Y_2) \leq R^4$, for n sufficiently large. Thus, $t_2/t_1 \geq R/2 \to \infty$ as $n \to \infty$. These facts imply that if $\alpha \in (1/2, 1)$, then for some $\delta \in (0, \infty)$

$$t_2^{2(1-\alpha)} - t_1^{2(1-\alpha)} \overset{4\alpha(1-\alpha)<1}{>} (1+\delta)4\alpha(1-\alpha)t_2^{2(1-\alpha)},$$

whereas if $\alpha = 1$, then

$$\ln \frac{t_2}{t_1} = \ln t_2 \left(1 - \frac{\ln t_1}{\ln t_2}\right) > \ln t_2 \left(1 - \frac{\ln R^4}{\ln R^5}\right) = \frac{1}{5} \ln t_2.$$

Finally if $\alpha \in (1, \infty)$, then $t_1^{2(1-\alpha)} - t_2^{2(1-\alpha)} > \frac{1}{2} t_1^{2(1-\alpha)}$, for n sufficiently large. The lemma follows. □

For $(Y_1, Y_2) \in W'$ and n sufficiently large we have $Y_{12} \vee 1_{+\varepsilon} \cdot (Y_1 + Y_2) \le R^4$. In this domain the definition of s gives

$$e^{h/2}(Y_1 - Y_2) - (Y_{12} \vee 1_{-\varepsilon} \cdot (Y_1 + Y_2)) \ge R^5 - R^4 > R^5/2. \tag{4.15}$$

4.3.3 Three regimes for integral L_2'

4.3.3 (a) The integral L_2', $\alpha \in (1/2, 1)$. Recalling the definitions of t_1 and t_2 and appealing to Lemma 4.2, we deduce the following lower bound

$$\begin{aligned}\int_{t_1}^{t_2} \left(\exp\left(\kappa t^{1-2\alpha} - \eta(Y_2)\right) - 1\right) dt &\ge \int_{t_1}^{t_2} \left(\kappa t^{1-2\alpha} - \eta(Y_2)\right) dt \\ &\ge 2\alpha(1+\delta)\kappa e^{h(1-\alpha)}(Y_1 - Y_2)^{2(1-\alpha)} - \gamma(Y_1 - Y_2)Y_2 e^{h(1-\alpha)}.\end{aligned}$$

Recall from (3.18) that $\kappa := 1_{-\varepsilon} \cdot \frac{\gamma}{4\alpha}((Y_1 + Y_2)^{2\alpha} - (Y_1 - Y_2)^{2\alpha})$. For any $\varepsilon > 0$ sufficiently small (in terms of δ), we have $1_{-\varepsilon} \cdot (1+\delta) > 1 + \delta/2$. Therefore,

$$\begin{aligned}&2\alpha(1 + \frac{\delta}{2})\kappa(Y_1 - Y_2)^{2(1-\alpha)} - \gamma(Y_1 - Y_2)Y_2 \\ &\ge \gamma(Y_1 - Y_2)\left(\frac{1+\delta/2}{2}\left(\left(\frac{Y_1 + Y_2}{Y_1 - Y_2}\right)^{2\alpha}(Y_1 - Y_2) - (Y_1 - Y_2)\right) - Y_2\right) \\ &\ge \gamma(Y_1 - Y_2)\left(\frac{1+\delta/2}{2}\left(\left(\frac{Y_1 + Y_2}{Y_1 - Y_2}\right)(Y_1 - Y_2) - (Y_1 - Y_2)\right) - Y_2\right) \\ &= \gamma(Y_1 - Y_2)\left(\frac{1+\delta/2}{2}\left(Y_1 + Y_2 - (Y_1 - Y_2)\right) - Y_2\right) \\ &= \gamma(Y_1 - Y_2)\left(\frac{2(1+\delta/2)}{2}Y_2 - Y_2\right) = \frac{\gamma\delta}{2}(Y_1 - Y_2)Y_2 := Q_1(Y_1, Y_2) > 0.\end{aligned} \tag{4.16}$$

We then deduce that

$$\frac{J_2}{e^{h(1-\alpha)}\exp\left(-\gamma(Y_1 + Y_2)\right)} \ge Q_1(Y_1, Y_2). \tag{4.17}$$

Recall that $h = R - y(p_1) - C + 2\ln 1_{-\varepsilon}$. This implies that

$$e^{h(1-\alpha)} = e^{R(1-\alpha)} e^{-(y(p_1) + C - 2\ln 1_{-\varepsilon})(1-\alpha)} \overset{Y_1 = e^{y(p_1)/2}}{=} \Omega(1) \cdot e^{R(1-\alpha)} \cdot Y_1^{-2(1-\alpha)}.$$

The above bounds imply that

$$L_2' = \Omega(1) \cdot e^{R(1-\alpha)} \int_{W'} \exp\left(-\gamma(Y_1+Y_2)\right) Q_1(Y_1,Y_2) \frac{Y_1^{2(1-\alpha)}}{Y_1^{-2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1.$$

Thus, for $\alpha \in (1/2, 1)$ we have

$$L_2 \geq L_2' = \Omega\left(e^{R(1-\alpha)}\right). \tag{4.18}$$

4.3.3 (b) The integral L_2', $\alpha = 1$. Note first that

$$\begin{aligned}\kappa &= 1_{-\varepsilon} \cdot \frac{\gamma}{4\alpha}((Y_1+Y_2)^{2\alpha} - (Y_1-Y_2)^{2\alpha}) = 1_{-\varepsilon} \cdot \frac{\gamma}{4}((Y_1+Y_2)^2 - (Y_1-Y_2)^2) \\ &= 1_{-\varepsilon} \cdot \gamma Y_1 Y_2.\end{aligned} \tag{4.19}$$

In this case, the integral in (4.14) is bounded below as follows:

$$\begin{aligned}
&\int_{t_1}^{t_2} (e^{\frac{\kappa}{t} - \eta(Y_2)} - 1) dt \geq \int_{t_1}^{t_2} (\frac{\kappa}{t} - \eta(Y_2)) dt \\
&\geq \frac{\kappa}{5} \ln\left(e^{\frac{h}{2}}(Y_1 - Y_2)\right) - \gamma Y_2 e^{-\frac{h}{2}} \int_0^{e^{\frac{h}{2}}(Y_1-Y_2)} dt \\
&= \frac{\kappa}{5} \ln\left(e^{\frac{h}{2}}(Y_1 - Y_2)\right) - \gamma Y_2 (Y_1 - Y_2) \\
&= \kappa \frac{h}{10} + \frac{\kappa}{5} \ln(Y_1 - Y_2) - \gamma Y_2 (Y_1 - Y_2) \\
&\overset{(4.19), Y_1 \geq Y_2}{\geq} 1_{-\varepsilon} \cdot \frac{\gamma}{10} Y_1 Y_2 \frac{h}{2} + 1_{-\varepsilon} \cdot \frac{\gamma}{5} Y_1 Y_2 \ln(Y_1 - Y_2) - \gamma Y_1 Y_2 \\
&= 1_{-\varepsilon} \cdot \frac{\gamma}{5} Y_1 Y_2 \left(\frac{h}{4} + \ln(Y_1 - Y_2) - 5 \cdot 1_{-\varepsilon}^{-1}\right) \\
&\overset{\varepsilon < 1/8}{\geq} 1_{-\varepsilon} \cdot \frac{\gamma}{5} Y_1 Y_2 \left(\frac{h}{4} + \ln(Y_1 - Y_2) - 6\right)
\end{aligned}$$

where the last inequality holds for n sufficiently large, if we put $\varepsilon = O(e^{-5 \log R})$ (cf. Lemma 2.1(iii)). In particular, if we let $(Y_1, Y_2) \in W'' := \{(Y_1, Y_2) \in W' \,:\, Y_1 - Y_2 > e^6\}$, then

$$\int_{t_1}^{t_2} (e^{\frac{\kappa}{t} - \eta(Y_2)} - 1) dt \geq 1_{-\varepsilon} \cdot \gamma Y_1 Y_2 \frac{h}{20}.$$

Also, recall that $h := R - y(p_1) - C + 2 \ln 1_{-\varepsilon}$. Since $y(p_1) < H$, it follows that for n sufficiently large we have $h \in (R/2, R]$. Combining this observation with the above lower bound, (4.14) yields

$$J_2 \geq 1_{-\varepsilon} \cdot \frac{h}{2} e^{-\gamma(Y_1+Y_2)} \frac{\gamma}{20} Y_1 Y_2 \geq \frac{R\gamma}{80} e^{-\gamma(Y_1+Y_2)} Y_1 Y_2.$$

Therefore

$$\begin{aligned}
L_2' &= \int_{W'} J_2 \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1 \geq \int_{W''} J_2 \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1 \\
&\geq \frac{R\gamma}{80} \int_e^{R^2} \int_1^{Y_1 - e} e^{-\gamma(Y_1+Y_2)} Y_1 Y_2 \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1 = \Omega(R).
\end{aligned}$$

4.3.3 (c) The integral L_2', $\alpha \in (1,\infty)$. Recall by (4.15) that for any $(Y_1,Y_2) \in W'$, we have

$$e^{h/2}(Y_1 - Y_2) > Y_{12} \vee 1_{-\varepsilon} \cdot (Y_1 + Y_2) + R^s,$$

for some $s = s(\alpha) \geq 5$. Since $Y_{12} \vee 1_{-\varepsilon} \cdot (Y_1+Y_2) \leq R^4$ when n is sufficiently large, we get

$$e^{h/2}(Y_1 - Y_2) > 2(Y_{12} \vee 1_{-\varepsilon} \cdot (Y_1 + Y_2)).$$

Using the third case in Lemma 4.2, the integral in (4.14) is bounded from below as follows:

$$\begin{aligned}
&\int_{t_1}^{t_2} \left(\exp\left(\kappa t^{1-2\alpha} - \eta(Y_2)\right) - 1\right) dt \geq \int_{t_1}^{t_2} \left(\kappa t^{1-2\alpha} - \eta(Y_2)\right) dt \\
&\geq \frac{\kappa}{4(\alpha-1)} (Y_{12} \vee (Y_1+Y_2))^{2(1-\alpha)} - \gamma(Y_1 - Y_2) Y_2 e^{h(1-\alpha)} \\
&\geq \frac{\kappa}{4(\alpha-1)} (Y_{12} \vee (Y_1+Y_2))^{2(1-\alpha)} - \gamma R^4 e^{h(1-\alpha)} \\
&= \frac{\kappa}{4(\alpha-1)} (Y_{12} \vee (Y_1+Y_2))^{2(1-\alpha)} - o(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
L_2' &= \int_{W'} J_2 \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1 \\
&\geq \frac{1}{4(\alpha-1)} \int_{W'} \kappa \left[(Y_{12} \vee (1_{-\varepsilon} \cdot (Y_1+Y_2))\right]^{2(1-\alpha)} e^{-\gamma(Y_1+Y_2)} \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1 \\
&\quad - o(1) \int_{W'} e^{-\gamma(Y_1+Y_2)} \frac{1}{Y_1^{2\alpha+1}} \frac{1}{Y_2^{2\alpha+1}} dY_2 dY_1 = \Omega(1).
\end{aligned}$$

4.4 Proof of growth rates for $\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})]$

We have now estimated the two summands that bound the main term of $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{P}_\alpha \cap D)]$ from below. Our findings are summarised as follows:

$$|L_1| = \Theta(1), \text{ and } L_2 = \begin{cases} \Omega\left(e^{R(1-\alpha)}\right) & \alpha \in (\frac{1}{2}, 1) \\ \Omega(R) & \alpha = 1 \\ \Omega(1) & \alpha \in (1,\infty) \end{cases}. \tag{4.20}$$

By (4.10) we have $V_1 = \Omega\left(I_n(L_1+L_2)\right)$, and $2I_n = \pi e^{R/2} = \Theta(n)$. Therefore,

$$V_1 = \begin{cases} \Omega\left(n^{1+2(1-\alpha)}\right) & \alpha \in (\frac{1}{2}, 1) \\ \Omega(nR) & \alpha = 1 \\ \Omega(n) & \alpha \in (1,\infty) \end{cases}.$$

As $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{P}_\alpha \cap D)] \geq V_1$, we finally deduce that

$$\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{P}_\alpha \cap D)] = \begin{cases} \Omega\left(n^{3-2\alpha}\right) & \alpha \in (\frac{1}{2}, 1) \\ \Omega(nR) & \alpha = 1 \\ \Omega(n) & \alpha \in (1,\infty) \end{cases}. \tag{4.21}$$

Combining (4.9) and (4.21), and recalling Corollary 2.7, we thus establish the desired growth rates for $\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})]$, completing the proof of (1.4).

5 Proof of Theorem 1.2

5.1 Proof of expectation asymptotics for $S^{ext}(\mathcal{P}_{\alpha,n})$

By Lemmas 2.5 and 2.6 it suffices to compute $\lim_{n\to\infty} n^{-1}\mathbb{E}[\tilde{S}_H^{ext}(\tilde{P}_\alpha \cap D)]$. Given a point $p \in D([0,H])$ we have

$$\begin{aligned}\mu_\alpha(B(p)\cap D([0,y(p)])) &\le \mu_\alpha(B^+(p)\cap D([0,y(p)]))\\ &= 2\cdot 1_{+\varepsilon}\cdot\beta\int_0^{y(p)} e^{\frac{y(p)}{2}+\frac{y}{2}}e^{-\alpha y}dy\\ &= 2\cdot 1_{+\varepsilon}\cdot\frac{2\beta}{2\alpha-1}e^{\frac{y(p)}{2}}(1-e^{(\frac12-\alpha)y(p)}).\end{aligned}$$

Similarly, we have the lower bound

$$\begin{aligned}\mu_\alpha(B(p)\cap D([0,y(p)])) &\ge \mu_\alpha(B^-(p)\cap D([0,y(p)]))\\ &= 2\cdot 1_{-\varepsilon}\cdot\frac{2\beta}{2\alpha-1}e^{\frac{y(p)}{2}}(1-e^{(\frac12-\alpha)y(p)}).\end{aligned}$$

Taking again $\varepsilon = O(e^{-5\log R})$, and $\gamma = 4\beta/(2\alpha-1)$, we then deduce that uniformly over all $p \in D([0,H])$

$$\mu_\alpha(B(p)\cap D([0,y(p)])) = \gamma\cdot e^{\frac{y(p)}{2}}(1-e^{(\frac12-\alpha)y(p)}) + o(1).$$

Therefore,

$$\mathbb{E}[\tilde{\xi}^{ext}(p,\tilde{\mathcal{P}}_{\alpha,D}\cup\{p\})] \sim \exp\left(-\gamma e^{\frac{y(p)}{2}}(1-e^{(\frac12-\alpha)y(p)})\right) \tag{5.1}$$

uniformly over all $p \in D([0,H])$. Hence, the Campbell-Mecke formula (4.1) yields

$$\begin{aligned}n^{-1}\mathbb{E}[\tilde{S}_H^{ext}(\tilde{P}_\alpha\cap D)] &= n^{-1}\beta\int_{-I_n}^{I_n}\int_0^H \mathbb{E}[\tilde{\xi}^{ext}((x,y),\tilde{\mathcal{P}}_{\alpha,D}\cup\{(x,y)\})]e^{-\alpha y}dxdy\\ &\sim \beta\cdot 2I_n n^{-1}\int_0^H e^{-\gamma e^{y/2}(1-e^{(\frac12-\alpha)y})}e^{-\alpha y}dy\\ &\sim \beta\cdot\pi e^{R/2}n^{-1}\int_0^\infty e^{-\gamma e^{y/2}(1-e^{(\frac12-\alpha)y})}e^{-\alpha y}dy\\ &= 2\alpha\int_0^\infty e^{-\gamma e^{y/2}(1-e^{(\frac12-\alpha)y})}e^{-\alpha y}dy := \mu \end{aligned}\tag{5.2}$$

as desired.

5.2 Proof of variance asymptotics for $S^{ext}(\mathcal{P}_{\alpha,n})$

The determination of variance asymptotics for $S^{ext}(\mathcal{P}_{\alpha,n})$ is handled by extending existing stabilization methods. We show that when the constants describing the tail behavior of the stabilization radius at a point p are allowed to grow exponentially fast with the height of p, as at (5.5) below, then one may nonetheless establish explicit variance asymptotics as $n\to\infty$, as shown in the analysis between (5.12) and (5.14) below.

We first require several auxiliary lemmas. For all $r > 0$ and $p := (x(p), y(p)) \in \mathbb{R}\times\mathbb{R}^+$ we let $B(p,r)$ denote the closed Euclidean ball of radius r centered at p.

The identity (2.6) implies that for $y(p)\in[0,H]$ we have

$$D(p) = \{p' \ : \ y(p') \le y(p),\ |x(p')-x(p)|_\Phi < (1+\lambda_n(p',p))e^{\frac12(y(p')+y(p)-R)}\}.$$

We set $s_n := s_n(p) := 1 + \lambda_n(p', p)$, where $\lambda_n(p', p) = o(1)$.

Put $p_0 := (0, y_0)$. We let d_0 be the Euclidean distance between p_0 and the point in $(\tilde{\mathcal{P}}_\alpha \cap D) \cap D(p_0)$ which is closest to p_0. Set $d_n := \mathrm{diam}(D(p_0))$ and note $d_n \le 2 s_n e^{y_0}$. Now we put

$$R^\xi := R^\xi(p_0, (\tilde{\mathcal{P}}_\alpha \cap D)) := \begin{cases} d_0 & (\tilde{\mathcal{P}}_\alpha \cap D) \cap D(p_0) \neq \{p_0\} \\ d_n & (\tilde{\mathcal{P}}_\alpha \cap D) \cap D(p_0) = \{p_0\} \end{cases}.$$

The extremality status of p_0 depends only on the point set $(\tilde{\mathcal{P}}_\alpha \cap D) \cap B(p_0, R^\xi)$ in the sense that points outside this set will not modify $\tilde{\xi}^{ext}(p_0, \tilde{\mathcal{P}}_\alpha \cap D)$. In other words,

$$\tilde{\xi}^{ext}(p_0, \tilde{\mathcal{P}}_\alpha \cap D) = \tilde{\xi}^{ext}\left(p_0, (\tilde{\mathcal{P}}_\alpha \cap D) \cap B(p_0, R^\xi)\right),$$

that is to say that R^ξ is *a radius of stabilization* for $\xi := \tilde{\xi}^{ext}$.

Clearly, for $t \in (d_n, \infty)$, we have $\mathbb{P}(R^\xi(p_0, \tilde{\mathcal{P}}_\alpha \cap D) \ge t) = 0$. We seek to control $\mathbb{P}(R^\xi(p_0, \tilde{\mathcal{P}}_\alpha \cap D) > t), t \in [0, d_n]$, as a function of both t and the height parameter y_0. Put $c_0 := \sqrt{3}\beta(1 - e^{-8\alpha})/\alpha$ and set $\phi(t) := \min\{\alpha t/4, c_0\sqrt{t/3}\}$ for $t \in (0, \infty)$. We assert there is a constant c_1 such that for $y_0 \in [0, H]$ we have

$$\mathbb{P}(R^\xi(p_0, \tilde{\mathcal{P}}_\alpha \cap D) \ge t) \le c_1 \exp\left(\frac{\alpha y_0}{2}\right) \exp(-\phi(t)), \quad t \in [0, \infty). \tag{5.3}$$

We first compute lower bounds on the μ_α probability content of the regions

$$R_t(p_0) := B(p_0, t) \cap D(p_0), \ t \in [2y_0, d_n].$$

Lemma 5.1. *Let $y_0 \in [8, H]$. For all n large we have*

$$\mu_\alpha(R_t(p_0)) \ge c_0\sqrt{t}, \ t \in [2y_0, d_n].$$

Proof. First assume $t \in [2y_0, e^{\frac{y_0}{2}}]$. Notice that $B(p_0, t)$ meets the positive x-axis at points $\pm\sqrt{t^2 - y_0^2}$ which have absolute value exceeding $\sqrt{3}t/2$ when $t \ge 2y_0$. In other words, we have $B(p_0, t) \supseteq [-\sqrt{3}t/2, \sqrt{3}t/2] \times [0, y_0]$. We also have $D(p_0) \supseteq [-e^{\frac{y_0}{2}}, e^{\frac{y_0}{2}}] \times [0, y_0]$, implying $R_t(p_0) \supseteq [-\sqrt{3}t/2, \sqrt{3}t/2] \times [0, y_0]$. Consequently, we have

$$\mu_\alpha(R_t(p_0)) \ge \sqrt{3}t\beta \int_0^8 e^{-\alpha y'} dy' \ge c_0 t \ge c_0\sqrt{t}.$$

Now assume $t \in [e^{\frac{y_0}{2}}, d_n]$. Since y_0 exceeds 8, we have $e^{y_0/2} \ge 2y_0$. As above, it follows that

$$R_t(p_0) \supseteq \left[-\frac{\sqrt{3}e^{\frac{y_0}{2}}}{2}, \frac{\sqrt{3}e^{\frac{y_0}{2}}}{2}\right] \times [0, y_0].$$

Hence

$$\mu_\alpha(R_t(p_0)) \ge \int_0^{y_0} \sqrt{3}e^{\frac{y_0}{2}} \beta e^{-\alpha y'} dy' \ge c_0 e^{\frac{y_0}{2}} > c_0\sqrt{t/3},$$

where the last inequality uses $t \le d_n \le 2s_n e^{y_0}$. Hence, $\mu_\alpha(R_t(p_0)) \ge c_0\sqrt{t/3}$, as desired. □

Now we note that $R^\xi(p_0, \tilde{\mathcal{P}}_\alpha \cap D) \ge t$ iff $d_0 \ge t$ which happens if $(\tilde{\mathcal{P}}_\alpha \cap D) \cap D(p_0) = \{p_0\}$. Lemma 5.1 shows that for $y_0 \in [8, H]$

$$\mathbb{P}(R^\xi(p_0, \tilde{\mathcal{P}}_\alpha \cap D) \ge t) \le \exp(-\mu_\alpha(R_t(p_0))) \le \exp(-c_0\sqrt{t/3}), \quad t \in [2y_0, d_n].$$

For $t \in [0, 2y_0]$ we have the trivial bound

$$\mathbb{P}(R^{\xi}(p_0, \tilde{\mathcal{P}}_\alpha \cap D) \geq t) \leq \exp\left(\frac{\alpha y_0}{2}\right) \exp\left(-\frac{\alpha t}{4}\right).$$

Put $\phi(t) := \min\{\alpha t/4, c_0\sqrt{t/3}\}$ for $t \in (0, \infty)$. Summarizing the above we have shown for $y_0 \in [8, H]$ that

$$\mathbb{P}(R^{\xi}(p_0, \tilde{\mathcal{P}}_\alpha \cap D) \geq t) \leq \exp\left(\frac{\alpha y_0}{2}\right) \exp(-\phi(t)), \quad t \in [0, \infty). \tag{5.4}$$

It remains to show that (5.4) holds for $y_0 \in [0, 8]$. Recall that the left-hand side of (5.4) vanishes for t in the range $t \in [d_n, \infty)$. If $y_0 \in [0, 8]$ and $t \in [0, d_n]$ then $R^{\xi} \leq e^8$. For c large enough we thus have

$$\mathbb{P}(R^{\xi}(p_0, \tilde{\mathcal{P}}_\alpha \cap D) \geq t) \leq c \exp\left(\frac{\alpha y_0}{2}\right) \exp(-\phi(t)), \quad t \in [0, \infty). \tag{5.5}$$

Thus we have shown (5.3) as desired.

Recall that $2I_n = \pi e^{R/2}$. Since $\tilde{\mathcal{P}}_\alpha \cap D$ is stationary with respect to the spatial x-coordinate it follows that for all $(x(p), y(p)) \in [-I_n, I_n] \times [0, H]$, we have

$$\mathbb{P}(R^{\xi}((x(p), y(p)), \tilde{\mathcal{P}}_\alpha \cap D) \geq t) \leq c \exp\left(\frac{\alpha y(p)}{2}\right) \exp(-\phi(t)), \quad t \in [0, \infty). \tag{5.6}$$

Given the bound (5.6), we now find asymptotics for $n^{-1}\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]$. In the remainder of this section we continue to abbreviate $\tilde{\xi}^{ext}$ by ξ. In the next lemma, we will bound the covariance of ξ with respect to $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times [0, R]$, namely

$$\begin{aligned} &c^{\xi}((x_1, y_1), (x_2, y_2)) \\ &= \mathbb{E}\left[\xi((x_1, y_1), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_1, y_1), (x_2, y_2)\}) \cdot \xi((x_2, y_2), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_1, y_1), (x_2, y_2\})\right] \\ &\quad - \mathbb{E}\left[\xi((x_1, y_1), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_1, y_1)\})\right] \cdot \mathbb{E}\left[\xi((x_2, y_2), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_2, y_2)\})\right]. \end{aligned}$$

Lemma 5.2. *There is a constant $c \in (0, \infty)$ such that for all $(x_1, y_1), (x_2, y_2) \in (-I_n, I_n] \times [0, H]$, we have*

$$c^{\xi}((x_1, y_1), (x_2, y_2)) \leq c\left(\exp\left(\frac{\alpha y_1}{2}\right) + \exp\left(\frac{\alpha y_2}{2}\right)\right) \exp\left(-\phi\left(\frac{|x_1 - x_2|}{3}\right)\right). \tag{5.7}$$

Proof. Write $M := \max\{R^{\xi}((x_1, y_1), \tilde{\mathcal{P}}_\alpha \cap D), R^{\xi}((x_2, y_2), \tilde{\mathcal{P}}_\alpha \cap D)\}$. Put $r := |x_1 - x_2|/3$ and define $E := \{M \leq r\}$. Since ξ is bounded by 1, we note that

$$\mathbb{E}\left[\xi((x_1, y_1), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_1, y_1), (x_2, y_2)\}) \cdot \xi((x_2, y_2), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_1, y_1), (x_2, y_2)\})\right]$$

differs from

$$\mathbb{E}\left[\xi((x_1, y_1), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_1, y_1), (x_2, y_2)\}) \cdot \xi((x_2, y_2), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x_1, y_1), (x_2, y_2)\}) \times \mathbf{1}(E)\right]$$

by at most $\mathbb{P}(E)$.

Notice that

$$\begin{aligned}&\mathbb{E}[\xi((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})\\&\qquad\cdot\xi((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})\times\mathbf{1}(E)]\\&=\mathbb{E}[\xi((x_1,y_1),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})\cap B((x_1,y_1),r))\\&\qquad\cdot\xi((x_2,y_2),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_2,y_2)\})\cap B((x_2,y_2),r)))\times(1-\mathbf{1}(E^c))].\end{aligned}$$

We consequently obtain

$$\begin{aligned}&|\mathbb{E}[\xi((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})\cdot\xi((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})]\\&\qquad-\mathbb{E}[\xi((x_1,y_1),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})\cap B((x_1,y_1),r))\\&\quad\cdot\xi((x_2,y_2),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_2,y_2)\})\cap B((x_2,y_2),r)))]|\\&\le 2\mathbb{P}(E^c).\end{aligned}$$

By independence we have

$$\begin{aligned}&|\mathbb{E}[\xi((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})\cdot\xi((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})]\\&\qquad-\mathbb{E}[\xi((x_1,y_1),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})\cap B((x_1,y_1),r))]\\&\quad\cdot\mathbb{E}[\xi((x_2,y_2),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_2,y_2)\})\cap B((x_2,y_2),r)))]|\\&\le 2\mathbb{P}(E^c).\end{aligned}$$

Likewise

$$\mathbb{E}[\xi((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})]\mathbb{E}[\xi((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_2,y_2)\})]$$

differs from

$$\begin{aligned}&\mathbb{E}[\xi((x_1,y_1),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})\cap B((x_1,y_1),r))]\\&\quad\cdot\mathbb{E}[\xi((x_2,y_2),((\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_2,y_2)\})\cap B((x_2,y_2),r)))]\end{aligned}\tag{5.8}$$

by at most $2\mathbb{P}(E^c)$. We conclude that

$$c^\xi((x_1,y_1),(x_2,y_2))\le 4\mathbb{P}\left(M\ge\frac{|x_1-x_2|}{3}\right).$$

The bound (5.3) completes the proof. □

Recall that $\tilde{\mathcal{P}}_\alpha$ is the Poisson point process on $\mathbb{R}\times[0,\infty)$ with intensity measure μ_α as at (2.15). Next, define $c^\xi((x_1,y_1),(x_2,y_2);\tilde{\mathcal{P}}_\alpha)$ analogously as in the definition of $c^\xi((x_1,y_1),(x_2,y_2);\tilde{\mathcal{P}}_\alpha\cap D)$. Note that $(\tilde{\mathcal{P}}_\alpha\cap D)\xrightarrow{\mathcal{D}}\tilde{\mathcal{P}}_\alpha$ as $n\to\infty$. The next lemma follows from stabilization methods; see for example [4], [20].

Lemma 5.3. *We have*

$$\lim_{n\to\infty}c^\xi((x_1,y_1),(x_2,y_2);\tilde{\mathcal{P}}_\alpha\cap D)=c^\xi((x_1,y_1),(x_2,y_2);\tilde{\mathcal{P}}_\alpha).\tag{5.9}$$

Now we may finally prove the asserted variance asymptotics at (1.6). Put

$$\begin{aligned}\sigma^2&=2\alpha\int_0^\infty\mathbb{E}[\xi((0,y_1),\tilde{\mathcal{P}}_\alpha)]e^{-\alpha y_1}dy_1\\&\quad+\ 2\alpha\beta\int_0^\infty\int_{-\infty}^\infty\int_0^\infty c^\xi((0,y_1),(z,y_2);\tilde{\mathcal{P}}_\alpha)e^{-\alpha y_2}dy_2dze^{-\alpha y_1}dy_1.\end{aligned}\tag{5.10}$$

By (2.19), it is enough to show that

$$\lim_{n\to\infty}\frac{\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha\cap D)]}{n}=\sigma^2. \tag{5.11}$$

Proof of (1.6). We have by (4.2)

$$\begin{aligned}
&\frac{1}{n}\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha\cap D)]\\
&=\frac{\beta}{n}\int_{-I_n}^{I_n}\int_0^H \mathbb{E}[\xi((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})^2]e^{-\alpha y_1}dy_1dx_1\\
&\quad+\frac{\beta^2}{n}\int_{-I_n}^{I_n}\int_0^H\int_{-I_n}^{I_n}\int_0^H c^\xi((x_1,y_1),(x_2,y_2))e^{-\alpha y_2}dy_2dx_2e^{-\alpha y_1}dy_1dx_1.
\end{aligned}\tag{5.12}$$

The first integral in (5.12) reduces to $\beta\int_0^H\mathbb{E}[(\xi((0,y_1),\tilde{\mathcal{P}}_\alpha\cap D)^2]e^{-\alpha y_1}dy_1$ by translation invariance of ξ in the spatial x coordinate. The stabilization of ξ shows for all y_1 that

$$\lim_{n\to\infty}\mathbb{E}[\xi((0,y_1),\tilde{\mathcal{P}}_\alpha\cap D)]=\mathbb{E}[\xi((0,y_1),\tilde{\mathcal{P}}_\alpha)].$$

By the dominated convergence theorem and using $2I_n\beta/n=2\alpha$ and $\xi^2=\xi$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{\beta}{n}\int_{-I_n}^{I_n}\int_0^H&\mathbb{E}[\xi((x_1,y_1),\tilde{\mathcal{P}}_\alpha\cap D)^2]e^{-\alpha y_1}dy_1dx_1\\
&=2\alpha\int_0^\infty\mathbb{E}[\xi((0,y_1),\tilde{\mathcal{P}}_\alpha)]e^{-\alpha y_1}dy_1.
\end{aligned}\tag{5.13}$$

Now we turn to the second integral in (5.12). By translation invariance in the spatial coordinate we have

$$\begin{aligned}
&\frac{\beta^2}{n}\int_{-I_n}^{I_n}\int_0^H\int_{-I_n}^{I_n}\int_0^H c^\xi((x_1,y_1),(x_2,y_2);\tilde{\mathcal{P}}_\alpha\cap D)e^{-\alpha y_2}dy_2dx_2e^{-\alpha y_1}dy_1dx_1\\
&=\frac{\beta^2}{n}\int_{-I_n}^{I_n}\int_0^H\int_{-I_n}^{I_n}\int_0^H c^\xi((0,y_1),(x_2-x_1,y_2);\tilde{\mathcal{P}}_\alpha\cap D)e^{-\alpha y_2}dy_2dx_2e^{-\alpha y_1}dy_1dx_1\\
&=\frac{\beta^2}{n}\int_{-I_n}^{I_n}\int_0^H\int_{-I_n-x_1}^{I_n-x_1}\int_0^H c^\xi((0,y_1),(z,y_2);\tilde{\mathcal{P}}_\alpha\cap D)e^{-\alpha y_2}dy_2dze^{-\alpha y_1}dy_1dx_1.
\end{aligned}$$

Let $v:=\beta x_1/\alpha n$, $dv:=\beta/(\alpha n)dx_1$. Then since $I_n=\alpha n/\beta$ the above becomes

$$=\alpha\beta\int_{-1}^1\int_0^H\int_{-I_n(1-v)}^{I_n(1-v)}\int_0^H c^\xi((0,y_1),(z,y_2);\tilde{\mathcal{P}}_\alpha\cap D)e^{-\alpha y_2}dy_2dze^{-\alpha y_1}dy_1dv.$$

For every $v\in[-1,1]$ we have by (5.7) that $c^\xi((0,y_1),(z,y_2);\tilde{\mathcal{P}}_\alpha\cap D)e^{-\alpha y_2}e^{-\alpha y_1}$ is dominated by an integrable function of (y_1,y_2,z). It follows by the dominated convergence theorem that for every $v\in[-1,1]$ we have

$$\begin{aligned}
\lim_{n\to\infty}&\int_0^H\int_{-I_n(1-v)}^{I_n(1-v)}\int_0^H c^\xi((0,y_1),(z,y_2);\tilde{\mathcal{P}}_\alpha\cap D)e^{-\alpha y_2}dy_2dze^{-\alpha y_1}dy_1\\
&\overset{(5.9)}{=}\int_0^\infty\int_{-\infty}^\infty\int_0^\infty c^\xi((0,y_1),(z,y_2);\tilde{\mathcal{P}}_\alpha)e^{-\alpha y_2}dy_2dze^{-\alpha y_1}dy_1.
\end{aligned}$$

The second integral in (5.12) thus converges to

$$2\alpha\beta \int_0^\infty \int_{-\infty}^\infty \int_0^\infty c^\xi((0,y_1),(z,y_2);\tilde{\mathcal{P}}_\alpha) e^{-\alpha y_2} dy_2 dz e^{-\alpha y_1} dy_1. \tag{5.14}$$

Notice that σ^2 is the sum of (5.13) and (5.14). This completes the proof of Theorem 1.2. □

6 Proof of Theorem 1.3

To prove (1.7) and (1.8), we first assert that it suffices to prove central limit theorems for the random variables $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$ and $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$, defined at (2.16) and (2.17), respectively. We prove this assertion for $\tilde{S}_n := \tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ as the proof for $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$ is identical.

Set S_n to be $S^{iso}(\mathcal{P}_{\alpha,n}) := \tilde{S}^{iso}(\tilde{\mathcal{P}}_{\alpha,n})$. Recall that S_n is determined by the Poisson process $\tilde{\mathcal{P}}_{\alpha,n}$ on D defined at (2.10) whereas $\tilde{S}_n$ is determined by $\tilde{\mathcal{P}}_\alpha \cap D$ defined at (2.16). By Lemma 2.2, the intensities of these two processes differ by $\epsilon_n = O(n^{-2\alpha})$. We can couple these two processes using a sprinkling argument. Let $\hat{\mathcal{P}}$ be the Poisson process on D with intensity equal to $\lambda(x,y) := \min\{\beta e^{-\alpha y}, \beta e^{-\alpha y} + \epsilon_n\}$ at $(x,y) \in D$ – in other words the minimum of the intensities of $\tilde{\mathcal{P}}_\alpha \cap D$ and $\tilde{\mathcal{P}}_{\alpha,n}$. Now, we define two other independent processes on D: $\hat{\mathcal{P}}_1$ of intensity $\beta e^{-\alpha y} - \lambda(x,y)$ at (x,y) and $\hat{\mathcal{P}}_2$ of intensity $\beta e^{-\alpha y} + \epsilon_n - \lambda(x,y)$ at (x,y). The union of $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_1$ is distributed as $\tilde{\mathcal{P}}_\alpha \cap D$, whereas the union of $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}_2$ is distributed as $\tilde{\mathcal{P}}_{\alpha,n}$. We will use the symbols $\tilde{\mathcal{P}}_\alpha \cap D$ and $\tilde{\mathcal{P}}_{\alpha,n}$ to denote the copies of these processes in the coupling space. For each n, we may define the nth coupling space to be the product of the spaces on which $\hat{\mathcal{P}}$, $\hat{\mathcal{P}}_1$, and $\hat{\mathcal{P}}_2$ are all defined. Let $\widehat{\mathbb{P}}_n$ denote the product probability measure on the coupling space.

Thus, for any $\alpha \in (1/2, \infty)$

$$\widehat{\mathbb{P}}_n \left(\tilde{\mathcal{P}}_{\alpha,n} \neq \tilde{\mathcal{P}}_\alpha \cap D \right) = \widehat{\mathbb{P}}_n \left(\hat{\mathcal{P}}_1 \cup \hat{\mathcal{P}}_2 \neq \emptyset \right) = O(1) \cdot R \cdot n^{1-2\alpha} = o(1).$$

This implies that on the coupling space we have $\tilde{\mathcal{P}}_{\alpha,n} = \tilde{\mathcal{P}}_\alpha \cap D$ with probability $\to 1$ as $n \to \infty$. Also, the coupling will allow us to assume that S_n and $\tilde{S}_n$ are defined on the same probability space.

Furthermore, by Lemmas 2.5 and 2.6 we have

$$|\mathbb{E}S_n - \mathbb{E}\tilde{S}_n| = o(1) \text{ and } \left| \mathrm{Var} S_n - \mathrm{Var}\tilde{S}_n \right| = o(n).$$

In particular, the former implies that $\mathbb{P}(S_n \neq \tilde{S}_n) = o(1)$. Henceforth, if $X_n, n \geq 1$, is a sequence of random variables with X_n defined on the nth coupling space, then by $X_n \xrightarrow{\widehat{\mathbb{P}}_n} 0$ we mean that for all $\epsilon > 0$ we have $\widehat{\mathbb{P}}_n(|X_n| \geq \epsilon) \to 0$ as $n \to \infty$.

Thus we have $|S_n - \mathbb{E}S_n - (\tilde{S}_n - \mathbb{E}\tilde{S}_n)| \xrightarrow{\widehat{\mathbb{P}}_n} 0$ as $n \to \infty$, whence

$$\left| \frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var} S_n}} - \frac{\tilde{S}_n - \mathbb{E}\tilde{S}_n}{\sqrt{\mathrm{Var} S_n}} \right| \xrightarrow{\widehat{\mathbb{P}}_n} 0$$

as well. If $X_n, n \geq 1$, and $Y_n, n \geq 1$, are sequences of random variables with $|X_n - Y_n| \xrightarrow{\widehat{\mathbb{P}}_n} 0$, if $\sup_n \mathbb{E}|Y_n| < \infty$, and if $\alpha_n, n \geq 1$, is a sequence of scalars with $\lim_{n\to\infty} \alpha_n = 1$, then $|X_n - \alpha_n Y_n| \xrightarrow{\widehat{\mathbb{P}}_n} 0$. Since $\lim_{n\to\infty} \sqrt{\mathrm{Var} S_n}/\sqrt{\mathrm{Var}\tilde{S}_n} = 1$ it follows that as $n \to \infty$ we have

$$\left| \frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}S_n}} - \frac{\tilde{S}_n - \mathbb{E}\tilde{S}_n}{\sqrt{\mathrm{Var}\tilde{S}_n}} \right| \xrightarrow{\widehat{\mathbb{P}}_n} 0.$$

Thus the asymptotic normality for $(\tilde{S}_n - \mathbb{E}\tilde{S}_n)/\sqrt{\mathrm{Var}\tilde{S}_n}$ implies the asymptotic normality of $(S_n - \mathbb{E}S_n)/\sqrt{\mathrm{Var}S_n}$, i.e., we have as $n \to \infty$

$$\mathbb{P}\left(\frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}S_n}} \leq x \right) \to \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

In the following sub-sections, we will show that $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ and $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$ satisfy a central limit theorem, the former for all $\alpha \in (1, \infty)$ and the latter for all $\alpha \in (1/2, \infty)$. These imply (1.7) and (1.8). On the other hand, in the final sub-section, we show that $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ does not satisfy a central limit theorem for $\alpha \in (1/2, 1)$. The above argument implies that $S^{iso}(\mathcal{P}_{\alpha,n})$ also does not satisfy a central limit theorem in the same range of α.

6.1 The central limit theorem for $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$

Consider the ball $B(p_2)$ centered at a point $p_2 \in D([0, H])$. We compute the maximum x-distance between p_2 and a generic point p in the intersection of $B(p_2) \cap D([0, H])$. This tells us the maximum x-distance $x_{\max} := x_{\max}^{ext}$ of the set given by the intersection of $B(p_2)$ with $D([0, H])$. Since both p_2 and p have heights at most $H = 4 \log R$, the inclusion $B(p_2) \subseteq B^+(p_2)$ at (2.11) implies that

$$x_{\max} = O(1) \cdot R^4.$$

We define a dependency graph $G_n := G_n^{ext} := (\mathcal{V}_n, \mathcal{E}_n)$ as follows. Firstly, we partition the interval $(-I_n, I_n]$ into $\Theta(n/R)$ consecutive intervals of equal length, which we enumerate $J_1, \ldots, J_{[2I_n/R]}$. For each $i = 1, \ldots, [2I_n/R]$, we set $C_i := J_i \times [0, R]$. The collection of axis-parallel rectangles $\{C_i\}_{i=1,\ldots,[2I_n/R]}$ partitions D. The vertex set $\mathcal{V}_n$ consists of the rectangles $C_1, ..., C_{[2I_n/R]}$. We put an edge (C_i, C_j) between any two rectangles whenever C_i and C_j are separated by a rectangle having x-distance at most $2x_{\max}$. Let $\mathcal{E}_n$ be the collection of all such edges. Put for all $i = 1, ..., [2I_n/R]$

$$Z_i := Z_{C_i}^{ext} := \sum_{p \in \tilde{\mathcal{P}}_{\alpha,D,H} \cap C_i} \tilde{\xi}^{ext}(p, \tilde{\mathcal{P}}_{\alpha,D}).$$

By the definition of $x_{\max}$, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint collections of rectangles in $\mathcal{V}_n$ such that no edge in $\mathcal{E}_n$ has one endpoint in $\mathcal{C}_1$ and the other endpoint in $\mathcal{C}_2$, then the random variables $\{Z_{C_i}, C_i \in \mathcal{C}_1\}$ and $\{Z_{C_i}, C_i \in \mathcal{C}_2\}$ are independent. Note that a rectangle C having x-side equal to $x_{\max}$ will have non-empty intersection with at most

$$\frac{x_{\max}}{2I_n/[2I_n/R]} = O(1) \cdot R^3 \tag{6.1}$$

rectangles from the collection $C_1, ..., C_{[2I_n/R]}$.

Thus $G_n^{ext} := (\mathcal{V}_n, \mathcal{E}_n)$ is a dependency graph for $Z_i, i = 1, ..., [2I_n/R]$. Now note that

$$\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) := \sum_{i=1}^{[2I_n/R]} Z_i^{ext}.$$

Furthermore,

$$\mathbb{E}|C_1 \cap (\tilde{\mathcal{P}}_\alpha \cap D)| = \beta \cdot 2I_n \cdot [\frac{2I_n}{R}]^{-1} \int_0^R e^{-\alpha y} dy = O(1) \cdot R. \tag{6.2}$$

Standard tail estimates for Poisson random variables give $\mathbb{P}\left(|C_1 \cap (\tilde{\mathcal{P}}_\alpha \cap D)| > R^2 \right) \leq e^{-R^2}$, for n sufficiently large. So

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(\mathrm{card}((\tilde{\mathcal{P}}_\alpha \cap D) \cap C_i) \leq R^2,\ 1 \leq i \leq [\frac{2I_n}{R}]) &= 1 - O\left(\frac{n}{R} \cdot e^{-R^2/2}\right) \\ &= 1 - o(n^{-15}). \end{aligned} \tag{6.3}$$

Define

$$A_n := \{\mathrm{card}((\tilde{\mathcal{P}}_\alpha \cap D) \cap C_i) \leq R^2 \text{ for all } 1 \leq i \leq [\frac{2I_n}{R}]\}.$$

The maximal degree D_n of the dependency graph G_n^{ext} satisfies $D_n = O(R^3)$. We also set $B_n := \max_{1 \leq i \leq [2I_n/R]} Z_{C_i}^{ext} \leq 2R$, $V_n := \mathrm{card}(\mathcal{V}_n) = [2I_n/R]$. Set $\sigma_n^2 := \mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)|A_n]$. Hölder's inequality gives

$$\begin{aligned} \mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)\mathbf{1}(A_n^c)] &\leq \mathbb{E}[(\tilde{S}_H^{ext}((\tilde{\mathcal{P}}_\alpha \cap D)))^2 \mathbf{1}(A_n^c)] \leq (\mathbb{E}[|\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)|^3])^{2/3} \mathbb{P}(A_n^c)^{1/3} \\ &\leq n^2 \cdot n^{-5} = n^{-3} \end{aligned}$$

and

$$\mathbb{E}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)\mathbf{1}(A_n^c)] \leq \mathbb{E}[|\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)|^2]^{1/2} \mathbb{P}(A_n^c)^{1/2} \leq n \cdot n^{-15/2} = o(1).$$

We thus conclude that

$$|\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)|A_n]| = o(1) \tag{6.4}$$

and

$$\left| \mathbb{E}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathbb{E}[\tilde{S}_H^{ext}((\tilde{\mathcal{P}}_\alpha \cap D))|A_n] \right| = o(1). \tag{6.5}$$

We have shown that $\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)] = \Theta(n)$ and thus also $\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)|A_n] = \sigma_n^2 = \Theta(n)$. The Baldi-Rinott central limit theorem for dependency graphs [3] gives

$$\begin{aligned} &\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)|A_n]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)|A_n]}} \leq x \,|A_n \right) - \Phi(x) \right| \\ &\leq 32(1+\sqrt{6}) \left(\frac{D_n^2 B_n^3 V_n}{\sigma_n^3} \right)^{1/2}. \end{aligned} \tag{6.6}$$

Since $\sigma_n = \Theta(n^{1/2})$, we have $D_n^2 B_n^3 V_n / \sigma_n^3 = o(1)$. This shows a central limit theorem for $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$ conditional on A_n.

To deduce a central limit theorem for $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$, we write

$$\begin{aligned}
&\mathbb{P}\left(\frac{\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]}} \le x\right) \\
&= \mathbb{P}\left(\frac{\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}\left[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)\right]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)]}} \le x | A_n\right) + o(1) \\
&\overset{(6.4)}{=} \mathbb{P}\left(\frac{\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}\left[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) | A_n\right]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D) | A_n]}} \le x + o(1) | A_n\right) + o(1).
\end{aligned}$$

Since $\tilde{S}_H^{ext}(\tilde{\mathcal{P}}_\alpha \cap D)$ conditional on A_n satisfies a central limit theorem by (6.6), the probability on the right-hand side converges to $\Phi(x)$. □

6.2 The central limit theorem for $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$: the regime $\alpha \in (1, \infty)$

The above approach turns out to be not strong enough for showing the asymptotic normality for the number of isolated vertices. For a certain range of α a dependency graph defined as above has high maximum degree making the bounds (6.6) of little use. We will instead prove a central limit theorem for $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ using a Poincaré-type inequality for Poisson functionals due to Last, Peccati and Schulte [17].

Let $\mathcal{P}$ denote a Poisson point process on a space $\mathbf{S}$ having intensity measure λ. Let F denote a functional on locally finite point sets in $\mathbf{S}$. Recall that for a point $p \in \mathbf{S}$ we defined the first order linear operator $\nabla_p F := F(\mathcal{P} \cup \{p\}) - F(\mathcal{P})$. Here, we will also use the *second order operator* $\nabla^2_{p_1,p_2} F := F(\mathcal{P} \cup \{p_1, p_2\}) - F(\mathcal{P} \cup \{p_1\}) - F(\mathcal{P} \cup \{p_2\}) + F(\mathcal{P})$. The functional F belongs to the *domain of* ∇ if

$$\mathbb{E}[F(\mathcal{P})^2] < \infty \text{ and } \mathbb{E}\int_{\mathbf{S}} (\nabla_p F(\mathcal{P}))^2 \lambda(dp) < \infty.$$

Theorem 1.1 of [17] uses these differential operators to approximate the normalised version of F by the standard normal N. For two real-valued random variables X and Y, let $d_W(X, Y)$ denote the Wasserstein distance between the measures on $\mathbb{R}$ induced by X and Y.

Theorem 6.1. *Let F be a functional defined on locally finite collections of points in $\mathbf{S}$. Assume F belongs to the domain of ∇ and satisfies $\mathbb{E}F = 0$ and $\mathrm{Var}F = 1$. If N is a standard normally distributed random variable, then*

$$d_W(F, N) \le \gamma_1 + \gamma_2 + \gamma_3,$$

where

$$\gamma_1 := 4\left[\int_{\mathbf{S}^3} \left(\mathbb{E}\left[(\nabla_{p_2} F)^2 (\nabla_{p_3} F)^2\right]\right)^{1/2} \left(\mathbb{E}\left[(\nabla^2_{p_1,p_2} F)^2 (\nabla^2_{p_1,p_3} F)^2\right]\right)^{1/2} \lambda^3(d(p_1, p_2, p_3))\right]^{1/2},$$

$$\gamma_2 := \left[\int_{\mathbf{S}^3} \mathbb{E}\left[(\nabla^2_{p_1,p_3} F)^2 (\nabla^2_{p_2,p_3} F)^2\right] \lambda^3(d(p_1, p_2, p_3))\right]^{1/2},$$

$$\gamma_3 := \int_{\mathbf{S}} \mathbb{E}|\nabla_p F|^3 \lambda(dp).$$

We will apply Theorem 6.1 on the conditional space of the event

$$E_n := (\tilde{\mathcal{P}}_\alpha \cap D) \cap D([\frac{R}{2}, R]) = \emptyset. \tag{6.7}$$

A calculation similar to the one in (6.2) shows that for any $\alpha \in (1, \infty)$ we have $\mathbb{P}(E_n) = 1 - O(n^{1-\alpha})$.

We shall apply Theorem 6.1 setting λ to be μ_α and letting

$$\mathbf{S} := D([0, \frac{R}{2}]),\ \mathcal{P} := \tilde{\mathcal{P}}_\alpha \cap D \text{ and } F := \frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)|E_n]}{\sqrt{\mathrm{Var}[\tilde{S}^{iso}(\tilde{\mathcal{P}}_{\alpha,D,H})|E_n]}}.$$

These ensure that on E_n, one has $\mathbb{E}[F|E_n] = 0$ and $\mathrm{Var}[F|E_n] = 1$. We will verify that F is on the domain of ∇ later on, using the estimate on γ_3. We will only check the second condition; the requirement that $\mathbb{E}F(\tilde{\mathcal{P}}_\alpha \cap D)^2 < \infty$ follows from our bounds on $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]$.

Set $\sigma_n^2 := \mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]$ and $\sigma_n'^2 := \mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)|E_n]$. The proof of the next lemma is postponed until Section B.

Lemma 6.2. *For any $\alpha \in (1, \infty)$, we have*

$$\lim_{n\to\infty} \frac{\sigma_n^2}{\sigma_n'^2} = 1 \text{ and } \lim_{n\to\infty} \frac{\mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)|E_n]}{\sigma_n'} = 0.$$

To apply Theorem 6.1 we shall bound $|\nabla_p F|$ by the number of points of $(\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p\}$ which are inside the hyperbolic ball around p having height at most H. By the inclusion-exclusion principle, the second order operator $\nabla_{p_1,p_2} F$ is proportional to the number of isolated points of $\tilde{\mathcal{P}}_\alpha \cap D$ which are contained in the intersection of the hyperbolic balls around p_1 and p_2 and having height at most H. Thus

$$\begin{aligned} |\nabla_p F| &\le \frac{1}{\sigma_n'} \cdot \Big(|(\tilde{\mathcal{P}}_\alpha \cap D) \cap B(p) \cap D([0,H])| + 1 \Big) \text{ and} \\ |\nabla_{p_1,p_2} F| &\le \frac{1}{\sigma_n'} \cdot |(\tilde{\mathcal{P}}_\alpha \cap D) \cap (B(p_1) \cap B(p_2)) \cap D([0,H])|. \end{aligned} \tag{6.8}$$

Given a Borel-measurable set $A \subset D([0,H])$, we have that $\mathrm{card}(\tilde{\mathcal{P}}_\alpha \cap D \cap A)$ is a Poisson-distributed random variable with parameter equal to the intensity measure of A. The next lemma, a consequence of Lemma 4.1, bounds these intensity measures for the sets A appearing in (6.8).

Lemma 6.3. *There exists a constant $\eta > 0$ depending on α and ν such that for all $p = (x(p), y(p)) \in D$ we have*

$$\mu_\alpha(B(p) \cap D([0,H])) \le \eta \cdot e^{\frac{y(p)}{2}}.$$

Hence,

$$\mu_\alpha((B(p_1) \cap B(p_2)) \cap D([0,H])) \le \eta \cdot e^{\frac{1}{2}(y(p_1) \wedge y(p_2))}.$$

Set $\lambda(p) := \eta \cdot e^{y(p)/2}$. Thus, $|\nabla_p F|$ is stochastically dominated from above by a random variable $X(p) + 1$, where $X(p)$ distributed as $\mathrm{Po}(\lambda(p))$. Analogously, $|\nabla_{p_1,p_2}|$ is stochastically dominated by a Poisson-distributed random variable $X(p_1, p_2)$ with parameter $\lambda(p_1, p_2) := \eta \cdot e^{(y(p_1) \wedge y(p_2))/2}$. We now bound γ_3, γ_2 and γ_1 in this order.

Lemma 6.4. *If $\alpha \in (1, \infty)$, then $\gamma_3 = o(1)$.*

Proof. For $p \in \mathbf{S}$,

$$\mathbb{E}|\nabla_p F|^3 \leq \frac{1}{{\sigma'_n}^3}\mathbb{E}[(X(p)+1)^3] = O(1)\cdot \frac{1}{{\sigma'_n}^3}\cdot e^{\frac{3y(p)}{2}}.$$

We deduce that

$$\int \mathbb{E}|\nabla_p F|^3\lambda(dp) = O\left(1\right)\frac{1}{{\sigma'_n}^3}\cdot\int_{-I_n}^{I_n}\int_0^{R/2} e^{\frac{3y}{2}-\alpha y}dydx = O\left(1\right)\frac{n}{{\sigma'_n}^3}\cdot\int_0^{R/2} e^{\frac{3y}{2}-\alpha y}dy.$$

Recall that $n = \nu e^{R/2}$ and $\sigma'_n = \Theta(n^{1/2})$ by the first part of Lemma 6.2 and (4.21). Therefore if $\alpha \in (1, 3/2]$ then

$$\gamma_3 = O\left(R\right)\frac{n}{{\sigma'_n}^3}\cdot n^{\frac{3}{2}-\alpha} = O(R)\cdot n^{1+\frac{3}{2}-\alpha-\frac{3}{2}} = O(R)\cdot n^{1-\alpha} = o(1).$$

If $\alpha \in (3/2, \infty)$, then $\gamma_3 = o(1)$. □

Let us point out that the bound on $\int \mathbb{E}|\nabla_p F|^3\lambda(dp)$ is also a bound on $\int \mathbb{E}|\nabla_p F|^2\lambda(dp)$ and thus F is in the domain of ∇.

Lemma 6.5. *If* $\alpha \in (1,\infty)$*, then* $\gamma_2 = o(1)$.

Proof. The second inequality in (6.8) implies

$$|\nabla_{p_1,p_2}F| \leq \frac{1}{\sigma'_n}\cdot \min\{|\tilde{\mathcal{P}}_\alpha\cap D\cap B(p_1)\cap D([0,R/2])|, |\tilde{\mathcal{P}}_\alpha\cap D\cap B(p_2)\cap D([0,R/2])|\}.$$

Now, we claim that there exists a constant $\gamma > 0$ such that if

$$|x(p_1)-x(p_2)|_\Phi > \gamma e^{\frac{H}{2}}\left(e^{\frac{y(p_1)}{2}} + e^{\frac{y(p_2)}{2}}\right). \tag{6.9}$$

then $B(p_1)\cap B(p_2)\cap D([0,H]) = \emptyset$. Indeed, for any $p \in D([0,H])$ we have $B(p)\cap D([0,H]) \subseteq B^+_H(p)$ where $B^+_H(p)$ is defined at (3.7). Now, Lemma 3.1 implies that there exist some constant $\gamma > 0$ such that $B^+_H(p_1)\cap B^+_H(p_2) = \emptyset$ if (6.9) holds. This implies that when we integrate with respect to $x(p_2)$ and $x(p_3)$, relative distances with respect to p_1 greater than this quantity have no contribution to the integral defining γ_2. In other words, p_2 (and p_3, respectively) contributes to this integral only if

$$|x(p_1)-x(p_2)|_\Phi \leq \gamma e^{\frac{H}{2}}\left(e^{\frac{y(p_1)}{2}} + e^{\frac{y(p_2)}{2}}\right) \leq 2\gamma e^{\frac{H}{2}} e^{\frac{y(p_1)\vee y(p_2)}{2}}.$$

Therefore, when we integrate over the choices of p_1, p_2, p_3 against the intensity measure $d\mu^3(p_1,p_2,p_3)$ (letting $x_i = x(p_i)$ and $y_i = y(p_i)$), we will get (using that $e^{y_1/2} + e^{y_2/2} \leq 2e^{(y_1\vee y_2)/2}$)

$$\begin{aligned}\gamma_2^2 \leq 4\gamma^2\cdot\frac{1}{{\sigma'_n}^4}\int_{-I_n}^{I_n}\int_{-I_n}^{I_n}\int_{-I_n}^{I_n}\int_{[0,R/2]^3}&\mathbb{E}\left[\,(X(p_1,p_2)X(p_1,p_3))^2\,\right]\\ &\times \mathbf{1}(|x_2-x_1|_\Phi \leq 2\gamma e^{H/2}e^{(y_1\vee y_2)/2})\mathbf{1}(|x_3-x_1|_\Phi \leq 2\gamma e^{H/2}e^{(y_1\vee y_3)/2})\\ &\times e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3dx_3dx_2dx_1.\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}\mathbb{E}\left[\,(X(p_1,p_2)X(p_1,p_3))^2\,\right] &\leq \mathbb{E}\left[\,(X(p_1,p_2)^4\,\right]^{1/2}\mathbb{E}\left[\,X(p_1,p_3))^4\,\right]^{1/2}\\ &= O(1)\cdot e^{y_1\wedge y_2}\cdot e^{y_1\wedge y_3}.\end{aligned} \tag{6.10}$$

Using this inequality and integrating first with respect to x_2 and x_3 we obtain

$$\begin{aligned}\gamma_2^2 &= O(1)\cdot e^H \cdot \frac{1}{\sigma_n'^4}\int_{-I_n}^{I_n}\int_{[0,R/2]^3} e^{(y_1\vee y_2)/2}\cdot e^{(y_1\vee y_3)/2} e^{y_1\wedge y_2}\cdot e^{y_1\wedge y_3} \\ &\qquad\qquad \times e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3dx_1 \\ &\overset{I_n=\Theta(n)}{=} O(1)\cdot e^H \frac{n}{\sigma_n'^4}\int_{[0,R/2]^3} e^{(y_1\vee y_2)/2}\cdot e^{(y_1\vee y_3)/2} e^{y_1\wedge y_2}\cdot e^{y_1\wedge y_3} \\ &\qquad\qquad \times e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3.\end{aligned} \tag{6.11}$$

To bound the triple integral in (6.11) for $\alpha \in (1,\infty)$, we will split the domain of integration into four sub-domains:

$$\begin{aligned} D_1 &:= \{(y_1,y_2,y_3)\in[0,R/2]^3 \ : \ y_1 \le y_2, y_3\}, \\ D_2 &:= \{(y_1,y_2,y_3)\in[0,R/2]^3 \ : \ y_2 \le y_1 \le y_3\}, \\ D_3 &:= \{(y_1,y_2,y_3)\in[0,R/2]^3 \ : \ y_3 \le y_1 \le y_2\}, \\ D_4 &:= \{(y_1,y_2,y_3)\in[0,R/2]^3 \ : \ y_2, y_3 \le y_1\}. \end{aligned}$$

We evaluate the integral in (6.11) on each of these four sub-domains. On D_1 we have:

$$\begin{aligned} &\int_{D_1} e^{(y_1\vee y_2)/2}\cdot e^{(y_1\vee y_3)/2} e^{y_1\wedge y_2}\cdot e^{y_1\wedge y_3}\cdot e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3 \\ &= \int_{D_1} e^{y_2/2}\cdot e^{y_3/2}\cdot e^{2y_1}\cdot e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3 \\ &\le \int_0^{R/2} e^{2y_1}\left(\int_{y_1}^{\infty} e^{(\frac{1}{2}-\alpha)y_2}dy_2\right)\left(\int_{y_1}^{\infty} e^{(\frac{1}{2}-\alpha)y_3}dy_3\right) e^{-\alpha y_1}dy_1 \\ &= O(1)\cdot \int_0^{R/2} e^{2y_1+(1-2\alpha)y_1-\alpha y_1}dy_1 \\ &= O(1)\cdot \int_0^{R/2} e^{3(1-\alpha)y_1}dy_1 = O(1). \end{aligned}$$

For the second sub-domain D_2, we get:

$$\begin{aligned} &\int_{D_2} e^{(y_1\vee y_2)/2}\cdot e^{(y_1\vee y_3)/2} e^{y_1\wedge y_2}\cdot e^{y_1\wedge y_3}\cdot e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3 \\ &= \int_{D_2} e^{y_1/2}\cdot e^{y_3/2}\cdot e^{y_2}\cdot e^{y_1}\cdot e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3 \\ &= \int_{D_2} e^{3y_1/2+y_2+y_3/2}\cdot e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3 \\ &\le \int_0^{R/2} e^{3y_1/2}\left(\int_0^{y_1} e^{y_2(1-\alpha)}dy_2\right)\left(\int_{y_1}^{\infty} e^{(\frac{1}{2}-\alpha)y_3}dy_3\right)\cdot e^{-\alpha y_1}dy_1 \\ &= O(1)\cdot \int_0^{R/2} e^{3y_1/2+(\frac{1}{2}-\alpha)y_1-\alpha y_1}dy_1 \\ &= O(1)\cdot \int_0^{R/2} e^{2(1-\alpha)y_1}dy_1 = O(1). \end{aligned}$$

The third sub-domain D_3 gives an identical result due to symmetry.

Finally, for the fourth sub-domain D_4 we get:

$$\int_{D_4} e^{(y_1 \vee y_2)/2} \cdot e^{(y_1 \vee y_3)/2} e^{y_1 \wedge y_2} \cdot e^{y_1 \wedge y_3} \cdot e^{-\alpha y_1} e^{-\alpha y_2} e^{-\alpha y_3} dy_1 dy_2 dy_3$$
$$= \int_{D_4} e^{y_1} e^{y_2 + y_3} \cdot e^{-\alpha y_1} e^{-\alpha y_2} e^{-\alpha y_3} dy_1 dy_2 dy_3$$
$$= \int_0^{R/2} e^{y_1} \left(\int_0^{y_1} e^{(1-\alpha) y_2} dy_2 \right) \left(\int_0^{y_1} e^{(1-\alpha) y_3} dy_3 \right) e^{-\alpha y_1} dy_1 = O(1).$$

Combining the integrals for each of the four sub-domains we obtain

$$\int_0^{R/2} \int_0^{R/2} \int_0^{R/2} e^{(y_1 \vee y_2)/2} \cdot e^{(y_1 \vee y_3)/2} e^{y_1 \wedge y_2} \cdot e^{y_1 \wedge y_3} \cdot e^{-\alpha y_1} e^{-\alpha y_2} e^{-\alpha y_3} dy_1 dy_2 dy_3 = O(1). \tag{6.12}$$

Substituting the bound (6.12) into (6.11) we deduce for $\alpha \in (1, \infty)$ that $\gamma_2 \leq O(1) \cdot e^H \cdot n^{1/2} / \sigma_n'^2 = o(1)$. This completes the proof of Lemma 6.5. □

Lemma 6.6. *If $\alpha \in (1, \infty)$, then $\gamma_1 = o(1)$.*

The proof of this lemma is almost identical to the proof of the previous lemma. We postpone it to Section C.

We now establish the central limit theorem at (1.7). Consider the random variable

$$\hat{S}_H^{iso} := \frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}\left[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | E_n \right]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | E_n]}}$$

on the conditional space E_n. Theorem 6.1 and Lemmas 6.4-6.6 yield

$$d_W \left(\hat{S}_H^{iso}, N \right) \leq \gamma_1 + \gamma_2 + \gamma_3 = o(1). \tag{6.13}$$

Recalling that $\mathbb{P}(E_n) = 1 - O(n^{1-\alpha})$, we have

$$\mathbb{P} \left(\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}{\sqrt{\mathrm{Var}[S_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}} \leq x \right)$$
$$= \mathbb{P} \left(\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}\left[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) \right]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}} \leq x | E_n \right) + o(1)$$
$$= \mathbb{P} \left(\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}\left[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | E_n \right]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | E_n]}} \leq x + o(1) | E_n \right) + o(1),$$

where the last equality follows by Lemma 6.2. Since the bound (6.13) shows that $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ satisfies a central limit theorem on E_n, the probability on the left-hand side of the above display converges to $\Phi(x)$. Thus the central limit theorem (1.7) holds.

6.3 The regime $\alpha \in (1/2, 1)$

We establish that $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ does not exhibit normal convergence for $\alpha \in (1/2, 1)$. We redefine A_n to be the event that $\mathrm{card}((\tilde{\mathcal{P}}_\alpha \cap D) \cap D([h_1, R])) = 0$, where now $h_1 := R/(2\alpha) + (\log \log R)/2\alpha$. That is, on the event A_n there are no points having height greater than h_1. An elementary calculation shows that $\mathbb{P}(A_n) = 1 - o(1)$.

For any $x \in \mathbb{R}$ we have

$$\mathbb{P}\left(\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}} \le x\right)$$
$$= \mathbb{P}\left(\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}} \le x | A_n\right) + o(1).$$

We are going to show that

$$\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | A_n] = o(\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]). \tag{6.14}$$

Since

$$\mathrm{Var}\left[\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}} | A_n\right] = \frac{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | A_n]}{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]} = o(1)$$

this implies that

$$\mathbb{P}\left(\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}} \le x | A_n\right)$$

cannot converge to $\Phi(x)$ and therefore

$$\frac{\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}{\sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]}}$$

cannot converge in distribution to a standard normally distributed random variable N.

We now show (6.14). We will bound $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | A_n]$ using the Poincaré inequality

$$\mathrm{Var} F \le \mathbb{E}\left[\int_{\mathbf{S}} (\nabla_p F(\mathcal{P}))^2 \lambda(dp)\right].$$

We put λ to be μ_α and set

$$F := \tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) | A_n,\ \mathbf{S} := D([0, h_1]),\ \text{and } \mathcal{P} := \tilde{\mathcal{P}}_\alpha \cap D.$$

By Lemma 6.3 and the discussion immediately after its statement, we have that $|\nabla_p F(\mathcal{P})|$ is stochastically bounded by $X(p) + 1$ where $X(p)$ is a Poisson-distributed random variable with parameter $\eta \cdot e^{y(p)/2}$. Hence,

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbf{S}} (\nabla_p F(\mathcal{P}))^2 \lambda(dp)\right] &\le \int_{\mathbf{S}} \mathbb{E}\left[(\nabla_p F(\mathcal{P}))^2\right] \lambda(dp) \\ &= O(1) \cdot n \int_0^{h_1} e^{(1-\alpha)y} dy \\ &= O(1) \cdot n \cdot e^{(1-\alpha)h_1}.\end{aligned}$$

Recalling $h_1 = R/2\alpha + (\log\log R)/\alpha$ and $n = \nu e^{R/2}$ we obtain

$$\int_{\mathbf{S}} \mathbb{E}\left[(\nabla_p F(\mathcal{P}))^2\right] \lambda(dp) = O(\log^{\frac{1-\alpha}{\alpha}} R) \cdot n^{1+(1-\alpha)/\alpha} \overset{\frac{1-\alpha}{\alpha}<2}{=} O(\log^2 R) \cdot n^{\frac{1}{\alpha}}.$$

We conclude that $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)|A_n] = O(\log^2 R) \cdot n^{1/\alpha}$. By Theorem 1.1 we have $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] = \Theta(1) \cdot n^{3-2\alpha}$. But $1/\alpha < 3 - 2\alpha$, for $\alpha \in (1/2, 1)$. Thus (6.14) follows, concluding the proof of Theorem 1.3. □

A Proof of Lemmas 2.1 and 2.2

Proof of Lemma 2.1. The expression for $\theta_R(r, r')$ is a consequence of the hyperbolic law of cosines at (2.1). We first prove (i). We compute:

$$\begin{aligned}\frac{\cosh r \cosh r' - \cosh R}{\sinh r \sinh r'} &= \frac{\frac{1}{4}(e^{r+r'} + e^{r-r'} + e^{r'-r} + e^{-(r+r')}) - \frac{1}{2}(e^R + e^{-R})}{\frac{1}{4}(e^{r+r'} - e^{r-r'} - e^{r'-r} + e^{-(r+r')})} \\ &= 1 + 2\frac{e^{r-r'} + e^{r'-r} - e^R - e^{-R}}{e^{r+r'} - e^{r-r'} - e^{r'-r} + e^{-(r+r')}} \\ &= 1 - 2e^{-(r+r'-R)}\left(\frac{1 - e^{r-r'-R} - e^{r'-r-R} + e^{-2R}}{1 - e^{-2r} - e^{-2r'} + e^{-2(r+r')}}\right) \\ &= 1 - x,\end{aligned}$$

where

$$x := 2e^{-(r+r'-R)} \cdot \frac{(1 - e^{r-r'-R})(1 - e^{r'-r-R})}{(1 - e^{-2r})(1 - e^{-2r'})}. \tag{A.1}$$

By definition of $\Delta(r, r')$ it suffices to bound $\frac{1}{2}e^{R/2} \arccos(1 - x)$ above and below. First, we remark that $r - r' - R > -2r'$ since $r + r' > R$. This implies that $(1 - e^{r-r'-R})/(1 - e^{-2r'}) \in (0, 1)$. Similarly, we have $(1 - e^{r'-r-R})/(1 - e^{-2r}) < 1$. Let $\varepsilon \in (0, 1)$. Since $r, r' \in [C, R]$, it follows that if $C := C(\varepsilon)$ is large enough then we have

$$\frac{1 - e^{r-r'-R}}{1 - e^{-2r'}} > \frac{1 - e^{-r'}}{1 - e^{-2r'}} > 1 - \varepsilon,$$

and

$$\frac{1 - e^{r-r'-R}}{1 - e^{-2r}} > 1 - \varepsilon.$$

With $s := 2e^{-(r+r'-R)}$, this shows that

$$s(1 - \varepsilon)^2 < x < s. \tag{A.2}$$

Taylor's expansion of $\arccos(\cdot)$ implies there exists a constant $K > 0$ such that

$$\sqrt{2v} - Kv^{3/2} < \arccos(1 - v) < \sqrt{2v} + Kv^{3/2}, \quad v \in (0, 1). \tag{A.3}$$

Replacing v with x, inequality (A.2) implies that

$$(1 - \varepsilon)\sqrt{2s} - Ks^{3/2} < \arccos(1 - x) < \sqrt{2s} + Ks^{3/2}.$$

Now, since $r + r' > R + C$ it follows that $s < 2e^{-C}$ and thus $s^{3/2} = s^{1/2}s < s^{1/2}2^{1/2}2^{1/2}e^{-C}$. If $C := C(\varepsilon)$ is large enough so that $2^{1/2}e^{-C} < \varepsilon/K$, we have

$$Ks^{3/2} \le \varepsilon\sqrt{2s}. \tag{A.4}$$

This yields

$$(1 - 2\varepsilon)\sqrt{2s} < \arccos(1 - x) < \sqrt{2s}(1 + \varepsilon) < \sqrt{2s}(1 + 2\varepsilon).$$

Note that

$$\frac{1}{2}e^{R/2}\sqrt{2s} = e^{R/2-(r+r'-R)/2} = e^{R-(r+r')/2} = e^{(y+y')/2}, \tag{A.5}$$

where we recall $y := R - r$ and $y' := R - r'$. So for $\varepsilon \in (0, 1/2)$ we obtain

$$(1-2\varepsilon)e^{(y+y')/2} < \frac{1}{2}e^{R/2}\arccos(1-x) < (1+2\varepsilon)e^{(y+y')/2}.$$

Replacing 2ε by ε, the inequality (2.3) follows. We now show (ii). The assumption $r, r' \in [R-H, R]$ implies that $|r-r'| \leq H$. Thus,

$$e^{r-r'-R} \leq e^{H-R}, \ \ e^{r'-r-R} \leq e^{H-R} \text{ and } e^{-2r} \leq e^{-2(R-H)}, \ \ e^{-2r'} \leq e^{-2(R-H)}.$$

The definition of x gives $x = 2e^{-(r+r'-R)}(1+\delta_n(r,r'))$, where $\delta_n(r,r') = o(1)$ uniformly over all $r, r' \in [R-H, R]$. Thus, (A.3) implies that

$$\begin{aligned}\arccos(1-x) &= 2e^{-\frac{1}{2}(r+r'-R)}(1+\delta_n(r,r'))^{1/2}\,(1+\Theta(x)) \\ &=: 2e^{-\frac{1}{2}(r+r'-R)}(1+\lambda_n(r,r')).\end{aligned} \tag{A.6}$$

The result then follows by (A.5).

We now prove (iii). To see this, recall from that C is chosen to satisfy $2^{1/2}e^{-C} < \varepsilon/K$. Thus, this implies that ε as a function of C can be selected such that $\varepsilon = \Theta(e^{-C})$. □

Proof of Lemma 2.2. The proof involves elementary calculations, included here for completeness. For the lower bound, we have

$$\begin{aligned}\bar{\rho}_{\alpha,n}(y) &> \frac{\alpha\big(e^{\alpha(R-y)} - e^{-\alpha(R-y)}\big)}{e^{\alpha R}+e^{-\alpha R}} = \frac{\alpha e^{\alpha(R-y)}}{e^{\alpha R}+e^{-\alpha R}} - \frac{\alpha e^{-\alpha(R-y)}}{e^{\alpha R}+e^{-\alpha R}} \\ &= \frac{\alpha e^{\alpha(R-y)}}{e^{\alpha R}} - \frac{\alpha e^{\alpha(R-y)}}{e^{\alpha R}} + \frac{\alpha e^{\alpha(R-y)}}{e^{\alpha R}+e^{-\alpha R}} - \frac{\alpha e^{-\alpha(R-y)}}{e^{\alpha R}+e^{-\alpha R}} \\ &= \frac{\alpha e^{\alpha(R-y)}}{e^{\alpha R}} + \alpha e^{\alpha(R-y)}\left(-\frac{1}{e^{\alpha R}} + \frac{1}{e^{\alpha R}+e^{-\alpha R}}\right) - \frac{\alpha e^{-\alpha(R-y)}}{e^{\alpha R}+e^{-\alpha R}} \\ &= \alpha e^{-\alpha y} - \alpha e^{\alpha(R-y)}\frac{e^{-\alpha R}}{e^{\alpha R}\big(e^{\alpha R}+e^{-\alpha R}\big)} - \frac{\alpha e^{-\alpha(R-y)}}{e^{\alpha R}+e^{-\alpha R}} \\ &> \alpha e^{-\alpha y} - \frac{\alpha e^{-\alpha y}}{e^{\alpha R}\big(e^{\alpha R}+e^{-\alpha R}\big)} - \frac{\alpha e^{-\alpha y}}{e^{\alpha R}+e^{-\alpha R}} \\ &> \alpha e^{-\alpha y} - \frac{2\alpha}{e^{\alpha R}} > \alpha e^{-\alpha y} - \frac{2\alpha}{e^{\alpha R}-2}.\end{aligned}$$

The upper bound is derived similarly:

$$\begin{aligned}\bar{\rho}_{\alpha,n}(y) &< \frac{\alpha e^{\alpha(R-y)}}{e^{\alpha R}-2} = \frac{\alpha e^{\alpha R}e^{-\alpha y}}{e^{\alpha R}-2} \\ &= \frac{\alpha(e^{\alpha R}-2+2)e^{-\alpha y}}{e^{\alpha R}-2} \\ &= \frac{\alpha(e^{\alpha R}-2)e^{-\alpha y}}{e^{\alpha R}-2} + \frac{2\alpha e^{-\alpha y}}{e^{\alpha R}-2} \\ &= \alpha e^{-\alpha y} + \frac{2\alpha e^{-\alpha y}}{e^{\alpha R}-2} \\ &< \alpha e^{-\alpha y} + \frac{2\alpha}{e^{\alpha R}-2}.\end{aligned}$$

□

B Proof of Lemma 6.2

For a point process $\tilde{\mathcal{P}}$ on D and a point $p \in D([0,R/2]) \cap \tilde{\mathcal{P}}$, we define $\hat{\xi}^{iso}(p,\tilde{\mathcal{P}})$ to be equal to 1 if and only if $B(p) \cap D([0,R/2]) \cap \{\tilde{\mathcal{P}} \setminus \{p\}\} = \emptyset$. In other words, $\hat{\xi}^{iso}(p,\tilde{\mathcal{P}})$ is equal to 1 precisely when $B(p)$ does not contain any other points of $\tilde{\mathcal{P}}$ of height at most $R/2$. Otherwise we put $\hat{\xi}^{iso}(p,\tilde{\mathcal{P}}) = 0$. For such p we have

$$\tilde{\xi}^{iso}(p,\tilde{\mathcal{P}}) \le \hat{\xi}^{iso}(p,\tilde{\mathcal{P}}). \tag{B.1}$$

We write $\xi(p,\tilde{\mathcal{P}})$ instead of $\tilde{\xi}^{iso}(p,\tilde{\mathcal{P}})$ and we write $\hat{\xi}(p,\tilde{\mathcal{P}})$ instead of $\hat{\xi}^{iso}(p,\tilde{\mathcal{P}})$. With this definition, we set

$$\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D) := \sum_{p \in \tilde{\mathcal{P}}_\alpha \cap D([0,H])} \hat{\xi}(p, \tilde{\mathcal{P}}_\alpha \cap D).$$

Observe that $\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ is distributed as $\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)$ conditional on E_n.

Thus

$$\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)|E_n] = \mathrm{Var}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)].$$

We will show that

$$|\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathrm{Var}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]| = o(n). \tag{B.2}$$

By (4.21) we have $\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] = \Omega(n)$ for $\alpha \in (1,\infty)$ and thus the first part of Lemma 6.2 will then follow.

With $E := \{(p_1,p_2) \in D : y(p_2) \le y(p_1) \le H\}$, using (4.2) we write the difference (B.2) as follows:

$$\begin{aligned}
&|\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathrm{Var}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]| \le \\
&2\beta^2 \cdot \left| \int_E (c^{\hat{\xi}}((x_1,y_1),(x_2,y_2)) - c^{\xi}((x_1,y_1),(x_2,y_2))) e^{-\alpha y_1} e^{-\alpha y_2} dx_2 dy_2 dy_1 dx_1 \right| \\
&+ \mathbb{E}\left[\sum_{p \in \tilde{\mathcal{P}}_\alpha \cap D} (\hat{\xi}(p,\tilde{\mathcal{P}}) - \xi(p,\tilde{\mathcal{P}})) \right].
\end{aligned}$$

Observe now that $B(p) \cap D([0,R/2]) \subseteq B^+_{R/2}(p)$ (cf. (3.7)). Furthermore, Lemma 3.1 implies that if $p_1, p_2 \in E$, then $B^+_{R/2}(p_1) \cap B^+_{R/2}(p_2) = \emptyset$, when

$$|x(p_2) - x(p_1)| >_\Phi 2e^{R/4}(e^{y(p_1)/2} + e^{y(p_2)/2}).$$

If this condition holds, we have $B(p_1) \cap B(p_2) \cap D([0,R/2]) = \emptyset$, which in turn implies that $c^{\hat{\xi}}(p_1,p_2) = 0$. As we did before (see Lemma 3.3), we set $Y_i = e^{y(p_i)/2}$ for $i = 1,2$ (we will be using this notation inside several integrals – there, we will be writing $Y_i = e^{y_i/2}$, for $i = 1,2$). This observation motivates us to split E into two sets:

$$E_1 := \{(p_1,p_2) \in E \ : \ 0 <_\Phi |x(p_2) - x(p_1)| \le_\Phi 2e^{\frac{R}{4}}(Y_1 + Y_2)\}$$

and its complement inside E. In particular, it will suffice to show

$$\left|\int_{E_1}(c^{\hat{\xi}}((x_1,y_1),(x_2,y_2))-c^{\xi}((x_1,y_1),(x_2,y_2)))e^{-\alpha y_1}e^{-\alpha y_2}dx_2dy_2dy_1dx_1\right|=O(n^{2-\alpha}) \quad \text{(B.3)}$$

and

$$\left|\int_{E\setminus E_1}c^{\xi}((x_1,y_1),(x_2,y_2))e^{-\alpha y_1}e^{-\alpha y_2}dx_2dy_2dy_1dx_1\right|=O(n^{2-\alpha}) \quad \text{(B.4)}$$

as on $E\setminus E_1$ the other covariance vanishes.

Let us first show (B.3). For any $p_1,p_2\in D([0,H])$, we write

$$\begin{aligned}
&c^{\hat{\xi}}(p_1,p_2)-c^{\xi}(p_1,p_2)\\
&=\mathbb{E}[\hat{\xi}(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})\hat{\xi}(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})\\
&\qquad\qquad-\xi(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})\xi(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})]\\
&-\Big(\mathbb{E}[\hat{\xi}(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1\})]\mathbb{E}[\hat{\xi}(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_2\})]\\
&\qquad\qquad-\mathbb{E}[\xi(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1\})]\mathbb{E}[\xi(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_2\})]\Big)
\end{aligned}$$

But

$$\begin{aligned}
&\hat{\xi}(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})\hat{\xi}(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})\\
&\qquad\qquad-\xi(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})\xi(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1,p_2\})\\
&\le\mathbf{1}(B(p_1)\cap(\tilde{\mathcal{P}}_\alpha\cap D)\cap D([R/2,R])\neq\emptyset)+\mathbf{1}(B(p_2)\cap\tilde{\mathcal{P}}_\alpha\cap D\cap D([R/2,R])\neq\emptyset).
\end{aligned}$$

In other words, if the left-hand side is 1, then $\tilde{\mathcal{P}}_\alpha\cap D$ has a point in $B(p_1)$ or in $B(p_2)$ that has height at least $R/2$. But by (2.11) we have

$$\mu(B(p_1)\cap D([\frac{R}{2},R]))\le\mu(B^+(p_1)\cap D([\frac{R}{2},R])).$$

Recall that $B^+(p_1)$ is the union of two disjoint sets – so its measure naturally splits into two terms. The first term is

$$\begin{aligned}
\mu(B^+(p_1)\cap D([\frac{R}{2},R-y(p_1)-C]))&=O(1)\cdot Y_1\cdot\int_{R/2}^{R}e^{y/2-\alpha y}dy\\
&=O(1)\cdot Y_1\cdot e^{(\frac{1}{2}-\alpha)R/2}\\
&=O(n^{-(\alpha-\frac{1}{2})})\cdot Y_1.
\end{aligned}$$

Now, the second term is (using $y(p_1)\le H$)

$$\begin{aligned}
\mu(B^+(p_1)\cap D([R-y(p_1)-C,R])))&=O(1)\cdot n\int_{R-H}^{R}e^{-\alpha y}dy\\
&=O(R^{4\alpha})\cdot n^{1-2\alpha}\\
&=o(n^{-(\alpha-\frac{1}{2})}).
\end{aligned}$$

Therefore, since $y(p_2)\le y(p_1)$ we deduce that

$$\mu(B(p_2)\cap D([R/2,R]))\le\mu(B(p_1)\cap D([\frac{R}{2},R]))=O(n^{-(\alpha-\frac{1}{2})})\cdot Y_1. \quad \text{(B.5)}$$

Using these upper bounds, we obtain:

$$\begin{aligned}
&\int_{E_1} \mathbb{E}\Big[\hat{\xi}((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})\hat{\xi}((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\}) \\
&\qquad -\xi((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})\xi((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1),(x_2,y_2)\})\Big] \\
&\quad\times e^{-\alpha y_1}e^{-\alpha y_2}dx_2dy_2dy_1dx_1 \\
&= O(1)\cdot n^{-(\alpha-\frac{1}{2})}\int_{-I_n}^{I_n}\int_0^H\int_0^{y_1}\int_{-I_n}^{I_n}\mathbf{1}\left(|x_2-x_1|_\Phi<2e^{R/4}(Y_1+Y_2)\right) \\
&\qquad\times Y_1e^{-\alpha y_1}e^{-\alpha y_2}dx_2dy_2dy_1dx_1 \\
&= O(1)\cdot n^{-(\alpha-\frac{1}{2})}\cdot e^{R/4}\cdot\int_{-I_n}^{I_n}\int_0^H\int_0^{y_1}Y_1(Y_1+Y_2)e^{-\alpha y_1}e^{-\alpha y_2}dy_2dy_1dx \\
&\overset{e^{R/4}=O(n^{1/2}),Y_2\le Y_1}{=} O(1)\cdot n^{1-\alpha}\int_{-I_n}^{I_n}\int_0^H\int_0^{y_1}Y_1^2e^{-\alpha y_1}e^{-\alpha y_2}dy_2dy_1dx_1 \\
&\overset{\alpha\ge 1}{=} O(1)\cdot n^{2-\alpha}.
\end{aligned} \tag{B.6}$$

Note also that for any $p\in D([0,H])$

$$\begin{aligned}
\mathbb{E}[\xi(p,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p\})] &= \mathbb{E}[\hat{\xi}(p,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p\})]\cdot\exp\left(-\mu(B(p)\cap D([\frac{R}{2},R]))\right) \\
&= \mathbb{E}[\hat{\xi}(p,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p\})]\left(1+O(n^{-(\alpha-\frac{1}{2})})\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big|\mathbb{E}[\hat{\xi}(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1\})]\mathbb{E}[\hat{\xi}(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_2\})]- \\
&\qquad\qquad \mathbb{E}[\xi(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1\})]\mathbb{E}[\xi(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_2\})]\Big| \\
&= O(n^{-(\alpha-\frac{1}{2})})\cdot\mathbb{E}\left[\xi(p_1,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_1\})\right]\mathbb{E}\left[\xi(p_2,(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{p_2\})\right] \\
&= O(n^{-(\alpha-\frac{1}{2})}).
\end{aligned}$$

So

$$\begin{aligned}
&\int_{E_1}\Big|\mathbb{E}\left[\hat{\xi}((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})\right]\mathbb{E}\left[\hat{\xi}((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_2,y_2)\})\right] \\
&\qquad -\mathbb{E}\left[\xi((x_1,y_1),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_1,y_1)\})\right]\mathbb{E}\left[\xi((x_2,y_2),(\tilde{\mathcal{P}}_\alpha\cap D)\cup\{(x_2,y_2)\})\right]\Big|\times \\
&\quad e^{-\alpha y_1}e^{-\alpha y_2}dx_2dy_2dy_1dx_1 \\
&= O(n^{-(\alpha-\frac{1}{2})})\cdot\int_{-I_n}^{I_n}\int_0^H\int_0^{y_1}\int_{-I_n}^{I_n}\mathbf{1}\left(|x_2-x_1|_\Phi<2e^{R/4}(Y_1+Y_2)\right) \\
&\qquad\times e^{-\alpha y_1}e^{-\alpha y_2}dx_2dy_2dy_1dx_1 \\
&= O(n^{-(\alpha-\frac{1}{2})})\cdot e^{R/4}\int_{-I_n}^{I_n}\int_0^H\int_0^H(Y_1+Y_2)e^{-\alpha y_1}e^{-\alpha y_2}dy_2dy_1dx_1 \\
&\overset{Y_1,Y_2\ge 1,Y_i=e^{y_i/2}}{=} O(n^{-(\alpha-\frac{1}{2})})\cdot e^{R/4}\int_{-I_n}^{I_n}\int_0^H\int_0^H Y_1\cdot Y_2Y_1^{-1-2\alpha}Y_2^{-1-2\alpha}dY_2dY_1dx_1 \\
&\overset{e^{R/4}=O(n^{1/2})}{=} O(n^{1+\frac{1}{2}-(\alpha-\frac{1}{2})})=O(n^{2-\alpha}).
\end{aligned} \tag{B.7}$$

Combining (B.6) and (B.7) we obtain (B.3) as desired.

Now we establish (B.4). In particular, Lemma 3.3(ii) implies that for any $(p_1, p_2) \in E \setminus E_1$, with $t = |x(p_1) - x(p_2)|_\Phi$ we have

$$\mu(\mathrm{S}^{\pm}_{p_1 p_2}) = O(1) \cdot t^{1-2\alpha}(Y_1 + Y_2)^{2\alpha} + n^{1-2\alpha} \cdot Y_1^{2\alpha}.$$

But $t > 2e^{R/4}(Y_1 + Y_2)$ which implies that $t^{1-2\alpha}(Y_1 + Y_2)^{2\alpha} = O(1) \cdot n^{-(\alpha-\frac{1}{2})}(Y_1 + Y_2)$. Since $Y_2 \leq Y_1 \leq R^2$, we deduce that

$$\mu(\mathrm{S}^{\pm}_{p_1 p_2}) = O(1) \cdot t^{1-2\alpha} Y_1 = O(R^2) \cdot n^{-(\alpha-1/2)} = o(1).$$

So by (3.3) we conclude that

$$c^{\xi}(p_1, p_2) = O(1) \cdot t^{1-2\alpha} Y_1 \mathbb{E}[\xi(p_1, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p_1\})] \mathbb{E}[\xi(p_2, (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{p_2\})].$$

Therefore, setting $t = |x_1 - x_2|$

$$\begin{aligned} &\left| \int_{E \setminus E_1} c^{\xi}((x_1, y_1), (x_2, y_2)) e^{-\alpha y_1} e^{-\alpha y_2} dx_2 dy_2 dy_1 dx_1 \right| \\ &\overset{t > 2e^{R/4}(Y_1 + Y_2) > e^{R/4}}{=} O(1) \cdot \int_{-I_n}^{I_n} \int_0^H \int_0^H \int_{e^{R/4}}^{I_n} t^{1-2\alpha} Y_1 e^{-\alpha y_1} e^{-\alpha y_2} dt dy_2 dy_1 dx_1 \\ &= O(1) \cdot n \cdot e^{2(1-\alpha)R/4} = O(n^{2-\alpha}), \end{aligned}$$

where we recall $Y_1 := e^{y_1/2}, \alpha \in (1, \infty)$. Thus (B.4) holds.

To finish the bound on $|\mathrm{Var}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]|$, we also need to bound $\left| \mathbb{E}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] \right|$, from which will conclude the second part of Lemma 6.2.

Applying the Campbell-Mecke formula (4.1), we get

$$\begin{aligned} &\left| \mathbb{E}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] \right| \\ &= \left| \mathbb{E} \sum_{p \in \tilde{\mathcal{P}}_\alpha \cap D([0,H])} \left(\hat{\xi}(p, \tilde{\mathcal{P}}_\alpha \cap D) - \xi(p, \tilde{\mathcal{P}}_\alpha \cap D) \right) \right| \\ &\leq \int_{-I_n}^{I_n} \int_0^H \mathbb{E}[\hat{\xi}((x,y), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x,y)\}) - \xi((x,y), (\tilde{\mathcal{P}}_\alpha \cap D) \cup \{(x,y)\})] e^{-\alpha y} dy dx. \end{aligned}$$

But by (B.5), for any $p \in D([0, H])$ we have

$$\mathbb{E}[\hat{\xi}(p, \tilde{\mathcal{P}}_\alpha \cap D) - \xi(p, \tilde{\mathcal{P}}_\alpha \cap D)] \leq \mu(B(p) \cap D([R/2, R])) = O(n^{-(\alpha - \frac{1}{2})}) \cdot e^{y(p)/2}.$$

Substituting this into the above integral we get

$$\left| \mathbb{E}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] \right| = O(1) \cdot n^{\frac{3}{2} - \alpha}.$$

Combining (B.3) and (B.4) with this, we deduce that $|\mathrm{Var}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]| = O(n^{2-\alpha}) \overset{\alpha \geq 1}{=} o(n)$, which shows (B.2).

Furthermore, since $\sigma_n' = \sqrt{\mathrm{Var}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)]} = O(n^{1/2})$ we obtain for $\alpha \in (1, \infty)$

$$\frac{\left| \mathbb{E}[\widehat{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] - \mathbb{E}[\tilde{S}_H^{iso}(\tilde{\mathcal{P}}_\alpha \cap D)] \right|}{\sigma_n'} = O(n^{\frac{3}{2} - \alpha - \frac{1}{2}}) = O(n^{1-\alpha}) = o(1),$$

which concludes the proof of the second part of the lemma. □

C Proof of Lemma 6.6

For γ_1 we have:

$$\gamma_1 = O(1)\cdot \int_{-I_n}^{I_n}\int_{-I_n}^{I_n}\int_{-I_n}^{I_n}\int_{[0,R/2]^3} \left(\mathbb{E}\left[(\nabla_{p_2}F)^2(\nabla_{p_3}F)^2\right]\right)^{1/2}\left(\mathbb{E}\left[(\nabla^2_{p_1,p_2}F)^2(\nabla^2_{p_1,p_3}F)^2\right]\right)^{1/2} \times e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3dx_3dx_2dx_1$$

$$= O(1)\cdot \int_{-I_n}^{I_n}\int_{-I_n}^{I_n}\int_{-I_n}^{I_n}\int_{[0,R/2]^3} \left(\mathbb{E}\left[(\nabla_{p_2}F)^2(\nabla_{p_3}F)^2\right]\right)^{1/2}\left(\mathbb{E}\left[(\nabla^2_{p_1,p_2}F)^2(\nabla^2_{p_1,p_3}F)^2\right]\right)^{1/2} \times \mathbf{1}(|x_2-x_1|_\Phi \le 2\gamma e^{H/2}e^{(y_1\vee y_2)/2})\mathbf{1}(|x_3-x_1|_\Phi \le 2\gamma e^{H/2}e^{(y_1\vee y_3)/2}) \times e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3dx_3dx_2dx_1.$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}\mathbb{E}\left[(\nabla_{p_2}F)^2(\nabla_{p_3}F)^2\right] &\le \mathbb{E}\left[(\nabla_{p_2}F)^4\right]^{1/2}\mathbb{E}\left[(\nabla_{p_3}F)^4\right]^{1/2}\\ &\le \frac{1}{\sigma_n'^4}\mathbb{E}\left[(X(p_2)+1)^4\right]^{1/2}\mathbb{E}\left[(X(p_3)+1)^4\right]^{1/2}\\ &= O(1)\cdot\frac{1}{\sigma_n'^4}e^{y_2+y_3},\end{aligned}$$

where the last equality follows by Lemma 6.3. Therefore,

$$\mathbb{E}\left[(\nabla_{p_2}F)^2(\nabla_{p_3}F)^2\right]^{1/2} = O(1)\cdot\frac{1}{\sigma_n'^2}e^{\frac{1}{2}(y_2+y_3)}. \tag{C.1}$$

Using (6.10) and (C.1) and integrating first with respect to x_2 and x_3, we get

$$\begin{aligned}\gamma_1^2 &= O(1)\cdot e^{2H}\frac{1}{\sigma_n'^4}\int_{-I_n}^{I_n}\int_{[0,R/2]^3} e^{y_2/2+y_3/2}e^{(y_1\vee y_2)/2}\cdot e^{(y_1\vee y_3)/2}\cdot e^{(y_1\wedge y_2)/2}\cdot e^{(y_1\wedge y_3)/2}\\ &\qquad\times e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3dx_1\\ &= O(1)\cdot e^{2H}\frac{n}{\sigma_n'^4}\int_{[0,R/2]^3} e^{y_2/2+y_3/2}e^{(y_1\vee y_2)/2}\cdot e^{(y_1\vee y_3)/2}\cdot e^{(y_1\wedge y_2)/2}\cdot e^{(y_1\wedge y_3)/2}\\ &\qquad\times e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3,\end{aligned} \tag{C.2}$$

where we use $I_n = \Theta(n)$ in the last equality. We will bound the integral in (C.2) by considering the four sub-domains we considered for the bound on γ_2. We start with D_1 on which $y_1 \le y_2, y_3$:

$$\begin{aligned}&\int_{D_1} e^{y_2/2+y_3/2}\cdot e^{(y_1\vee y_2)/2}\cdot e^{(y_1\vee y_3)/2}e^{(y_1\wedge y_2)/2}\cdot e^{(y_1\wedge y_3)/2}\cdot e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3\\ &\quad= \int_{D_1} e^{y_2/2+y_3/2}e^{y_2/2}\cdot e^{y_3/2}\cdot e^{y_1}\cdot e^{-\alpha y_1}e^{-\alpha y_2}e^{-\alpha y_3}dy_1dy_2dy_3\\ &\quad= \int_{D_1} e^{(1-\alpha)y_1}e^{(1-\alpha)y_2}e^{(1-\alpha)y_3}dy_2dy_3dy_1\\ &\quad\le \int_{[0,R/2]^3} e^{(1-\alpha)y_1}e^{(1-\alpha)y_2}e^{(1-\alpha)y_3}dy_2dy_3dy_1 \overset{\alpha\ge 1}{=} O(1).\end{aligned}$$

On D_2, where $y_2 \le y_1 \le y_3$ we have:

$$\begin{aligned}
&\int_{D_2} e^{y_2/2+y_3/2} \cdot e^{(y_1 \vee y_2)/2} \cdot e^{(y_1 \vee y_3)/2} e^{(y_1 \wedge y_2)/2} \cdot e^{(y_1 \wedge y_3)/2} \cdot e^{-\alpha y_1} e^{-\alpha y_2} e^{-\alpha y_3} dy_1 dy_2 dy_3 \\
&= \int_{D_2} e^{y_2/2+y_3/2} \cdot e^{y_1/2} \cdot e^{y_3/2} \cdot e^{y_2/2} \cdot e^{y_1/2} \cdot e^{-\alpha y_1} e^{-\alpha y_2} e^{-\alpha y_3} dy_1 dy_2 dy_3 \\
&= \int_{D_2} e^{(1-\alpha) y_1} e^{(1-\alpha) y_2} e^{(1-\alpha) y_3} dy_2 dy_3 dy_1 \\
&\le \int_{[0,R/2]^3} e^{(1-\alpha) y_1} e^{(1-\alpha) y_2} e^{(1-\alpha) y_3} dy_2 dy_3 dy_1 = O(1).
\end{aligned}$$

By symmetry, integration on D_3 gives the same upper bound. Finally, on D_4 where $y_2, y_3 \le y_1$ we get

$$\begin{aligned}
&\int_{D_4} e^{y_2/2+y_3/2} \cdot e^{(y_1 \vee y_2)/2} \cdot e^{(y_1 \vee y_3)/2} e^{(y_1 \wedge y_2)/2} \cdot e^{(y_1 \wedge y_3)/2} \cdot e^{-\alpha y_1} e^{-\alpha y_2} e^{-\alpha y_3} dy_1 dy_2 dy_3 \\
&= \int_{D_4} e^{y_2/2+y_3/2} \cdot e^{y_1} e^{y_2/2+y_3/2} \cdot e^{-\alpha y_1} e^{-\alpha y_2} e^{-\alpha y_3} dy_1 dy_2 dy_3 \\
&= \int_{D_4} e^{(1-\alpha) y_1} e^{(1-\alpha) y_2} e^{(1-\alpha) y_3} dy_2 dy_3 dy_1 \\
&\le \int_{[0,R/2]^3} e^{(1-\alpha) y_1} e^{(1-\alpha) y_2} e^{(1-\alpha) y_3} dy_2 dy_3 dy_1 = O(1).
\end{aligned}$$

Combining these four upper bounds into (C.2) we obtain $\gamma_1 = O(1) \cdot e^H n^{1/2} / {\sigma'_n}^2 = o(1)$ as desired, where the last equality follows since $\sigma'_n = \Theta(n^{1/2})$. This completes the proof of Lemma 6.6. □

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